Reconsidering Nash: the Nash equilibrium is inconsistent

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Abstract

This paper tries to rekindle the revolution in economic theory that Von Neumann and Morgenstern intended to unleash, but that was smothered by Nash. The assumption of rational, maximising behaviour by the participants in a social exchange economy leads to a set of interdependent maximum problems. The appropriate treatment of this kind of problem is provided by the theory of simultaneous maximisation, systematically studied for the first time by Bruno de Finetti. Cooperative game theory is a branch of simultaneous maximisation. Non-cooperative game theory errs in deriving the first-order conditions and yields conflicting outcomes for equivalent problems; a prime example is the Cournot equilibrium's differing from the Bertrand equilibrium. The analysis implies that large parts of positive economics must be revised, that normative economics is impossible, and that a stronger assumption is needed for economics to make determinate predictions of how money makes the world go round.

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JEL: C61, C70, C72, D43

> To treat *variables as constants* is the characteristic vice of the unmathematical economist.
>
> **F. Y. Edgeworth (1881, p. 127)**

1. Introduction

Economic theory assumes that the participants in a social exchange economy are rational, optimising agents. Consumers maximise their utilities, producers maximise their profits; they all do so subject to whatever constraints are present. Many textbooks on microeconomic theory contain appendices dealing with the mathematical theory of constrained maximisation of one function (for example, Malinvaud, 1972; Varian, 1992), and several textbooks on optimisation have been written especially for economists (for example, Dixit, 1976, 1990; Léonard and Van Long, 1992).

In economics the typical case is that the actions of each agent affect the outcomes for all participants. The theory thus results in a mathematical problem in which a number of functions with the same list of arguments must be simultaneously maximal, a *simultaneous maximum problem* for short. Von Neumann and Morgenstern (1947, Section I.2) argued that, at the time of their writing, mathematical economics had not dealt adequately with this type of problem. They constructed game theory to remedy the error, confining their analysis to the case of discrete decision variables that may take only a finite number of values. A decade earlier De Finetti (1937a,b), motivated by the work of Pareto, had considered the problem of simultaneously maximising a number of smooth functions of continuous variables.

Because economic theory takes some simultaneous maximum problem as its model of the world, you would expect textbooks on optimisation for economists, like Dixit (1976, 1990) and Léonard and Van Long (1992), to pay ample attention to simultaneous maximisation, and the economic literature to be full of applications of the technique.

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Nothing is less true. Yes, *cooperative* game theory is true to simultaneous maximisation. But the dominating branch of game theory, *non-cooperative* game theory, is not.

In this paper I argue that non-cooperative game theory is mathematically flawed. The error appears to be incorrect conditioning on endogenous variables in deriving the first-order conditions for simultaneous maximum problems. I use the oligopoly problem, the Prisoner's Dilemma, and dynamic general equilibrium models with imperfect competition and rational expectations to illustrate the error. I point out some major implications of the analysis and suggest a way for economic theory to move ahead.

2. Simultaneous maximisation

In this section I briefly review the mathematical theory of simultaneous maximisation, a close relative of the constrained maximisation of a single function. The treatment of simultaneous maximisation follows that of De Finetti (1937a,b).[2] As I do not present new mathematical results, I keep the exposition simple. I take the decision variables to be continuous and use classical differential calculus rather than the theory of convex sets in the more modern approach. Functions are real-valued and continuously differentiable of the degree required for the results to be valid. I consider only first-order conditions. In the subsequent sections I shall show that non-cooperative game theory, and therefore also a large part of economic theory, even fails to arrive at the correct first-order conditions of the maximum problems that it postulates; in this circumstance, second-order conditions are of secondary importance. I neglect border extremes and the questions of existence and uniqueness, too.

2.1. Maximising one function subject to a constraint

As a prelude to the simultaneous maximisation of several functions, consider the constrained maximisation of a single function. To have a concrete example, consider the problem of budget-constrained utility maximisation. Let q_j be the quantity of good j and p_j its price, $j = 1,\dots,J$. The utility υ of some consumer is an increasing function $u(\cdot)$ of the basket of goods $\mathbf{q}$, $\upsilon = u(q_1,\dots,q_J)$, with $u_j := \partial u/\partial q_j > 0$ for all j. The consumer wants maximal utility at the given prices, while she has no more than some fixed amount E of money to spend. The mathematical problem is

$$\begin{aligned} &\max_{q_j} u(q_1,\dots,q_J), \qquad j = 1,\dots,J, \\ &\text{subject to } \sum_k^J p_k q_k = E. \end{aligned} \tag{1}$$

There are several ways of solving the problem, for example, by using the budget constraint to eliminate one of the arguments from the utility function and proceeding with the unconstrained maximisation of the resulting function of $J-1$ variables, or by applying the method of Lagrange. Here I follow De Finetti (1937a,b) in deriving the first-order conditions. To a first-order approximation the changes in the consumer's utility and budget are

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_J \\ p_1 & p_2 & \cdots & p_J \end{bmatrix} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \\ \vdots \\ \mathrm{d}q_J \end{bmatrix} = \begin{bmatrix} \mathrm{d}\upsilon \\ 0 \end{bmatrix}. \tag{2}$$

If the matrix of first-order derivatives of the maximand and constraint has full rank, it is always possible to find a solution of (2) with a positive value of $\mathrm{d}\upsilon$. A necessary condition for utility to be maximal subject to the budget constraint is thus that the matrix have deficient rank. Then all its square submatrices of order 2 are singular, which yields the $J-1$ independent conditions

$$\frac{u_1}{p_1} = \frac{u_2}{p_2} = \cdots = \frac{u_J}{p_J}. \tag{3}$$

These conditions and the budget constraint define the utility-maximising quantities q_j^* as functions of the prices and the budget, $q_j^* = G_j(p_1,\dots,p_J,E)$, $j = 1,\dots,J$.

[2] Nieuwenhuis (2011) gives translations of these articles.

REMARK 1 All three methods of solving (1) lead to the same outcome. In particular, the method of elimination yields the same outcome, whichever variable you choose to eliminate.

2.2. *Maximising several functions simultaneously*

2.2.1. *Definition of simultaneous maximality*

Let us now turn to the simultaneous maximisation of several functions,

$$\max_{a_j} w_i(a_1,\ldots,a_J), \qquad i=1,\ldots,I,\ j=1,\ldots,J, \tag{4}$$

where $J \geq I$. De Finetti (1937a) gives a number of examples of problems that can be cast in this mould, some in geometry, others in physics. The question arises, "What does it mean to maximise several functions simultaneously? Will raising the value of one function not conflict with raising the values of the other functions?" The answer is, "Not necessarily." In general, there are many points from which you may travel in a direction such that the value of each function increases. Such points cannot be a solution of the problem. Consider the following definition of simultaneous maximality:

DEFINITION 1 (SIMULTANEOUS MAXIMALITY) *The functions $w_i(\cdot)$, $i=1,\ldots,I$, with arguments $a_j, j=1,\ldots,J$, $J \geq I$, are simultaneously maximal at some point $\mathbf{a}^*$ if it is not possible to increase the values of all functions simultaneously by moving away from $\mathbf{a}^*$.*

This definition is used implicitly by De Finetti (1937a) in deriving the first-order conditions for problems like (4). It is a natural generalisation of maximality from the case of one function to the case of several functions. Simultaneous maximality in this sense is equivalent to (weak) Pareto optimality. As a matter of fact, the motivation for De Finetti's article was to clarify the notion of optimum he had encountered in the writings of Pareto. As Pareto optimality has acquired a normative connotation in economics, I prefer the neutral term "simultaneous maximality" stemming from Zaccagnini (1947).

2.2.2. *The first-order conditions*

Elementary linear algebra suffices for stating the first-order conditions for simultaneous maximality. Consider the first-order approximation of the changes in the maximands,

$$\mathbf{W}\,\mathrm{d}\mathbf{a} = \mathrm{d}\boldsymbol{\omega}, \tag{5}$$

where $\mathbf{W} := [\partial w_i/\partial a_j]$, the $I \times J$ matrix of partial first-order derivatives of the maximands. Obviously the set of functions cannot be simultaneously maximal at some point $\mathbf{a}^*$ if, and only if, the space spanned by the columns of $\mathbf{W}(\mathbf{a}^*)$ contains a strictly positive vector. If the matrix $\mathbf{W}$ has full rank, it is always possible to find a solution of (5) with a strictly positive vector $\mathrm{d}\boldsymbol{\omega}$. Therefore a necessary condition for the functions $w_i(\cdot)$, $i=1,\ldots,I$, to be simultaneously maximal at some point $\mathbf{a}^*$ is that the matrix $\mathbf{W}(\mathbf{a}^*)$ have deficient rank. The condition requires that the determinants of all square submatrices of $\mathbf{W}$ of order I be zero, which yields $J-I+1$ independent scalar equations. Hence, in general the condition defines a variety of $J-(J-I+1)=I-1$ dimensions in J-dimensional $\mathbf{a}$-space, which is or contains the simultaneous maximum. For $I=1$ the condition is equivalent to the familiar first-order conditions that the partial first-order derivatives vanish.

Deficient rank of the matrix $\mathbf{W}$ is only a necessary, not a sufficient condition for the non-existence of a solution of (5) with a strictly positive vector $\mathrm{d}\boldsymbol{\omega}$. For obtaining the supplementary condition, consider a left eigenvector of $\mathbf{W}$ with eigenvalue zero:

$$\boldsymbol{\lambda}'\mathbf{W} = 0.$$

When the rank of $\mathbf{W}$ is $I-1$, $\boldsymbol{\lambda}$ is unique up to a multiplicative constant (let $\mathbf{W}_I$ denote any square submatrix of order I of $\mathbf{W}$; $\boldsymbol{\lambda}$ is proportional to the vector of cofactors of the elements of some column of $\mathbf{W}_I$). If two elements of $\boldsymbol{\lambda}$ have opposite signs, a strictly positive vector $\mathrm{d}\boldsymbol{\omega}$ exists such that $\boldsymbol{\lambda}'\mathrm{d}\boldsymbol{\omega}=0$. The existence of such a vector $\boldsymbol{\lambda}$ is a necessary and sufficient condition for the existence of a solution of (5) with strictly positive vector $\mathrm{d}\boldsymbol{\omega}$: the functions are not simultaneously maximal. If $\boldsymbol{\lambda}$ is positive or negative (some elements may be zero), no strictly positive vector $\mathrm{d}\boldsymbol{\omega}$ exists such that $\boldsymbol{\lambda}'\mathrm{d}\boldsymbol{\omega}=0$. Now the necessary conditions for a simultaneous maximum are fulfilled.

More generally, when the rank of $\mathbf{W}$ is $I-k$, $I \geq k \geq 1$, a k-dimensional subspace of I-dimensional vectors $\boldsymbol{\lambda}$ exists such that $\boldsymbol{\lambda}'\mathbf{W} = 0$. A necessary and sufficient condition for the existence of a solution of (5) with strictly positive vector $\mathrm{d}\boldsymbol{\omega}$ is the non-existence of a positive vector $\boldsymbol{\lambda}$ in this space. Let me state this result formally:

THEOREM 1 *Let $w_i(\cdot)$, $i = 1, \ldots, I$, be a set of functions in J variables a_j, $j = 1, \ldots, J$, $J \geq I$, with $\mathbf{W}(\mathbf{a})$ the $I \times J$ matrix of its partial first-order derivatives. The set of functions cannot be simultaneously maximal at some point $\mathbf{a}^*$ if, and only if, the orthogonal complement of the space spanned by the columns of $\mathbf{W}(\mathbf{a}^*)$ does not contain a positive vector.*

REMARK 2 Let $\mathbf{a}^*$ be a point of the simultaneous maximum where the rank of $\mathbf{W}$ is $I-1$ and the left eigenvector $\boldsymbol{\lambda}$ of $\mathbf{W}$ with eigenvalue zero is *strictly* positive. Normalise $\boldsymbol{\lambda}$ such that one of its elements, for example the kth, equals 1. Define $c_i := w_i(\mathbf{a}^*)$, $i = 1, \ldots, I$. Then c_k is the maximum of $w_k(\cdot)$ subject to the constraints $w_i(\cdot) = c_i$, $i \neq k$, with Lagrange multipliers λ_i.

EXAMPLE 1 (MAXIMAL UTILITY AT MINIMAL EXPENDITURE) A consumer wants maximal utility at minimal expenditure.[3] Define the two maximands $w_1(\mathbf{q}) := u(\mathbf{q})$, $w_2(\mathbf{q}) := -\mathbf{p}'\mathbf{q}$. The matrix of first-order derivatives is

$$\mathbf{W} = \begin{bmatrix} u_1 & u_2 & \cdots & u_J \\ -p_1 & -p_2 & \cdots & -p_J \end{bmatrix}.$$

All its square submatrices of order 2 must be singular, which again yields the $J-1$ independent conditions (3). As for each 2×2 submatrix of $\mathbf{W}$ the cofactors of the elements of either column have the same sign, the first-order conditions are met with in all points satisfying (3). Figure 1 sketches the solution. It is easy to convert the simultaneous maximum problem into the constrained maximum problem with one of the maximands set to a predetermined (feasible) value. Algebraically, all you have to do is to add the constraint thus obtained to the first-order conditions (3). Geometrically, the solution of the constrained maximum problem is the intersection of the constraint with the one-dimensional variety defined by the condition that $\mathbf{W}$ have deficient rank.

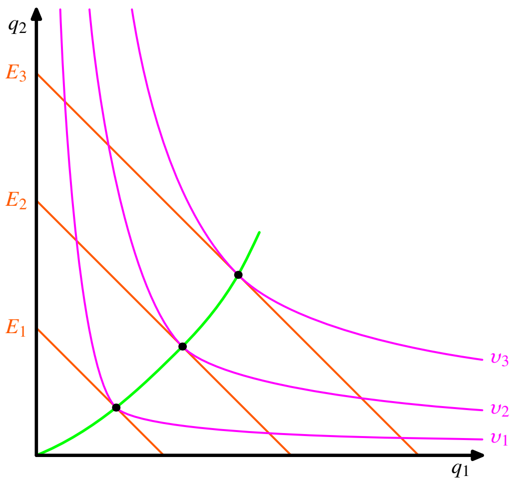

Figure 1: Maximal utility at minimal expenditure

2.2.3. The tangent space

Geometrically, the variety defined by the condition that $\mathbf{W}$ have deficient rank is the set of points in which the level surfaces of the maximands have a $(J-I+1)$-dimensional tangent space in common. For every vector of variations $\mathrm{d}\mathbf{a}$ in the tangent space,

$$\mathbf{W}\,\mathrm{d}\mathbf{a} = 0.$$

[3]Cf. Section 2.1 and Zaccagnini (1951, p. 218).

Restate the i-th equation as the condition that the total derivative of $w_i(\cdot)$ with respect to a_i equal zero. In scalar notation the result is

$$\sum_j w_{ij}\alpha_{ji} = 0, \qquad i = 1,\dots,I, \tag{6}$$

where $\alpha_{ji} := \mathrm{d}a_j/\mathrm{d}a_i$ denotes the marginal rate of substitution between a_j and a_i along some common tangent line to the level curves of the maximands at some point of the simultaneous maximum (the "own" marginal rates of substitution α_{ii} equal 1).

If $I = J$, the matrix $\mathbf{W}$ is square, and a first-order condition is that its determinant vanishes: $|\mathbf{W}| = 0$. Suppose that the rank of $\mathbf{W}$ is $I-1$; then the right eigenvector with eigenvalue zero is unique up to a multiplicative constant. This eigenvector is proportional to the vector of cofactors of the elements of some row of $\mathbf{W}$. It follows that

$$\alpha_{ji} = |\mathbf{W}_{ij}|^* / |\mathbf{W}_{ii}|^*, \qquad i,j = 1,\dots,I, \tag{7}$$

with $|\mathbf{W}_{ij}|^*$ the cofactor of w_{ij}. Substitute (7) for α_{ji} into (6) and multiply by $|\mathbf{W}_{ii}|^*$ to obtain

$$\sum_j w_{ij}|\mathbf{W}_{ij}|^* = 0, \qquad i = 1,\dots,I. \tag{8}$$

The left-hand side is the determinant of the matrix $\mathbf{W}$ expanded in terms of the elements of row i and their cofactors: (8) represents I equivalent versions of the condition $|\mathbf{W}| = 0$. All I^2 marginal rates of substitution α_{ji} derive from a single vector d$\mathbf{a}$ and satisfy

$$\alpha_{jk}\alpha_{ki} = \alpha_{ji}, \qquad i,j,k = 1,\dots,I. \tag{9}$$

The matrix $\mathbf{A} := [\alpha_{ji}]$ has rank 1, and all its diagonal elements are 1: each column $\boldsymbol{\alpha}_{\cdot i}$ of $\mathbf{A}$ is a differently scaled version of the vector in the nullspace of $\mathbf{W}$.

Similarly, if $I < J$, you may derive $J - I + 1$ independent J-dimensional vectors spanning the tangent space from the condition that all square submatrices of order I have rank $I - 1$.

2.3. Constrained simultaneous maximisation

De Finetti (1937b) considers simultaneous maximum problems with constraints on the variables and gives a number of examples, one of which in economics. Their solution is a straightforward extension of the solutions of (1) and (4), as the following example shows.

2.3.1. The problem

Let the functions $w_i(\cdot)$ be defined by

$$w_i(p_i, q_i) := p_i q_i - c_i(q_i), \qquad i = 1,\dots,I.$$

All $2I$ arguments must take positive values. I assume that an interior solution exists, and dismiss the non-binding positivity constraints right from the start. There are I independent other constraints, which the arguments must satisfy with equality:

$$q_i = g_i(p_1,\dots,p_I), \qquad i = 1,\dots,I, \tag{10}$$

or, equivalently,

$$p_i = f_i(q_1,\dots,q_I), \qquad i = 1,\dots,I. \tag{11}$$

Define $\boldsymbol{\Phi} := [\varphi_{ij}] = [\partial \ln f_i / \partial \ln q_j]$, $\mathbf{X} := [\chi_{ij}] = [\partial \ln g_i / \partial \ln p_j]$, where $\boldsymbol{\Phi}(\mathbf{q}^*) \equiv \big(\mathbf{X}(\mathbf{p}^*)\big)^{-1}$ when $\mathbf{q}^* \equiv \mathbf{g}(\mathbf{p}^*)$.

Consider the following problem:

$$\max_{p_i \text{ or } q_i} w_i(p_i, q_i), \qquad i = 1,\dots,I, \qquad \text{subject to (10) or (11).} \qquad \text{Problem PQ}$$

Although the maximands do not have arguments in common, Problem PQ is still a simultaneous maximum problem because the arguments are interrelated through the constraints. You can write the problem in the form of (4) by using the constraints to eliminate I of the variables. There are 2^I such problems; I consider only two of them.

Eliminating the qs from $w_i(\cdot)$ by means of (10) yields

$$\max_{\ln p_i} v_i(\ln p_1,\ldots,\ln p_I), \qquad i=1,\ldots,I, \qquad \text{Problem P}$$

where $v_i(\ln p_1,\ldots,\ln p_I) := p_i g_i(p_1,\ldots,p_I) - c_i\{g_i(p_1,\ldots,p_I)\}$. Define $\mathbf{V} := [v_{ij}] = [\partial v_i/\partial \ln p_j]$. Eliminating the ps from $w_i(\cdot)$ by means of (11) yields

$$\max_{\ln q_i} u_i(\ln q_1,\ldots,\ln q_I), \qquad i=1,\ldots,I, \qquad \text{Problem Q}$$

where $u_i(\ln q_1,\ldots,\ln q_I) := f_i(q_1,\ldots,q_I)\,q_i - c_i(q_i)$. Define $\mathbf{U} := [u_{ij}] = [\partial u_i/\partial \ln q_j]$. There holds $\mathbf{U} \equiv \mathbf{V\Phi}$ (and $\mathbf{V} \equiv \mathbf{UX}$).

REMARK 3 Problem PQ, Problem P and Problem Q are mathematically equivalent versions of one and the same problem: eliminating some of the variables with the help of equalities does not change the problem at all, just as in the case of maximising one function. You can always go back to the original problem, or change over from Problem P to Problem Q by applying a nonsingular transformation of variables.

2.3.2. The first-order conditions: equivalence of alternative formulations

For any positive variable x, write $\tilde{x}$ for $\mathrm{d}\ln x$. Define $mc_i := \partial c_i/\partial q_i$, $m_i := (p_i - mc_i)/p_i$, $i=1,\ldots,I$. Let $\hat{\mathbf{m}}$ be the diagonal matrix with the vector $\mathbf{m}$ as its main diagonal and $\mathbf{I}$ the identity matrix of the appropriate dimension (here I).

Each function $w_i(\cdot)$ has two arguments, $\ln p_i$ and $\ln q_i$. There is a common set of I constraints, (10) or (11), leaving I degrees of freedom. An infinitesimal variation $\tilde{q}_i$ affects $\ln p_i$ by $\tilde{p}_i = \varphi_{ii}\tilde{q}_i$; simultaneously it affects $\ln p_j$ ($j \neq i$) as well, by $\tilde{p}_j = \varphi_{ji}\tilde{q}_i$. Define the infinitesimal variation in the value of $w_i(\cdot)$ induced by the infinitesimal variations $\tilde{p}_i$ and $\tilde{q}_i$ to be $p_i q_i\,\mathrm{d}\omega_i$. A little algebra yields the system of equations

$$\begin{bmatrix} \hat{\mathbf{m}} & \mathbf{I} \\ \mathbf{\Phi} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}} \\ \tilde{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathrm{d}\boldsymbol{\omega} \\ \mathbf{0} \end{bmatrix}.$$

The first set of I equations expresses how the maximands $w_i(\cdot)$ vary with their two arguments, and the latter set of I equations expresses that the constraints must remain satisfied, all in a first-order approximation. If, at some point, the system has a solution with a strictly positive vector $\mathrm{d}\boldsymbol{\omega}$ on the right-hand side, then you can raise the values of all maximands: the functions are not simultaneously maximal at that point. A necessary condition for such a solution not to exist is that the $2I \times 2I$ matrix on the left-hand side be singular:

$$\begin{vmatrix} \hat{\mathbf{m}} & \mathbf{I} \\ \mathbf{\Phi} & -\mathbf{I} \end{vmatrix} = 0. \tag{12}$$

You may use the second set of equations to eliminate $\tilde{\mathbf{p}}$ from the first set, reducing the problem to one in I-dimensional $\ln \mathbf{q}$-space:

$$(\hat{\mathbf{m}} + \mathbf{\Phi})\tilde{\mathbf{q}} = \mathrm{d}\boldsymbol{\omega}.$$

This is the system of equations $\mathbf{U}\tilde{\mathbf{q}} = \mathrm{d}\boldsymbol{\omega}$ for Problem Q. The necessary condition $|\mathbf{U}| = 0$ for a simultaneous maximum reads

$$|\hat{\mathbf{m}} + \mathbf{\Phi}| = 0. \tag{13}$$

(13) is equivalent to (12). Alternatively, you may use the second set of equations to eliminate $\tilde{\mathbf{q}}$ from the first set, reducing the problem to one in I-dimensional $\ln \mathbf{p}$-space:

$$(\hat{\mathbf{m}}\mathbf{X} + \mathbf{I})\tilde{\mathbf{p}} = \mathrm{d}\boldsymbol{\omega}.$$

This is the system of equations $\mathbf{V}\tilde{\mathbf{p}} = \mathrm{d}\boldsymbol{\omega}$ for Problem P. An elegant way of writing the necessary condition $|\mathbf{V}| = 0$ for a simultaneous maximum, parallelling (13), is

$$\left|\hat{\mathbf{m}}^{-1} + \mathbf{X}\right| = 0. \tag{14}$$

REMARK 4 The necessary condition $|\mathbf{V}| = 0$ is equivalent to $|\mathbf{U}| = 0$ because $\mathbf{U} \equiv \mathbf{V\Phi}$ and $\mathbf{\Phi}$ is nonsingular: the solution of Problem PQ is the same, whichever variables you choose to eliminate (see also Remark 1 in Section 2.1, on the solution of the constrained maximum problem treated there).

2.3.3. Interpretation

I have stated and solved Problem PQ free from an interpretation in order to prevent non-mathematical reasoning from playing any role in the derivations. The interpretation I have in mind will be clear, though: it is a model of a heterogeneous oligopoly, in which p_i is the price of good i and q_i its quantity, $\mathbf{g}(\cdot)$ is the system of demand functions with $\mathbf{X}$ the matrix of price elasticities and $\mathbf{f}(\cdot)$ the system of inverse demand functions with $\mathbf{\Phi}$ the matrix of quantity elasticities, $c_i(\cdot)$ is the cost function and $w_i(\cdot)$ the profit function of firm i. Please note that I have derived the solution of Problem PQ without any additional assumption on the behaviour of the firms next to the one of profit maximisation, in perfect agreement with the view of Stigler (1964, p. 44):

> A satisfactory theory of oligopoly cannot begin with assumptions concerning the way in which each firm views its interdependence with its rivals. If we adhere to the traditional theory of profit-maximizing enterprises, then behavior is no longer something to be assumed but rather something to be deduced. The firms in an industry will behave in such a way, given the demand- and supply-functions (including those of rivals), that their profits will be maximized.

2.3.4. The tangent space

At each point of the variety defined by (13), a vector of relative variations $\tilde{\mathbf{q}}$ exists such as to leave the value of each maximand unchanged:

$$(\hat{\mathbf{m}} + \mathbf{\Phi})\tilde{\mathbf{q}} = \mathbf{0}.$$

In scalar notation the i-th equation of this system appears to be equivalent to

$$mc_i = p_i(1 + \sum_j \varphi_{ij}\xi_{ji}), \qquad i = 1, \ldots, I, \tag{15}$$

where mc_i is the marginal cost of Firm i (generally dependent on q_i) and where $\xi_{ji} := \tilde{q}_j/\tilde{q}_i$. Analogously to α_{ji} in (7), ξ_{ji} equals $|\mathbf{U}_{ij}|^*/|\mathbf{U}_{ii}|^*$, and each column $\boldsymbol{\xi}_{\cdot i}$ of the matrix $\mathbf{\Xi} := [\xi_{ji}]$ spans the nullspace of $\mathbf{U}$. The elasticities of substitution ξ_{ji} are functions of all quantities, involving parameters of the cost functions and inverse demand functions.

In the same way you may derive for Problem P

$$mc_i = p_i\Big(1 + \big(\sum_j \chi_{ij}\psi_{ji}\big)^{-1}\Big), \qquad i = 1, \ldots, I, \tag{16}$$

where $\psi_{ji} := \tilde{p}_j/\tilde{p}_i = |\mathbf{V}_{ij}|^*/|\mathbf{V}_{ii}|^*$.

(15) and (16) are equivalent, which implies that the ψs and ξs are related in a way that guarantees the equality of the right-hand sides of (15) and (16). Knowing that $\mathbf{\Phi}$ and $\mathbf{X}$ are inverses of one another, you may easily verify that

$$\xi_{ji} = \sum_\ell \chi_{j\ell}\psi_{\ell i} \Big/ \sum_\ell \chi_{i\ell}\psi_{\ell i}, \qquad i, j = 1, \ldots, I, \tag{17}$$

yields the desired result. A similar, equivalent set of relationships expresses the ψs in terms of the ξs and the elements of the matrix $\mathbf{\Phi}$,

$$\psi_{ji} = \sum_\ell \varphi_{j\ell}\xi_{\ell i} \Big/ \sum_\ell \varphi_{i\ell}\xi_{\ell i}, \qquad i, j = 1, \ldots, I. \tag{18}$$

The equations (15) and (16) will return in Section 4.1.3, with a rather different (and, in fact, erroneous) interpretation of the elasticities ξ_{ji} and ψ_{ji}.

2.4. Simultaneous maximisation in economic theory

The notion of simultaneous maximum appears in economics for the first time in Edgeworth's (1881) *Mathematical Psychics*: it is the famous *contract curve*—which is in fact an inequality-constrained version of a simultaneous maximum. But the notion is best known from the work of Pareto, whence the name of Pareto optimum. Pareto (1909, Figure 50, p. 355) represents the contract curve graphically in a figure nowadays called the "Edgeworth Box." As remarked by Hildenbrand (1993), it would be historically correct to speak of "the Edgeworth optimum in the Pareto Box."

EXAMPLE 2 (THE CONTRACT CURVE) Consider an exchange economy with two men and two goods. Let a_j^0, $j = 1,2$, be the shares of the economy's endowments owned initially by the first man, the other one owning the complements. The functions $w_i(a_1,a_2)$, $i = 1,2$, are the men's utility functions. The men may exchange some quantities of the goods among themselves to better their positions. Consider the problem

$$\max_{a_i} w_i(a_1,a_2), \qquad i = 1,2,$$
$$\text{subject to } w_i(a_1,a_2) \geq w_i(a_1^0,a_2^0), \qquad i = 1,2.$$

The inequality restrictions express that exchange is voluntary: neither man may be forced to enter an exchange that worsens his position. The part of the unconstrained simultaneous maximum satisfying the inequality constraints, $[C_1C_2]$ in Figure 2, is the solution of the problem, the *contract curve*.

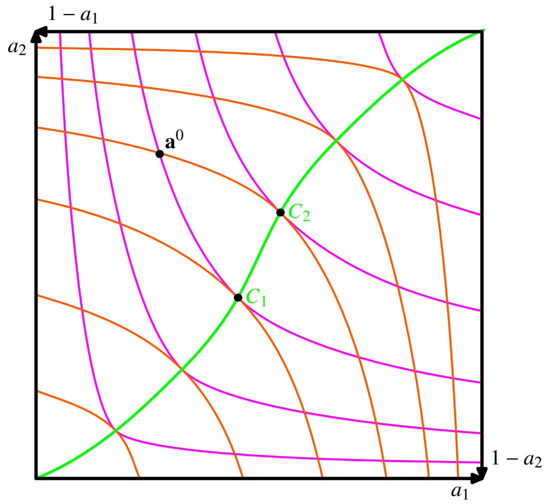

Figure 2: The contract curve

Edgeworth and Pareto both wrote on the theory of duopoly without invoking the notion of simultaneous maximum.[4] Apparently they were not fully aware of the notion's significance and general applicability in economics.

The first to see the importance of the notion of simultaneous maximum, which he attributes to Pareto, is Bruno de Finetti. In two articles, De Finetti (1937a,b), he sketches the mathematical theory of maximising several functions simultaneously and gives a number of examples of its application, in geometry, physics and economics. In De Finetti (1940), he applies the notion of simultaneous maximum in his study of hedging the risk of a set of insurances when determining the optimal retention levels, which yield the best way of reinsuring parts of the insurances so as to reduce the risk (as measured by the variance of profit) within the desired limits while minimising the loss of mean profit.[5] The articles seem not to have received the attention they deserve, maybe because until recently they have been available only in Italian. One reference to the work of De Finetti is by Zaccagnini (1947); this article is available in English as Zaccagnini (1951). Zaccagnini uses the technique of simultaneous maximisation to solve the oligopoly problem and to derive Edgeworth's contract curve. Much later, Smale (1975, 1974b) and others have revisited the subject of simultaneous maximisation. Smale applies simultaneous maximisation to the study of general economic equilibrium using a calculus approach (for example Smale, 1974a, 1976b).

Most of the work mentioned above deals with differentiable functions of continuous variables; the sole exception is the elementary example in Section 2 of De Finetti (1937a). In fact, a branch of simultaneous maximisation with variables that can take only a finite number of values is well-known in economics, be it under a different name: it

[4]Still, Pareto was on the right track by arguing that, in deriving the first-order conditions, the quantities supplied by the firms had to be considered as variables in *both* revenue functions.

[5]Only recently has this work been recognised as anticipating Markowitz (1952).

is *game theory* as conceived by Von Neumann and Morgenstern (1947), that is, the branch of it nowadays called "cooperative game theory." In game theory the interpretation of problem (4) is the following: there is a number of players indexed by i, each with a payoff function $w_i(\cdot)$ and a decision variable a_i, $i = 1, \ldots, I$; all players want maximal payoff.

Reading Von Neumann and Morgenstern (1947, pp. 10–12), one gets the impression that the authors themselves are not aware of their creating a theory of simultaneous maximisation. Nevertheless, their notion of solution, denoted by "solution," for games with variables that can assume only a finite number of values is closely related to the simultaneous maximum (or Pareto optimum); the difference is that in game theory the relation of "dominance" replaces the "larger than" relation in simultaneous maximisation. Example 2 clarifies the difference. All points not belonging to the unconstrained simultaneous maximum are said to be "dominated" by some point(s) of the unconstrained simultaneous maximum, as both men are better off in the latter point(s). Nothing new as yet. However, the requirement of a point not being dominated is stronger than the requirement of it being part of the simultaneous maximum. Some points of the unconstrained simultaneous maximum correspond to exchanges that one of the men rejects, because he would be worse off than in the initial position. By assumption the man is able to block such exchanges through the property rights on his original holdings. The points concerned are also said to be "dominated" by some point on the contract curve, in particular by one of the end points, $\mathbf{C}_1$ and $\mathbf{C}_2$ in Figure 2, where one man is just as well off as in the initial position. (Compare Luce and Raiffa (1957, Section 6.2, p. 118) on the *negotiation set*.)

The relation between the "solution" of a game and the simultaneous maximum, or Pareto optimum, was soon pointed out by two reviewers of the *Theory of Games and Economic Behavior*. Kaysen (1946–1947, p. 7) and Guilbaud (1951, p. 54–5) mention the contract curve of Edgeworth (1881) as an example of the "solution" in a simple case. Guilbaud, in particular, takes great care in explaining how the "solution" generalises the Pareto optimum. Still, when judged by the later development of game theory, it seems that the true meaning of Von Neumann and Morgenstern's work has not been appreciated fully. Maybe their choice of a discrete framework is to blame. In any case, a calculus approach to simultaneous maximum problems facilitates the comparison with other (unconstrained and constrained) maximum problems and highlights the similarities. It also helps at exposing a serious flaw in non-cooperative game theory—the task I take up in the next section.

3. Non-cooperative game theory

3.1. The Nash equilibrium

Non-cooperative game theory advocates that "first-order conditions" of a simultaneous maximum problem be derived by varying variables in one place while holding them constant in other places. Let $\mathbf{a}_{-\mathrm{i}}$ be the $(I-1)$-dimensional vector obtained from the I-dimensional vector $\mathbf{a}$ by deleting its ith element. Non-cooperative game theory attacks the simultaneous maximum problem (4), with $I \equiv J$ (just for simplicity), by splitting it into the I conditional maximum problems

$$\max_{a_i} w_i(a_i \,|\, \mathbf{a}_{-\mathrm{i}}), \qquad i = 1, \ldots, I.$$

The union of the first-order conditions of the conditional maximum problems is

$$w_{ii} = 0, \qquad i = 1, \ldots, I. \tag{19}$$

Its solution is the famous Nash equilibrium, the fundamental "solution concept" of non-cooperative game theory proposed by Nash (1951).

3.2. Interpretation

How does Nash (1951) attempt to justify this procedure? He considers the simultaneous maximum problem in its interpretation of a game, with $w_i(\cdot)$ the payoff function of player i and a_i her decision variable (usually called "action" or "strategy" in game theory). He says that, in contrast to Von Neumann and Morgenstern (1947), he intends to give a model for the case that the players in a game are unable to communicate and cooperate, in other words, unable to co-ordinate their actions. According to Nash, each player considers the actions of the other players as given in that situation. He defines an "equilibrium" as a point where each player's action maximises her payoff *if the actions of the other players are held fixed* (Nash, 1951, p. 287), and writes: "Thus each player's strategy is optimal against those of the others." No player would want to deviate from the Nash equilibrium, because it would be unprofitable to do so. Therefore the "equilibrium" is called "self-enforcing."

3.3. Discussion

What to think of the new "solution concept" for a simultaneous maximum problem and its justification? Does it yield a valid model of the behaviour of players who are unable to cooperate? Definitely not. No matter what the interpretation of the functions and variables is, the derivation of (19) contains an elementary error. The partial cross derivatives $\partial w_i / \partial a_j$ are absent from the "first-order conditions" not because they equal zero, but because each a_j has been varied in only one place—in the jth maximand—while being kept constant in all other places. Being the result of such a faulty derivation, (19) does not even satisfy a *necessary* condition for being a correct model of the world.

An alternative wording of the fault is the following. According to non-cooperative game theory each player builds her own submodel of the world, in which she treats the actions of the other players as exogenous and only her own action as endogenous. The various submodels rest upon conflicting assumptions, so that their union will be an inconsistent model. It will be possible to derive contradictions from the model, proving its inconsistency. As we shall shortly see, theorists have indeed stumbled upon contradictions, amazingly without recognising them as such.

Nash does not change the simultaneous maximum problem in any way: he does not add constraints on the variables, he does not modify the payoff functions, nor does he abandon the assumption of rational, optimising behaviour. He acknowledges this himself when he states (Nash, 1951, p. 286; italics added): "The non-cooperative idea will be *implicit*, rather than explicit, below." Without any change in the problem, the solution remains the same. No player can take the actions of the other players as given in maximising her payoff, as all actions are to be derived from the postulate of rational, optimising behaviour. In other words, *model consistency requires that each player, rather than condition on what the other players do, condition on what they are out for*.[6]

Kaysen (1946–1947, p. 1–2), one of the reviewers of the "Theory of Games and Economic Behavior," aptly describes the essence of the problem dealt with by (cooperative) game theory. I paraphrase his description here, borrowing many of his words and sentences.

In economic theory the device of parametrisation *plays a fundamental role. Entities which are in fact variables of the system (for example, prices) are taken as given for certain economic actors, who respond accordingly; these responses, in turn, affect the values of the parameters taken as given. However, this parametrisation device is inappropriate in certain cases, of which the duopoly problem is the best known example. It is in part the existence of such problems that has led Von Neumann and Morgenstern to seek an entirely new approach to economic problems. But an even more important motive is a fundamental doubt on their part as to the validity of the parametrisation device in nearly all economic situations. The new approach is to be found in the theory of games of strategy. In contrast to a game of chance, a game of strategy—such as chess or bridge—is one in which the personal moves of the players determine the outcome of the game.*

And what is the theory of games of strategy about? As Kaysen (1946–1947, p. 2) says,

> ... The theory of such games of strategy deals precisely with the actions of several agents, in a situation in which all actions are interdependent, and where, in general, there is no possibility of what we called parametrisation, that would enable each agent (player) to behave as if the actions of the others were given. In fact, it is this very lack of parametrisation which is the essence of a game.

I fully agree. With the introduction of the non-cooperative equilibrium, Nash (1951) has deprived game theory of its essence and has frustrated Von Neumann and Morgenstern's attempt to free economics from the error of incorrect conditioning on endogenous variables. This is all the more regrettable, as the impact of non-cooperative game theory on economics has grown enormously. According to Van Damme (1990, p. 1040; my translation), "You may look upon economic theory as a row of dominoes that, one by one, topple under the force of non-cooperative game theory." Modern textbooks on microeconomics pay hardly any attention to cooperative game theory at all. In the textbook of Varian (1992), Chapter 15, titled "Game Theory," deals only with non-cooperative game theory. In Kreps (1990) the term "cooperative game theory" is mentioned on only one page, and the item "non-cooperative game theory" of the index refers the reader to the item "game theory." It seems fair to say that nowadays, *to treat* variables as constants *is the characteristic vice of the mathematical economist*.

[6]The issue of model consistency will return in Section 4.3, where I discuss rational expectations.

4. Three examples

I shall support my rejection of the Nash equilibrium with three examples of the notion's failure, in the theory of oligopoly, in the Prisoner's Dilemma, and in dynamic general equilibrium models with imperfect competition and rational expectations.

4.1. Example 1: The oligopoly problem

The first occurrence of a Nash equilibrium in economic theory is probably the equilibrium for a duopoly proposed by Cournot in 1838 (Cournot, 1927). If the publication of Cournot's *Recherches sur les principes mathématiques de la théorie des richesses* marks the beginning of mathematical economics, the Nash equilibrium has always been part of the body of economic theory. It is therefore just fit that the oligopoly problem serves as the first example to illustrate how inadequate the notion of Nash equilibrium is.

Let us return to the constrained simultaneous maximum problem discussed in Section 2.3, Problem PQ, and let us see what non-cooperative game theory has to say about it. Non-cooperative game theory maintains the fundamental assumption that the firms are rational profit maximisers. In flat contradiction to the view of Stigler (1964, p. 44) as cited in Section 2.3.3, however, the theory moves assumptions concerning the way in which each firm views its interdependence with its rivals to the centre of the stage. There are two polar cases. One case corresponds to Problem Q, with the firms as quantity setters; it takes us back to Cournot. The other case corresponds to Problem P, with the firms as price setters; it takes us back to Bertrand.[7]

Cournot (1927) studies an oligopoly model with one homogeneous good. Because the firms cannot charge different prices for perfect substitutes, Cournot is naturally led to the case that the firms use their quantities as instruments. Avoiding this limiting case by assuming that the goods produced by different firms are imperfect substitutes, as in Problem PQ, allows us to study the case of price-setting firms in complete parallel to the case of quantity-setting firms.

4.1.1. Cournot oligopoly

Cournot (1927) assumes that each firm sets a quantity for its product, taking the quantities of its rivals as given. He thus replaces Problem Q by

$$\max_{\ln q_i} u_i(\ln q_i \,|\, \ln \mathbf{q}_{-\mathrm{i}}), \qquad i = 1, \ldots, I.$$

The union of the first-order conditions of the conditional maximum problems is

$$mc_i = p_i(1+\varphi_{ii}), \qquad i = 1, \ldots, I. \tag{20}$$

Its solution is the famous Cournot equilibrium, the oldest example of a Nash equilibrium in mathematical economics.

4.1.2. Bertrand oligopoly

Bertrand (1883) criticises Cournot by arguing that the firms would unite, or at least collude and try to set a common price in order to obtain monopoly-like profits. He next argues that, if they could not do so, they would set a price for their product rather than choose a quantity. Effectively he replaces Problem P by

$$\max_{\ln p_i} v_i(\ln p_i \,|\, \ln \mathbf{p}_{-\mathrm{i}}), \qquad i = 1, \ldots, I.$$

The union of the first-order conditions of the conditional maximum problems is

$$mc_i = p_i(1+\chi_{ii}^{-1}), \qquad i = 1, \ldots, I. \tag{21}$$

Its solution is the famous Bertrand equilibrium, another early example of a Nash equilibrium in mathematical economics.

[7] In fact, there is also a number of intermediate cases, where some firms use their quantities and other firms use their prices as instruments. You may analyse them by using *mixed* demand functions (see Chavas, 1984). In general, all 2^I Nash equilibria are different from one another.

4.1.3. Conjectural variations oligopoly

The approach of Bowley and Frisch

Bowley (1924, p. 38), studying the oligopoly problem with a homogeneous good and quantity-setting firms, suggests that the appropriate first-order conditions are that, instead of the *partial* derivative, the *total* derivative of each firm's profit function with respect to the firm's decision variable equal zero. In the present context, with the natural logarithms of the quantities as the decision variables, this approach leads to

$$mc_i = p_i(1+\sum_j \varphi_{ij}\xi_{ji}), \qquad i=1,\ldots,I, \tag{22}$$

where $\xi_{ji} := \tilde{q}_j/\tilde{q}_i$. With price-setting firms and the natural logarithms of the prices as the decision variables it leads to

$$mc_i = p_i\Big(1+\big(\sum_j \chi_{ij}\psi_{ji}\big)^{-1}\Big), \qquad i=1,\ldots,I, \tag{23}$$

where $\psi_{ji} := \tilde{p}_j/\tilde{p}_i$. Bowley says that to solve (22), you need to know each q_j as a function of q_i, that these functions depend on what Firm i thinks the others are likely to do, and that there are no well-defined functions from which to derive ξ_{ji}. Frisch (1933, translated as Frisch, 1951) baptises $\mathrm{d}q_j/\mathrm{d}q_i$ the *conjectural variation*, and he calls entities like ξ_{ji} and ψ_{ji} *conjectural elasticities*, with the interpretation that they express the relative change in the action of Firm j that Firm i thinks is induced by an infinitesimal relative change in its own action. He even introduces a special symbol for what he calls "conjectural differentiation" (see Frisch, 1951, p. 31).

The Cournot equilibrium results from (22) when the matrix $\Xi := [\xi_{ij}]$ of conjectural elasticities in quantity space equals the identity matrix; similarly the Bertrand equilibrium results from (23) when the matrix $\Psi := [\psi_{ij}]$ of conjectural elasticities in price space equals the identity matrix. In these cases, all *proper* (that is, for $i \neq j$) conjectural elasticities are equal to zero. When each firm considers his own action as a variable and the actions of the other firms as constants given by the actual situation, Frisch (1951) says that the firms act according to a system of *autonomous adaptation*. He speaks of *conjectural adaptation* in the general case of non-zero proper conjectural elasticities.

Frisch (1951) also distinguishes the case of what he calls *superior adaptation*. Here the assumption is that one group of firms acts in an autonomous manner, that is, their proper conjectural elasticities are all equal to zero. Next to this group there is another group of firms, which know that the firms in the first group act in an autonomous manner and also know the first group's profit functions. Now the firms in the second group do not need to guess the reactions of the firms in the first group, they can *derive* the "true" reactions from the "first-order conditions" for maximal profits of those firms. Conjectural considerations play a role only among the firms of the second group. Frisch (1951) says that the firms of the second group act under a system of superior adaptation. The *leader-follower duopoly* of Von Stackelberg (1934) corresponds to the special case that both groups consist of only one firm. Another special case is that the first group consists of a number of small firms, "the fringe," while the second group consists of one dominant firm, the leader.

Consistent conjectural variations

For the quantity-setting oligopoly, the "first-order condition" (20) gives q_i as an implicit function of $\mathbf{q}_{-\mathrm{i}}$. It is not a reduced-form relationship, as it relates one endogenous variable to other endogenous variables. The solution for q_i goes under the name of *reaction function* in the literature, with the interpretation that it tells Firm i how to adjust its output volume when the other firms vary their output volumes. The intersection of the reaction functions is the Cournot equilibrium. In general the derivatives of the reaction functions are non-zero, whereas in establishing its reaction function each firm has in fact conjectured that the other firms will *not* react but keep their output volumes constant. This observation has led Fellner (1949) to his famous dictum that in a Cournot equilibrium the firms are right (that is, about the quantities in equilibrium) for the wrong reason. Or, to quote a criticism of the Cournot equilibrium from an unexpected quarter (Mayberry, Nash and Shubik, 1953),

> The older approach often depended on the assumption of a specific "conjectural behavior" (Fellner, 1949). For example, one obtains the Cournot solution by presuming that each producer chooses his new production rate on the assumption that his competitor's production rate will remain fixed. The solution is then that situation where the producers' policies do not impel them to any changes in their rates of production. The great difficulty with

> these hypothetical rules of behavior is their multiplicity; in general, too, they require the producers to act in a rather shortsighted manner. In other words, if producer A could count on producer B behaving according to the hypothesis, he could generally do better for himself by departing from this pattern.

In the beginning of the 1980s, several authors have attempted to select a specific solution of the oligopoly problem as the real one by imposing the condition that the conjectural variations be *consistent*, in the sense that they coincide with the derivatives of the reaction functions at the equilibrium defined by those conjectures.[8] Bresnahan (1981) and Perry (1982) obtain encouraging results. For example, in a homogeneous oligopoly with constant marginal cost the consistency-of-conjectures condition selects the Bertrand equilibrium. Others (for example, Laitner, 1980; Boyer and Moreaux, 1983; Kamien and Schwartz, 1983), however, point to limitations of the approach: in more general models constant consistent conjectural variations need not exist, and linear consistent conjectural variations functions allow almost any point to be an equilibrium.

4.1.4. Example: homogeneous duopoly

A simple case of the oligopoly problem, a homogeneous duopoly, is perfectly suited for illustrating the various notions and their interrelationships, and also the method of simultaneous maximisation.

The model

Two firms i $(= 1,2)$ produce a homogeneous good. They have identical quadratic cost functions,

$$c_i(q_i) = c(q_i) = c_1 q_i + \frac{1}{2} c_2 q_i^2, \qquad i = 1,2, \tag{24}$$

where q_i is the quantity produced by Firm i. The subscripts 1 and 2 to the coefficients do not refer to the firms, but this notation will not cause any confusion. Adding fixed costs c_0 affects only the labels of the iso-profit curves. The derivative of $c(q_i)$ is the linear marginal-cost function, $mc(q_i) = c_1 + c_2 q_i$. I assume $c_1 > 0$. The coefficient c_2 may be positive (increasing marginal cost), zero (constant marginal cost) or negative (decreasing marginal cost), and I shall indicate how the value of c_2 impacts the outcomes.

The inverse demand function, too, is linear:

$$p = d_0 - d(q_1 + q_2), \qquad d_0 > c_1, d > 0. \tag{25}$$

The condition $d_0 > c_1$ ensures that the price of the good exceeds marginal cost when the firms produce nothing at all.

Profit υ_i is revenue minus cost, $\upsilon_i := pq_i - c(q_i)$. Use (24) and (25) to write the profit of Firm i as a function of both quantities,

$$\upsilon_i = u_i(q_i, q_j) = \big(d_0 - d(q_i + q_j)\big) q_i - c_1 q_i - \frac{1}{2} c_2 q_i^2, \qquad i \neq j.$$

The firms choose their quantities so as to obtain maximal profits. This assumption yields the maximum problems

$$\max_{q_i} u_i(q_1, q_2), \qquad i = 1,2, \tag{26}$$

as the model of the world. The maximum problems are interdependent because both quantities occur as the arguments of both profit functions: (26) is an example of a simultaneous maximum problem.

The choice of a homogeneous duopoly has the slight disadvantage that I cannot deal with the Bertrand equilibrium directly. From (21) it follows that in the limiting case of perfect substitutes ($\chi_{ii} \to -\infty$), $mc_i = p$ in the Bertrand equilibrium.

With product homogeneity and identical cost functions, the firms' profit functions exhibit symmetry in the way that the quantities enter them; therefore the "equilibria" and solution will exhibit symmetry as well. The assumption is not necessary for obtaining explicit results, but it eases the analysis without affecting its essence.

[8] Theil (1968, p. 332) doubted the existence of consistent conjectural variations.

The profit of Firm i is zero if $q_i = 0$ or else if $p - c(q_i)/q_i = 0$ (price equals average cost). The latter condition may be written as

$$q_i = -\frac{2d}{2d+c_2}\big(q_j - k\big), \qquad k := \frac{d_0 - c_1}{d} > 0,$$

which I call Firm i's *Zero-profit line*. The profit of a firm is negative above and to the right of its Zero-profit line. The line intersects the q_j-axis in the point $B_i = [0,1]k$ (it will appear convenient to express all quantities as multiples of k) and the q_i-axis in the point $Z_i = [q_{Z_i}, 0]$, $q_{Z_i} = \big(2d/(2d+c_2)\big)k$. The intersection of the Zero-profit lines is the Zero-profit point $Z = [q_Z, q_Z]$, $q_Z = \big(2d/(4d+c_2)\big)k$.

"Equilibria"

The functional forms of the cost functions and inverse demand function make it convenient to differentiate with respect to the quantities themselves rather than the log-quantities, and to study the conjectural variations rather than the ditto elasticities. The early literature takes the conjectural variations as free parameters, their values representing alternative modes of behaviour. In our simple model, constant conjectural variations lead to linear reaction functions. A symmetric conjectural variations equilibrium is the solution of

$$\begin{bmatrix} p - d(1+x)q_1 - mc(q_1) & 0 \\ 0 & p - d(1+x)q_2 - mc(q_2) \end{bmatrix} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{27}$$

where $x := \mathrm{d}q_j/\mathrm{d}q_i$ is the common conjectural variation. Each firm produces up to the point where its marginal cost equals its conjectured marginal revenue. The general expression of Firm i's reaction function in the symmetric conjectural variations duopoly is

$$q_i = -\frac{d}{(2+x)d + c_2}\big(q_j - k\big).$$

The line intersects the q_j-axis in the point $B_i = [0,1]k$ and the q_i-axis in the point $[d/(2+x)d + c_2, 0]k$. The intersection of the reaction functions is the "equilibrium" $[q_E, q_E]$, $q_E = \{d/\big((3+x)d + c_2\big)\}k$.

The Bertrand equilibrium results when $x = -1$: each firm produces up to the point where its marginal cost equals the price of the good. Solving $p - mc(q_i) = 0$ for q_i yields $q_i = -\big(d/(d+c_2)\big)(q_j - k)$, which may be called Firm i's *Bertrand reaction function* in quantity space. The marginal cost of Firm i exceeds the price of the good above and to the right of its Bertrand reaction function. The line intersects the q_i-axis in the point $B'_i = [q_{B'_i}, 0]$, $q_{B'_i} = \big(d/(d+c_2)\big)k$. The intersection of the Bertrand reaction functions is the Bertrand equilibrium $B = [q_B, q_B]$, $q_B = \big(d/(2d+c_2)\big)k$.

The Cournot equilibrium results when $x = 0$, and is the solution of

$$\begin{bmatrix} p - dq_1 - mc(q_1) & 0 \\ 0 & p - dq_2 - mc(q_2) \end{bmatrix} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{27.C}$$

Each firm i produces up to the point where its marginal cost equals its marginal revenue *at given value of* q_j. Solving $p - dq_i - mc(q_i) = 0$ for q_i yields $q_i = -\big(d/(2d+c_2)\big)(q_j - k)$, which is Firm i's *Cournot reaction function*. The marginal cost of Firm i exceeds the marginal revenue *at given value of* q_j to the right of its Cournot reaction function. The line intersects the q_i-axis in the point $C_i = [q_B, 0]$, which is the projection of B on the q_i-axis. C_i is halfway between Z_i and the origin. The intersection of the Cournot reaction functions is the Cournot equilibrium $C = [q_C, q_C]$, $q_C = \big(d/(3d+c_2)\big)k$.

The Collusive equilibrium results when $x = 1$: each firm produces up to the point where its marginal cost equals the joint marginal revenue. Solving $p - 2dq_i - mc(q_i) = 0$ for q_i yields $q_i = -\big(d/(3d+c_2)\big)(q_j - k)$, which may be called Firm i's *Collusive reaction function*. The marginal cost of Firm i exceeds the joint marginal revenue above and to the right of its Collusive reaction function. The line intersects the q_i-axis in the point $Col_i = [q_C, 0]$, which is the projection of C on the q_i-axis. The intersection of the Collusive reaction functions is the Collusive equilibrium $Col = [q_{Col}, q_{Col}]$, $q_{Col} = \big(d/(4d+c_2)\big)k$.

Consistency of the conjectural variations in the sense of Bresnahan (1981) and Perry (1982) imposes the restriction $x = -d/\big((2+x)d + c_2\big)$. The consistent conjectural variation is the larger root of the implied quadratic equation in x,

$$x = \frac{-(2d+c_2) + \sqrt{c_2(4d+c_2)}}{2d}.$$

The intersection of the consistent reaction functions is the Consistent Conjectures Equilibrium $CCE = [q_{CCE}, q_{CCE}]$,

$$q_{CCE} = \frac{4d + c_2 - \sqrt{c_2(4d + c_2)}}{2(4d + c_2)} k.$$

$c_2 = 0$ (constant marginal cost) yields $x = -1$, so that CCE coincides with the (symmetric) Bertrand equilibrium B. It may be verified that $c_2 > 0$ (increasing marginal cost) implies $-1 < x < 0$, so that the Consistent Conjectures Equilibrium is in between the Bertrand equilibrium and Cournot equilibrium. For negative values of c_2 not exceeding $4d$ in absolute value, no real solution for x exists.

The Stackelberg equilibrium S^C with the Follower (arbitrarily put in the first position) as a Cournot duopolist is the solution of

$$\begin{bmatrix} p - dq_F - mc(q_F) & 0 \\ -dq_L & p - dq_L - mc(q_L) \end{bmatrix} \begin{bmatrix} \mathrm{d}q_F \\ \mathrm{d}q_L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{28}$$

The "first-order condition" for the Follower is just his Cournot reaction function. The Leader derives x_{FL} $(= \mathrm{d}q_F/\mathrm{d}q_L)$ from it, which he uses to solve his own maximum problem. In a Stackelberg equilibrium the reaction function of the Follower is tangent to an iso-profit curve of the Leader. More generally, if the reaction function of the Follower is $q_F = -r_F(q_L - k)$, the first-order condition for maximal profit of the Leader conditional on this reaction function yields the line

$$q_L = -\frac{d}{(2 - r_F)d + c_2}\big(q_F - k\big).$$

The intersection of these two lines is the Stackelberg equilibrium S,

$$S = [q_F^S, q_L^S] = \Big[\frac{r_F\big(d(1 - r_F) + c_2\big)}{2d(1 - r_F) + c_2} k, \frac{d(1 - r_F)}{2d(1 - r_F) + c_2} k\Big].$$

Please note that $c_2 = 0$ (constant marginal cost) implies $q_L^S = k/2 = q_B$, independently of the value of r_F, and $q_F^S = r_F k/2$. If, moreover, the Follower is a Cournot duopolist, $r_F = r_C := d/(2d + c_2) = 1/2$, and substituting this value for r_F in the expression for q_F^S yields $q_F^C = k/4 = q_{Col}$.

Table 1: Summary: abscissa of "equilibria" and other special points

Point	General formula	Special cases		
		$c_2 = d/2$	$c_2 = 0$	$c_2 = -d/2$
	$\times k$			
B_j	1	1	1	1
Z_i	$2d/(2d + c_2)$	4/5	1	4/3
B'_i	$d/(d + c_2)$	2/3	1	2
Z	$2d/(4d + c_2)$	4/9	1/2	4/7
B, C_i	$d/(2d + c_2)$	2/5	1/2	2/3
CCE	$\big(4d + c_2 - \sqrt{c_2(4d + c_2)}\big)/2(4d + c_2)$	1/3	1/2	x
C, Col_i	$d/(3d + c_2)$	2/7	1/3	2/5
Col	$d/(4d + c_2)$	2/9	1/4	2/7
q_F^C	$\big(r_C\big(d(1 - r_C) + c_2\big)\big)/\big(2d(1 - r_C) + c_2\big)$	22/85	1/4	$-2/3$
q_L^C	$d(1 - r_C)/\big(2d(1 - r_C) + c_2\big)$	6/17	1/2	2

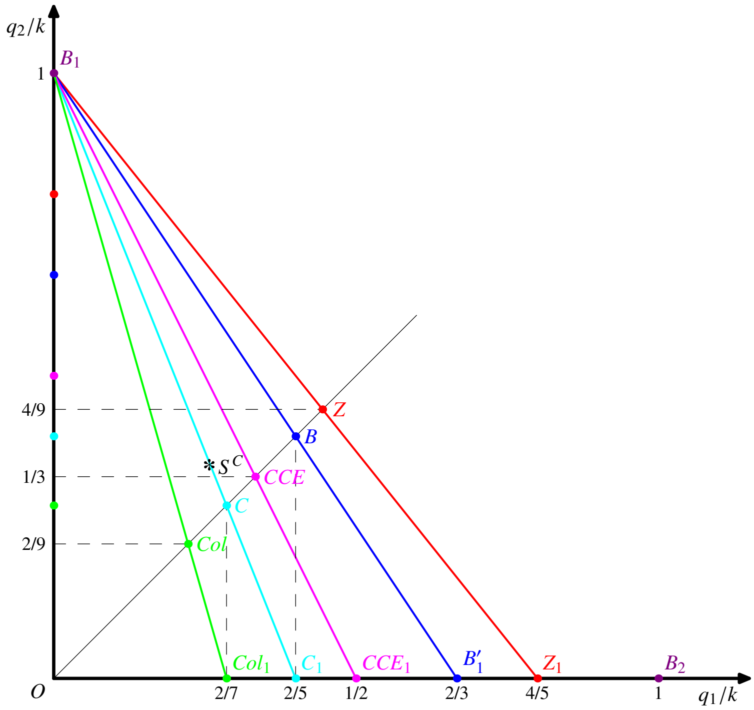

Figure 3: Zero-profit line, reaction functions and "equilibria" when $c_2 = d/2$ (increasing marginal cost)

Notes: $[B_1Z_1]$ is the Zero-profit line, $[B_1B'_1]$ the Bertrand reaction function, $[B_1CCE_1]$ the Consistent reaction function, $[B_1C_1]$ the Cournot reaction function, and $[B_1Col_1]$ the Collusive reaction function of Firm 1. B_2 is the begin point of the similar lines for Firm 2. Z is the Zero-profit point, B the Bertrand equilibrium, CCE the Consistent Conjectures Equilibrium, C the Cournot equilibrium, Col the Collusive equilibrium, and S^C the Stackelberg equilibrium with Firm 1 as the Cournot Follower.

Table 1 summarises the results obtained thus far. The second column contains the analytical expressions for the abscissa of the "equilibria" and other special points (the end points of Firm i 's Zero-profit line and reaction functions, the Zero-profit point, and the begin point of Firm j's Zero-profit line and reaction functions). The positions of these points (except B_j) and the slopes of the lines depend only on the ratio c_2/d, that is, on the slope of the marginal-cost function relative to the slope of the inverse demand function. The third, fourth and fifth column give the values of the abscissa for alternative values of the ratio c_2/d. Figure 3 draws the Zero-profit line and reaction functions of Firm 1 when $c_2 = d/2$ (increasing marginal cost); for Firm 2 there is a similar set of lines emanating from B_2 (not drawn). The symmetric "equilibria" are on the 45°-line. The Stackelberg equilibrium with Firm 1 as the Cournot Follower is indicated by the asterisk on the Cournot reaction function $[B_1C_1]$. When the ratio decreases, all end points move away from the origin. As the begin point B_1 of the lines stays put at $[0,1]$, all lines become less steep and all "equilibria" move away from the origin, too. When $c_2 = 0$ (constant marginal cost), B'_i and Z_i coincide with B_j ($i \neq j$). Now the Zero-profit lines, Bertrand reaction functions and Consistent reaction functions of both firms coincide with $[B_1B_2]$, which is the Bertrand equilibrium line.

The solution

The correct treatment of the simultaneous maximum problem shows what is wrong with the "equilibria" of Cournot, Bertrand and Von Stackelberg. Denote the matrix of first-order derivatives of the profit functions by $\mathbf{U}$. Consider the first-order approximation $\mathbf{U}\mathrm{d}\mathbf{q} = \mathrm{d}\boldsymbol{\upsilon}$ to the changes in the profits,

$$\begin{bmatrix} p - dq_1 - mc(q_1) & -dq_1 \\ -dq_2 & p - dq_2 - mc(q_2) \end{bmatrix} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \end{bmatrix} = \begin{bmatrix} \mathrm{d}\upsilon_1 \\ \mathrm{d}\upsilon_2 \end{bmatrix}$$

(please compare the matrix on the left-hand side with the ones in (27.C) and (28)). The profit functions may be (weakly) simultaneously maximal at points $\mathbf{q}^*$ where the space spanned by the columns of $\mathbf{U}(\mathbf{q}^*)$ does not contain a strictly positive vector. The (necessary and sufficient) conditions for such a vector not to exist consist of two parts. First, $\mathbf{U}$ must be singular:

$$
\begin{aligned}
|\mathbf{U}| = 0 = & \big(p - mc(q_1)\big)\big(p - mc(q_2)\big) \\
& - dq_1\big(p - mc(q_2)\big) \\
& - dq_2\big(p - mc(q_1)\big).
\end{aligned} \tag{29}
$$

It is easily verified that the variety defined by $|\mathbf{U}| = 0$ passes through the points B_1, B, B_2, C_1, Col and C_2.[9] Second, the elements of the left eigenvector of $\mathbf{U}$ with eigenvalue zero may not have different signs (zero counting as neither positive nor negative). This eigenvector is proportional to either column of the matrix $\mathbf{U}^*$ of cofactors of $\mathbf{U}$,

$$
\mathbf{U}^* = \begin{bmatrix} p - dq_2 - mc(q_2) & dq_2 \\ dq_1 & p - dq_1 - mc(q_1) \end{bmatrix}.
$$

Hence, the second part of the first-order conditions rules out all points in the positive quadrant beyond the Cournot reaction functions (seen from the origin) as elements of the simultaneous maximum. Let us first consider the conditions explicitly at some special points.

EXAMPLE 3 (COURNOT EQUILIBRIUM) At the Cournot equilibrium C the matrix of first-order derivatives is

$$
\mathbf{U} = -dq_C \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
$$

This matrix is not singular: the Cournot equilibrium does not belong to the simultaneous maximum. Please note that the partial cross derivatives $\partial u_i / \partial q_j$, which are absent from the "first-order conditions" (27.C) defining the Cournot equilibrium, are *not* equal to zero.

EXAMPLE 4 (STACKELBERG EQUILIBRIUM) When $c_2 = 0$, the matrix of first-order derivatives in the Stackelberg equilibrium S^C is

$$
\mathbf{U} = -dq_F^C \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}
$$

(the Follower is in the first position). This matrix is not singular: the Stackelberg equilibrium S^C does not belong to the simultaneous maximum. Please note that the partial cross derivative $\partial u_F / \partial q_L (= \partial u_1 / \partial q_2)$, which is absent from the "first-order conditions" (28) defining the Stackelberg equilibrium, is *not* equal to zero.

EXAMPLE 5 (BERTRAND EQUILIBRIUM) At the Bertrand equilibrium B the matrix of first-order derivatives is

$$
\mathbf{U} = -dq_B \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},
$$

which is singular. The corresponding matrix of cofactors is

$$
\mathbf{U}^* = -dq_B \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.
$$

The elements of either column of $\mathbf{U}^*$ have opposite signs: the first-order conditions for a simultaneous maximum are not fulfilled at B. A small move from B in the direction of the origin raises the values of both profit functions.

[9] At B_i, $q_i = 0$ and $p - mc(q_i) = 0$; at B, $q_1 = q_2 = q_B$ and $p - mc(q_B) = 0$; at C_i, $q_j = 0$ and $p - dq_i - mc(q_i) = 0$; at Col, $q_1 = q_2 = q_{Col}$ and $p - 2dq_{Col} - mc(q_{Col}) = 0$.

EXAMPLE 6 (COLLUSION) The matrix of first-order derivatives at the Collusive equilibrium *Col* is

$$\mathbf{U} = dq_{Col}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

which is singular. The corresponding matrix of cofactors is

$$\mathbf{U}^* = dq_{Col}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The elements of either column of $\mathbf{U}^*$ have the same sign: the first-order conditions are fulfilled at *Col*.

Let us next consider the general nature of the condition (29). It appears that the shape of the variety defined by the quadratic form $|\mathbf{U}| = 0$ is governed by the ratio c_2/d. The shapes separate into two classes, corresponding to the sign of c_2, with a special case in between when c_2 equals zero.

For $\mathbf{c_2 > 0}$, the quadratic form represents a hyperbola with *Col* and *B* as its vertices.

For $\mathbf{c_2 = 0}$, the quadratic form represents two lines, one through C_1 and C_2 and the other through B_1 and B_2.

For $\mathbf{c_2 < 0}$, the quadratic form represents an ellipse with the line segment $[ColB]$ as its minor axis.

When $c_2 \geq -d$, there are two segments in the positive quadrant, one running from B_1 through B to B_2 and the other running from C_1 through *Col* to C_2. We already know that the sign condition on the elements of the columns of $\mathbf{U}^*$ is satisfied at the latter segment but not at the former.

Figure 4 and Figure 5 draw the parts of the curve $|\mathbf{U}| = 0$ in the positive quadrant for the case $c_2 = d/2$ (increasing marginal cost) and $c_2 = -d/2$ (decreasing marginal cost). They also sketch three pairs of iso-profit curves and the Zero-profit lines. At the Bertrand equilibrium B the iso-profit curves of the firms have a tangent line in common, but

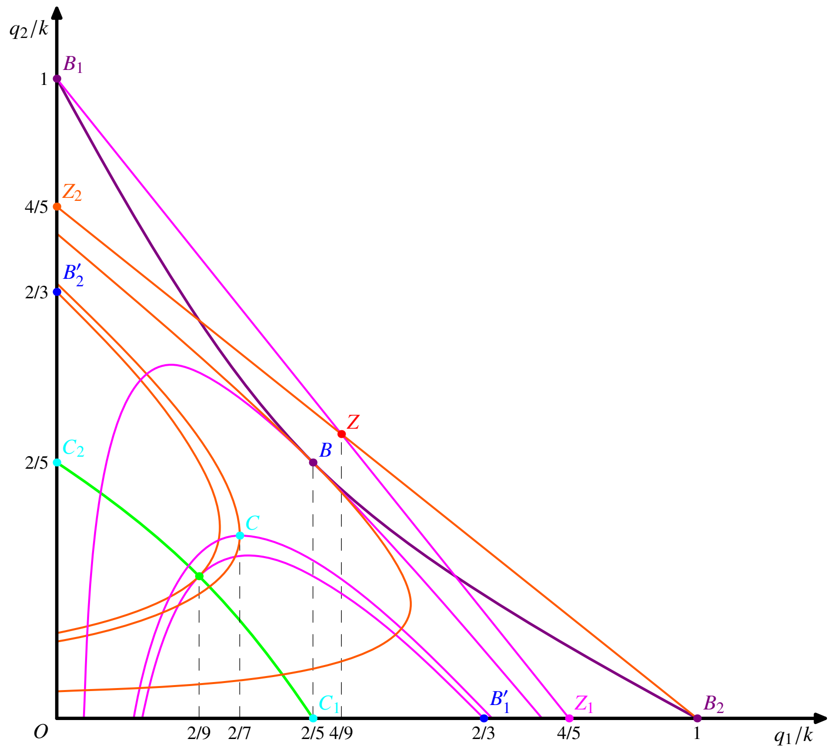

Figure 4: Solution and “equilibria” when $c_2 = d/2$ (increasing marginal cost).

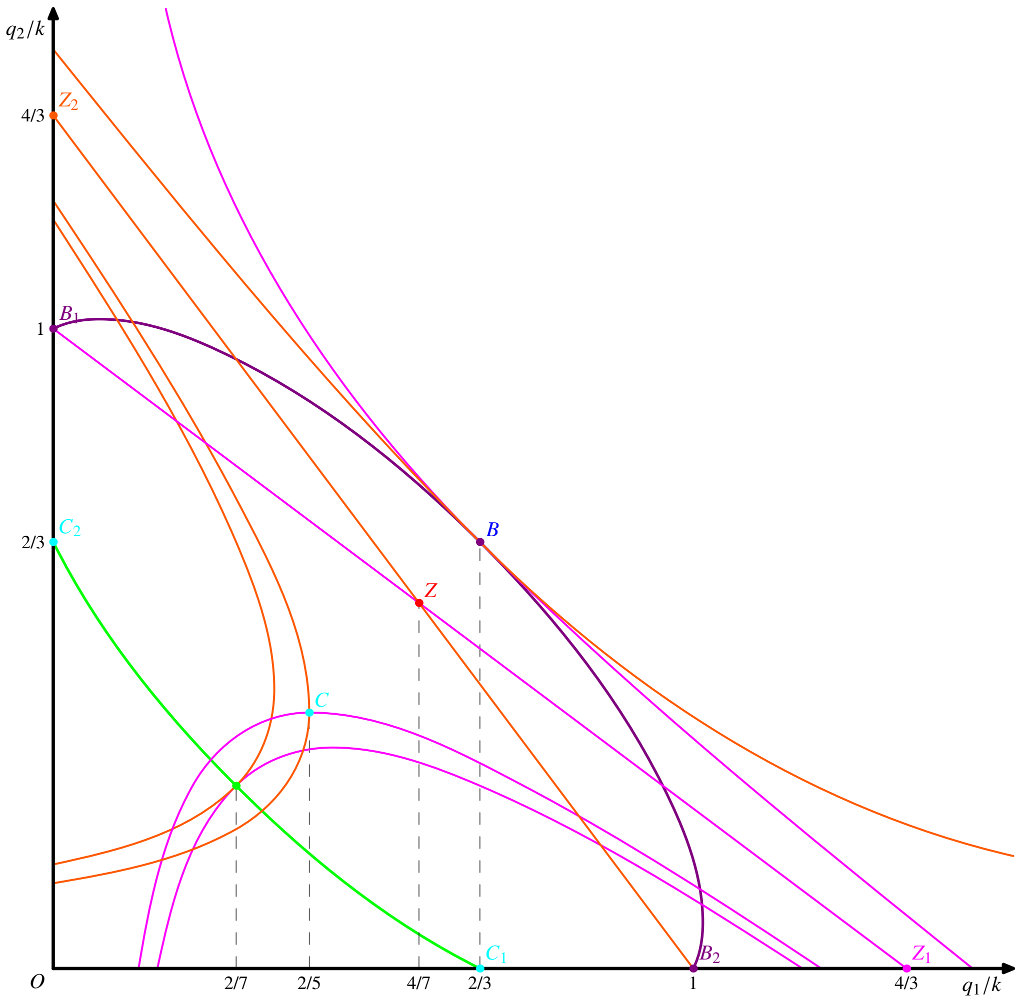

Figure 5: Solution and "equilibria" when $c_2 = -d/2$ (decreasing marginal cost).

still they intersect; a move from B towards the origin raises the values of both profit functions.[10] At other points of the curve through B_1, B and B_2, on the two segments beyond the intersections with the Zero-profit lines as seen from B, the iso-profit curves do not intersect but touch one another, and still there are directions in which the values of both profit functions increase. At the Cournot equilibrium C the iso-profit curves are perpendicular to one another. In all points of the simultaneous maximum, the curve through C_1 and C_2, the iso-profit curves touch one another, and there is no direction in which the values of both profit functions increase.

Figure 6 is for the special case $c_2 = 0$ (constant marginal cost). Now the iso-profit curves through the symmetric Bertrand equilibrium B share the line segment $[B_1B_2]$, which is the Bertrand equilibrium line, coinciding with the Zero-profit lines, Bertrand reaction functions and Consistent reaction functions. Remember that the first-order conditions yield the "conjectural variations" $x_{ji} := \mathrm{d}q_j/\mathrm{d}q_i$ as $|\mathbf{U}_{ij}|^*/|\mathbf{U}_{ii}|^*$. When $c_2 = 0$, $x_{ji} = -1$ in points on $[B_1B_2]$, and $x_{ji} = q_j/q_i$ in points on $[C_1C_2]$. The arrows departing from B and $Col = [¼, ¼]$ represent the "conjectural variations" at these points.

REMARK 5 The problem contains relationships between the quantities produced and the profits earned by the two firms. It allows answering the question, "Which combinations of quantities yield simultaneously maximal profits to

[10] B is special in that the common tangent line coincides with the tangent line to B_1BB_2.

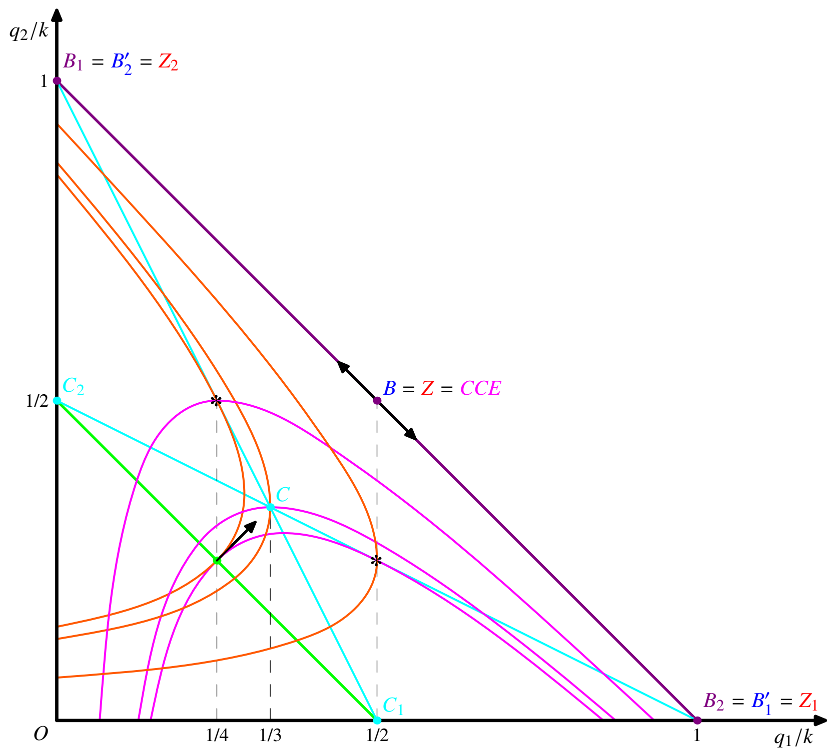

Figure 6: Solution and "equilibria" when $c_2 = 0$ (constant marginal cost)

Notes to Figures 4–6: The curve through B_1, B and B_2 solves $|U| = 0$, but is not part of the simultaneous maximum. The curve through C_1 and C_2 solves $|U| = 0$ and is the simultaneous maximum (or Pareto optimum); its midpoint is the Collusive equilibrium. $[B_iZ_i]$ is the Zero-profit line of Firm i and Z the Zero-profit point. In Figure 6, Z coincides with the *symmetric* Bertrand equilibrium B and Consistent Conjectures Equilibrium CCE, and $[B_1B_2]$ coincides with the Bertrand equilibrium line, Zero-profit lines, Bertrand reaction functions and Consistent reaction functions. $[B_iC_i]$ is the Cournot reaction function of Firm i and C the Cournot equilibrium. The asterisks are the Stackelberg equilibria. The arrows represent "conjectural variations."

the firms?" The quantities being the elementary choice variables, the answer to the question how the firms arrive at a particular combination of quantities is simply, "By choosing it." How the absence of opportunities for communication and co-ordination affects the outcome is a question that surpasses the boundaries of the problem as it has been posed.

4.1.5. The oligopoly problem in retrospect

For decades after Bertrand's (1883) criticism of Cournot (1838) the economics profession has hotly debated the solution of the oligopoly problem. Von Stackelberg (1934, Ch. 5) reviews many of the early contributions. Some argue that the quantities are the appropriate decision variables, others argue that the prices are. The introduction of the non-cooperative equilibrium by Nash (1951), which makes clear that the Cournot equilibrium and the Bertrand equilibrium are instances of a Nash equilibrium, has put an end to the debate. As Bresnahan (1981, p. 934) says,

> No attempt to decide among Bertrand and Cournot, and their more modern competitors can be based on mathematical correctness. (...) Oligopoly models are examples of what game theorists call Nash equilibrium. In them, every firm maximizes profits given the actions of all other firms. The mathematics does not care whether "actions" are defined to be prices (Bertrand), quantities (Cournot), or any other variables.

Contributions comparing the merits of the various solutions begin to appear, starting with Mayberry et al. (1953). Some articles focussing on the comparison of prices or quantities as strategic variables are Hathaway and Rickard (1979), Singh and Vives (1984), Cheng (1985), Vives (1985), Friedman (1988), and Allen (1992). Recent additions to this literature are Amir and Jin (2001), Tanaka (2001), Hsu and Wang (2005) and Tasnádi (2006).

I dispute the correctness of the view expressed by Bresnahan (1981). Every Nash equilibrium is mathematically flawed because of incorrect conditioning on endogenous variables. As pointed out above, non-cooperative game theory treats each argument as a variable in one associated maximand and simultaneously as a constant in all other maximands of the same simultaneous maximum problem. Such a severe error must manifest itself in the results somehow, so why has it not been discovered long ago? The answer is that the error does manifest itself in contradictions, but that—amazingly—these have not been recognised as such.

One contradiction is the very fact that the Cournot equilibrium does not coincide with the Bertrand equilibrium. Actually the absence of invariance under nonsingular transformations of variables is an intrinsic property of the Nash equilibrium. By contrast, the solution of any maximum problem is invariant under such transformations. In Section 2.3 I have already indicated that Problem PQ, Problem P and Problem Q are mathematically equivalent versions of one and the same problem, related to each other by nonsingular transformations of variables, and shown that their first-order conditions are indeed equivalent.

You may use the absence of invariance to argue that, contrary to a widely held belief, the Nash equilibrium is not "self-enforcing." Consider first the game in quantity space. According to non-cooperative game theory the firms then arrive at the Cournot equilibrium. The Cournot quantities would constitute an equilibrium because it is not profitable for any firm to change its quantity unilaterally: it is "self-enforcing." Here the textbook argument stops, without considering the corresponding prices. The firms must, however, also post prices for their products. The Cournot prices follow from substitution of the Cournot quantities into the inverse demand functions (11). The choice of the Cournot prices is equivalent to the choice of the Cournot quantities. The Cournot quantities constituting an equilibrium in the sense of non-cooperative game theory, the corresponding prices must constitute an equilibrium as well. In general, however, the Cournot prices differ from the prices at the Bertrand equilibrium: they do not constitute a Nash equilibrium in price space.

Following this train of thought, at the Cournot equilibrium the firms note that it is profitable to change the prices of their products unilaterally, until they arrive at the Bertrand equilibrium. They obtain the corresponding quantities by substituting the Bertrand prices into the ordinary demand functions (10). Then they note that the Bertrand quantities do not constitute a Nash equilibrium in quantity space. Now they are tempted to change their quantities unilaterally, until they arrive at the Cournot equilibrium. The reasoning can go on in this way indefinitely. Thus the property of being "self-enforcing" proves to be fragile, depending on the arbitrary choice of a particular frame of reference. Would anyone want to maintain that the Nash equilibrium is self-enforcing, given the absence of invariance under nonsingular transformations of variables?

This objection against the Nash equilibrium may be stated differently in terms of the conjectural elasticities. The Cournot equilibrium (20) corresponds to the case that all (proper) conjectural elasticities ξ_{ji} in (22) equal zero, and the Bertrand equilibrium (21) to the case that all (proper) conjectural elasticities ψ_{ji} in (23) equal zero. The Cournot prices not coinciding with the Bertrand equilibrium implies that the conjectural elasticities ψ_{ji} are *not* equal to zero at the Cournot equilibrium. Therefore a plea that the conjectural elasticities ξ_{ji} in quantity space be zero, like in Daughety (1985), is at the same time a plea for specific non-zero conjectural elasticities ψ_{ji} in price space.

For a heterogeneous duopoly Salant (1984), assuming that equilibrium exists, gives conditions on the conjectural variations that ensure the equivalence of the equilibria in the price and quantity models. Restated in terms of the conjectural elasticities his results are

$$\xi_{ji} = (\chi_{ji} + \chi_{jj}\psi_{ji})/(\chi_{ii} + \chi_{ij}\psi_{ji}), \qquad i, j = 1, 2\ (i \neq j), \tag{30}$$

and, equivalently,

$$\psi_{ji} = (\varphi_{ji} + \varphi_{jj}\xi_{ji})/(\varphi_{ii} + \varphi_{ij}\xi_{ji}), \qquad i, j = 1, 2\ (i \neq j). \tag{31}$$

Now is the time to refer back to the discussion of the tangent space in Section 2.3.4. The reader will have noted that (22) equals (15), and that (23) equals (16). Likewise, he will now note that (30) and (31) are just the special cases of (17) and (18) for $I = 2$. It thus appears that the conditions Salant (1984) imposes on the conjectural variations translate the simultaneous maximum in quantity space to the simultaneous maximum in price space, or the other way around. Still, like most students of the oligopoly problem, little is Salant (1984) aware of the true nature of the conjectural elasticities: ξ_{ji} is just the elasticity of substitution between q_j and q_i along the common tangent line to the iso-profit curves of the firms at some point of the simultaneous maximum. The term "conjectural elasticity" is a misnomer because nothing needs to be conjectured: ξ_{ji} results from the first-order conditions as $|\mathbf{U}_{ij}|^*/|\mathbf{U}_{ii}|^*$, and the ξs satisfy restrictions like the αs in (9) do. These "consistency conditions" need not be imposed separately: they follow from treating the oligopoly problem correctly, that is, as a simultaneous maximum problem. And yes, the mathematics does not care whether "actions" are defined to be prices or quantities, because it yields equivalent outcomes for equivalent problems.

According to Bowley (1924) there are no well-defined functions from which to derive the differential quotients $\mathrm{d}q_j/\mathrm{d}q_i$. In fact, there are I sets of (implicit) functions fulfilling his needs—the iso-profit functions of the firms. If only Bowley had been aware of this fact, and that at an equilibrium the differential quotients had to be equal for all iso-profit functions, he would have found the solution to the oligopoly problem (see also Zaccagnini, 1951, pp. 239–40).

REMARK 6 I have introduced the constraints, (10) or (11), as basic elements of Problem PQ. Actually the system of (inverse) demand functions stems, issues of aggregation apart, from the problems of utility maximisation by budget-constrained consumers. The prices facing the consumers occur in the problems of technology-constrained profit maximisation by the firms, while for each good the sum of the quantities consumed equals the quantity produced. It thus appears that the problems of utility maximisation and profit maximisation are interdependent: they constitute a simultaneous maximum problem. Therefore it is not correct to solve them separately. Section 2.3.2 presents the correct first-order conditions for Problem PQ, but the problem itself does not correctly represent a world with utility maximising consumers and profit maximising firms.

4.2. *Example 2: The Prisoner's Dilemma*

The simplest simultaneous maximum problem consists of two maximands, each of which is a function of the same two variables, each of which can take only two values. I state and solve such a mathematical problem free from any interpretation whatsoever, and then give two interpretations. In one interpretation the problem is the "Prisoner's Dilemma," the name under which it has become famous in economics and far beyond.

4.2.1. *The mathematical problem and its solution*

Let ω_i, $i = 1,2$, be functions $w_i(\cdot)$ of two discrete variables, a_1 and a_2, each of which can take only two values: $a_1 = T$ or B, $a_2 = L$ or R. The values of the functions are

$$\begin{aligned} w_1(T,R) = w_2(B,L) &= 0, \\ w_1(B,R) = w_2(B,R) &= 1, \\ w_1(T,L) = w_2(T,L) &= 2, \\ w_1(B,L) = w_2(T,R) &= 3. \end{aligned}$$

The four points and the corresponding function values are presented in Table 2, the *payoff matrix*; the figure in the South-West corner of a cell is the value of $w_1(\cdot)$, the figure in the North-East corner is the value of $w_2(\cdot)$. Consider the problem of simultaneously maximising ω_1 and ω_2,

$$\max_{a_i} w_i(a_1,a_2), \qquad i = 1,2. \tag{32}$$

A pairwise comparison of the four points reveals that by travelling from (B,R) to (T,L) you increase the values of $w_1(\cdot)$ and $w_2(\cdot)$, but that by moving away from any of the other three points, you decrease either $w_1(\cdot)$ or $w_2(\cdot)$, or you decrease both $w_1(\cdot)$ and $w_2(\cdot)$. In these three points ω_1 and ω_2 are simultaneously maximal.

A nonsingular transformation of variables does not affect the solution of the problem. For example, let a_1 and a_2 determine the two discrete variables b_1 and b_2, each of which can take only two values: $b_1 = T'$ or B', $b_2 = L'$ or R'. Consider the transformation that maps (T,L) to (T',L'), (B,L) to (T',R'), (T,R) to (B',L'), and (B,R) to (B',R'). The maximands ω_1 and ω_2 can now be written as functions $v_1(\cdot)$ and $v_2(\cdot)$ of b_1 and b_2. The new payoff matrix, Table 3, results from Table 2 by interchanging the Top-Right and Bottom-Left cells. In b-space the point (B',R') is the only one not in the solution set of the simultaneous maximum problem.

4.2.2. *Two interpretations*

I shall now give two interpretations of this simple simultaneous maximum problem. Needless to say, the solution is independent of whichever interpretation I choose.

Table 2: Payoff matrix of the original problem

	L	R
T	2 2	3 0
B	0 3	1 1

Table 3: Payoff matrix of the transformed problem

	L'	R'
T'	2 2	0 3
B'	3 0	1 1

The starving gourmand's dilemma

The first interpretation attaches a concrete meaning to the payoff matrix. A starving but choosy gourmand enters a restaurant. The four meals on the menu card are: (1) *Toast with caviar* (50gr); (2) a frugal helping of *Fish and chips*; (3) a generous helping of *Spaghetti bolognese*; (4) *Spare ribs*, as much as you can eat. Table 4 tells how the meals are arranged on the menu card. The gourmand evaluates each meal with respect to quantity and taste. The scores for quantity are the entries in the South-West corners of the cells of Table 2. The starving gourmand wants the values of both properties, quantity and taste, to be as high as possible. He may then discard *Fish and chips* from further consideration, because by choosing *Spaghetti bolognese* he is better off in both respects. He must still make a choice from the three remaining meals. This choice requires a trade-off between quantity and taste. Without such a trade-off, you can only say that *Fish and chips* is not the meal that satisfies him best. Note that the arrangement of the meals on the menu card does not affect the solution. For example, if you interchange *Spare ribs* and *Toast with caviar* on the menu card, the solution set consists of the same meals as it did before. Mathematically this change of positions corresponds to the nonsingular transformation of variables that translates Table 2 into Table 3.

Table 4: The menu card

	L	R
T	Spaghetti bolognese	Toast with caviar
B	Spare ribs	Fish and chips

The Prisoner's Dilemma

The problem is best known in its interpretation of the "Prisoner's Dilemma," originating from A. W. Tucker. On the suspicion of having jointly committed a crime, two men are kept prison in separate cells. The District Attorney is convinced that they are guilty, but hard evidence is lacking; he cannot convict them without obtaining a confession from at least one of them. He makes the following proposition to each prisoner separately. They may choose between *Confessing* (C) and *Not Confessing* (NC). If neither of them confesses, he will sentence them to one year imprisonment on some trumped-up charge. If both confess, he will sentence them both to two years imprisonment. If only one of them confesses, he will release that man immediately, while he will sentence the other man to three years imprisonment. Each suspect wants to minimise the number of years he will have to spend in jail. Transform the minimum problems into the problems of maximising the number of years not in jail over the next three years; for each suspect these are the entries in Table 2, the figure in the South-West corner of a cell relating to Prisoner 1. For Prisoner 1 (2), *Confessing* C corresponds to the choice of B (R). The analysis above has taught us that the point $(C,C) \equiv (B,R)$ is the *only* one *not* in the solution set of the simultaneous maximum problem. The prisoners can avoid this point (that is, they will not both have to spend two years in jail) by not both confessing. This is all you can learn from an analysis of the mathematical problem. In particular, how the prisoners succeed in not both confessing is a question that surpasses the boundaries of the problem as it has been posed: there are no other, more elementary decision variables determining which combination of Confessing and Not Confessing will prevail.

4.2.3. The treatment by non-cooperative game theory

Consider next what non-cooperative game theory has to say about the problem. Nash (1951) replaces the simultaneous maximum problem by the two conditional maximum problems $\max_{a_i} w_i(a_i \mid a_j)$, $i, j = 1, 2$, $i \neq j$. To find the Nash equilibrium, proceed as follows:

- Compare the first maximand's values only column-wise, varying a_1 while holding a_2 constant; associate with each a_2 the value of a_1 that makes ω_1 maximal; this relationship is the first reaction function;
- Similarly, compare the second maximand's values only row-wise, varying a_2 while holding a_1 constant; associate with each a_1 the value of a_2 that makes ω_2 maximal; this relationship is the second reaction function;
- Determine the intersection of the reaction functions; this point is the Nash equilibrium.

In our example, $w_1(B, a_2)$ exceeds $w_1(T, a_2)$ for every value of a_2, and $w_2(a_1, R)$ exceeds $w_2(a_1, L)$ for every value of a_1: Nash (1951) has "succeeded" in singling out as *the* equilibrium the *single* point that is *not* in the solution set of the simultaneous maximum problem, (B, R). It is hard to think of a better way to illustrate the conflict between mathematical optimisation theory and non-cooperative game theory.

4.2.4. The Prisoner's Dilemma in retrospect

In the first interpretation, the simultaneous maximum problem is the decision problem of one man, who has control over both arguments. The problem is an incomplete model, because it does not pin down one meal as the man's choice. You may try to "refine" the solution by adding some assumption, one that does not conflict with the assumptions you have already made. Never will the refinement yield *Fish and chips* as the optimal choice.

In the second interpretation, however, the simultaneous maximum problem is the union of the decision problems of two men, each of which has control over just one of the two arguments: it is a game, in particular a game in which the players cannot co-ordinate their actions. With this interpretation, Nash (1951) postulates that the Nash equilibrium is the appropriate solution. Although non-cooperative game theory has gained world-wide acceptance, it is fraught with weaknesses:

- The solution of a mathematical problem does not vary with its interpretation. You cannot tell from merely looking at (32) whether it is a cooperative or non-cooperative game; it may even not be a game at all. Ask a mathematician to solve the maximum problems for you. Don't tell him the meanings of the variables and subscripts. He does not need to know. I trust that his answer will be the Pareto optimal set.
- By splitting the simultaneous maximum problem into two conditional maximum problems, Nash (1951) ignores the interdependence of the maximum problems when he derives the "first-order conditions."
- Nash (1951) splits the simultaneous maximum problem into two conditional maximum problems by treating the arguments inconsistently: as a variable in one maximand and simultaneously as a constant in the other maximand. Because a variable is always equal to itself, when it is varied in one place it must be varied in *all* places where it occurs.
- The inconsistent treatment of the arguments enables one to derive contradictory results. In particular, observe that the Nash equilibrium of the simultaneous maximum problem corresponding to Table 3 is (T', L'), different from the image (B', R') of (B, R), which is the Nash equilibrium of the equivalent simultaneous maximum problem corresponding to Table 2. Once again, "being self-enforcing" appears a fragile property, depending on the arbitrary choice of a particular frame of reference.

To recapitulate: Nash's (1951) postulate that each player considers the actions of the other players as given if they are unable to cooperate amounts, mathematically, to a maltreatment of the simultaneous maximum problem.

By pointing out that (C, C) is not a solution of the simultaneous maximum problem I do not want to suggest that, to quote Aumann (1987, p. 469), "the Prisoner's Dilemma does not represent a real social problem that must be dealt with." In real life the co-ordination of the actions of several agents cannot be achieved costlessly and instantaneously. In this regard the mathematical problem that is known under the name of "Prisoner's Dilemma" is a poor model of reality. All you can learn from it is what the optimal choices are, disregarding all difficulties that in real life may arise in implementing a desirable outcome.

4.3. Example 3: Rational expectations

Multi-agent economic models are systems of simultaneous equations, reflecting the fact that the outcomes of the economic process result from interactions between the agents. The way in which these models are usually derived is suspect, though. For each agent separately, a (constrained) maximum problem is postulated and solved, in which only the agent's *own* decision variables are treated as endogenous. All first-order conditions are assembled into one system and a number of *new* constraints is added, the market equilibrium conditions, saying that total demands equal total supplies. If it were not apparent right from the start, then it should become clear at this stage that actually the maximum problems of the individual agents constitute a simultaneous maximum problem. The separate treatment of the maximum problems involves the risk of incorrect conditioning on endogenous variables.

There is one part of economic theory that does not fall prey to the error of incorrect conditioning, because it assumes that all agents condition on the *same* endogenous variables. I mean the theory of general competitive equilibrium, which postulates that all agents take the prices as given. At arbitrarily given prices the system will generally not be in equilibrium. General competitive equilibrium theory investigates the conditions for the existence of equilibrium prices and quantities. The general competitive equilibrium, if it exists, appears to be a Pareto optimum, so that there is no conflict with the theory of simultaneous maximisation. These results, beautiful as they are, leave economics without a theory of how prices come about (Arrow, 1959, p. 47). With price-taking behaviour as maintained hypothesis, the one logically consistent way out is to assume that all agents know the world they live in, are able to solve the general-equilibrium model for the equilibrium prices just like the applied general-equilibrium economist is able to, and effectuate all exchanges at the equilibrium prices.[11]

Trying to formulate a theory of price formation, you are naturally led to consider the possibility that some agents have market power and are able to set prices to their own advantage. You then leave the theory of general competitive equilibrium and enter the world of models of imperfect competition. Now you are likely to be in trouble, as incorrect conditioning is the essence of most models of imperfect competition. Non-cooperative game theory is used extensively in specifying such models, and you arrive at some Nash equilibrium. In general the equations defining a Nash equilibrium do not constitute the first-order conditions for the underlying simultaneous maximum problem: the Nash equilibrium is not Pareto optimal. Therefore all agents may be better off, in a payoff maximising sense, by making systematic errors in solving the model, in such a way that they end up in some point that is Pareto superior to the Nash equilibrium. It is not in their own interest, that is, not rational, to solve the model correctly.

This observation is made by Benassy (1993) in a somewhat more complicated context. Take a multi-period model with forward-looking agents, who condition on the expected future values of a number of variables in deriving the optimal choices for the current period. No complications arise in case the expectations concern only variables that are exogenous to the model at hand. But consider the case that some expectations relate to endogenous variables. The model itself yields the endogenous variables as functions of the exogenous variables. Therefore the model imposes restrictions on the expectations of endogenous variables that the agents may maintain consistently: the expected values of endogenous variables must coincide with the model solutions for these variables at the expected values of the exogenous variables.[12] Such expectations are called "model consistent" or "rational."

Benassy (1993) constructs an intertemporal micro-based macroeconomic model that allows for imperfect competition and accommodates both rational and non-rational expectations. The prices are set by the agents on the supply sides of the markets. The equilibrium is a Nash equilibrium. In the extreme case of perfect competition the first theorem of welfare holds, so that the general competitive equilibrium with rational expectations is Pareto optimal. As soon as you allow for a slight degree of imperfect competition, however, the outcomes under rational expectations are not Pareto optimal, that is, they can be dominated by outcomes under "irrational" expectations. Remember that, by definition, the rational expectations are the model consistent ones. Irrational expectations (of endogenous variables), being different from the rational ones, do not coincide with the model solutions for these variables: maintaining irrational expectations is like making errors in solving the model. Benassy (1993) proves the existence of irrational-expectations schemes that Pareto-dominate the rational expectations, or, stated differently, he proves that the agents may be better off in a payoff maximising sense by making systematic errors in solving the model.

[11] Waving the issue of plausibility away, you are still left wondering why the agents would trade along a straight line from the initial point to a point of the simultaneous maximum. The more realistic exchange processes with price adjustment as studied by Smale (1976a) allow for curvilinear paths, or *trade curves*.

[12] That is, under certainty equivalence.

Note that in the model with imperfect competition and rational expectations, the notion of rationality differs from one place to the other. In forming their expectations the agents are able to account for all interdependencies in the economy, as (wrongly) described by the equations defining the Nash equilibrium. At the optimisation stage, however, in deriving the “first-order conditions” that constitute the model, they have *not* been able to account for the interdependencies between their maximisation problems. The result of Benassy (1993) looks like a paradox only as long as you adhere to the view that the way prices are set in his model corresponds to rational behaviour. As soon as you acknowledge that the model of price-setting behaviour involves incorrect conditioning on endogenous variables, you will recognize the result as a contradiction, proving an inconsistency among the assumptions.

5. Non-cooperative game theory in retrospect

I hope that the argument and examples above have convinced the reader that non-cooperative game theory does not describe rational, optimising behaviour. The essential element in the definition of the Nash equilibrium of a game is that each player, in determining the “optimal” value of her own action, conditions on the *endogenous* actions of the other players. Thus Nash (1951) replaces the simultaneous maximum problem that constitutes the game by a number of conditional maximum problems; the solution of the union of the “first-order conditions” constitutes the “equilibrium.” This way of dealing with a simultaneous maximum problem is wrong in an elementary way. It treats the arguments inconsistently when deriving the “first-order conditions,” by holding each argument constant in all maximands but the one associated with it. And next, it treats all arguments as variables in the resulting system of partial first-order derivatives. In general the result is not optimal, as is evident from the theory of simultaneous maximisation developed by De Finetti (1937a,b).

Many economic models are the first-order conditions for a number of constrained maximum problems, interlinked by a number of identities (the market equilibrium conditions) that are added *only after the optimisation stage*. This way of model building is not generally correct: one may not introduce new constraints after the derivation of first-order conditions, one must take all constraints into account right from the start. The presence of the identities means that the constrained maximum problems are interdependent and constitute in fact a simultaneous maximum problem. It thus appears that such models suffer from the same weakness as the Nash equilibrium does. Reading Chapter I of Von Neumann and Morgenstern (1947), and especially their Section 2, I cannot help but getting the feeling that precisely this feature is the error in economic model building that they objected to and sought to remedy, by constructing the theory of games. Apparently Von Neumann and Morgenstern did not know the theory of simultaneous maximisation of De Finetti (1937a,b). They even denied explicitly the logical possibility of maximising several functions at once. Still, as shown in Section 2.4, their “solution” of a game is intimately related to the simultaneous maximum.

My reading of Nash (1951) is that, by enunciating the notion of non-cooperative equilibrium, he has promoted incorrect conditioning in mathematical economics from vice to virtue. Under the influence of his work game theory has contributed to the perpetuation of the very flaw it was designed to repair.

Finally, let me emphasise that my criticism of non-cooperative game theory is purely formal and has nothing to do with the empirical performance of the theory. As a matter of fact, non-cooperative game theory does have a poor empirical record. Because the proponents of the theory think that it correctly describes the players as rational and optimising, they have taken the conflict between theory and observations as evidence against (full) rationality and have begun to devise models of bounded rationality (for a survey of the burgeoning literature, see Conlisk, 1996). As I have shown, non-cooperative game theory errs in the solution of the simultaneous maximum problem that is defined by the assumption of rational, optimising behaviour. So, if the predictions of non-cooperative game theory would agree well with observations, we would know for sure that the players are *not* rational, optimising agents. My proof that non-cooperative game theory is mathematically flawed keeps alive the hope that the behaviour of the players may still be characterised as rational and optimising.

6. Some consequences

6.1. How to model non-cooperative behaviour

Nash (1951) motivates his notion of solution by arguing that it applies when the players of a game are unable to cooperate. The need to abandon non-cooperative game theory leaves us without a model for such a situation. There is, however, no need of a new "solution concept" to deal with this case. The gap may be filled in a quite ordinary way, at least in principle. The activities of communication and co-ordination take time, and probably require the input of other scarce resources as well. Some people have specialised in co-ordinating the activities of others, and some industries are in the business of connecting people. The economist faces the task of explaining the observed levels of such activities, along with those of the other activities. To this end he must incorporate in his model a number of variables that explicitly represent the activities involved in co-ordination and communication, specify the values they may take, and account for their effects on the payoffs of the players. Thus he obtains a new simultaneous maximum problem, possibly with (in)equality constraints that express limitations on the opportunities for cooperation. In the solution the optimal values of the co-ordinating activities are determined jointly with those of the other activities. It is quite likely that different levels of the co-ordinating activities are optimal at different times and different places, very much in the spirit of Coase's (1937) ideas. The Nash Program of finding "non-cooperative" justifications for "cooperative" (i.e., Pareto optimal) outcomes starts from the wrong end.

6.2. The need to revise positive economics

The widespread use of non-cooperative game theory means that large parts of economic theory are obsolete, being based on a mathematically flawed theory. The main exception is general competitive equilibrium theory, which assumes that all agents condition on the *same* endogenous variables—the prices of the goods—and arrives at an equilibrium that is Pareto optimal. Mathematical consistency is, however, no guarantee for empirical relevance: imperfect competition appears a real-life phenomenon. As non-cooperative game theory has become the standard tool for modelling non-price-taking behaviour, fields like the theory of Industrial Organisation and general-equilibrium theory with imperfect competition will have to be reconsidered.

6.3. The impossibility of normative economics

Economists are fond of welfare analysis, eager to discover an outcome that is not Pareto optimal. And when they find one, they are ready to suggest a remedy—usually the introduction of a tax or subsidy—to better the outcome for at least someone without harming anyone. It thus seems as if economists, even though they have based their model on the assumption of rational, optimising behaviour by completely informed players, have a superior insight into how the economy works and how the outcome can be improved. In fact the assumption means that the model is some simultaneous maximum problem, the solution of which is the set of Pareto optimal points. Any other outcome witnesses an error in the reasoning from assumptions to conclusions. The remedy is to rectify the error.

Postulating a social-welfare function to account for distributional concerns amounts to introducing a new player—a social planner, say—in the game. The social planner disposes of a set of instruments (for example, tax rates and other parameters of the tax system) that he wants to set so as to achieve maximal social welfare. Commonly the other players are assumed to take the actions of the social planner as given in making their own decisions. The assumption involves incorrect conditioning on endogenous variables, and leads to a specific case of the conjectural variations oligopoly with the social planner as the Stackelberg leader and all other players as followers. The correct approach to the extended model is to treat the social planner on a par with the other players in the game.

6.4. The need of a stronger assumption

The assumption of rational, optimising behaviour by all agents in an economy yields some simultaneous maximum problem as the model of the world. It leaves us with a large solution set, the simultaneous maximum. The assumption is not strong enough to predict a specific point as the outcome of the economic process. Reality, however, does not seem to have the same degree of indeterminateness. We are in need of a stronger assumption, one that gives us more definite predictions. What could this assumption be like?

Simultaneous maximisation is like maximising a weighted sum of functions without specifying the weights. The choice of specific weights would select one point of the simultaneous maximum as *the* solution, thus resolving the indeterminateness. The arbitrary choice of weights would, however, not be a satisfactory way out; it would just move the indeterminateness to a different place. The task before us, then, is to come up with an economic theory of the weights. Making the weights endogenous, dependent on economic variables, amounts to specifying new maximands in such a way that the maximisation of their unweighted sum is the principle that describes the working of the economic system.

References

Allen, B., 1992. Price and quantity competition in homogeneous duopoly markets. Economics Letters 38, 417–22.

Amir, R., Jin, J., 2001. Cournot and Bertrand equilibria compared: Substitutability, complementarity and concavity. International Journal of Industrial Organization 19, 303–17.

Arrow, K.J., 1959. Toward a theory of price adjustment, in: The Allocation of Economic Resources: Essays in Honor of Bernard Francis Haley. Stanford University Press, Stanford.

Aumann, R.J., 1987. Game theory, in: Eatwell, J., Milgate, M., Newman, P. (Eds.), The New Palgrave: A Dictionary of Economics. The Macmillan Press Ltd, London and Basingstoke, pp. 460–82.

Benassy, J.–P., 1993. Imperfect competition and the suboptimality of rational expectations. European Economic Review 37, 1315–30.

Bertrand, J., 1883. Revue de la "Théorie mathématique de la richesse sociale" et des "Recherches sur les principes mathématiques de la théorie des richesses". Journal des Savants 67, 499–508.

Bowley, A.L., 1924. The Mathematical Groundwork of Economics. The Clarendon Press, Oxford.

Boyer, M., Moreaux, M., 1983. Consistent versus non-consistent conjectures in duopoly theory: Some examples. Journal of Industrial Economics 32, 97–110.

Bresnahan, T.F., 1981. Duopoly models with consistent conjectures. American Economic Review 71, 934–45.

Chavas, J.–P., 1984. The theory of mixed demand functions. European Economic Review 24, 321–44.

Cheng, L., 1985. Comparing Bertrand and Cournot equilibria: A geometric approach. Rand Journal of Economics 16, 146–52.

Coase, R.H., 1937. The nature of the firm. Economica 4, 386–405.

Conlisk, J., 1996. Why bounded rationality? Journal of Economic Literature 34, 669–700.

Cournot, A.A., 1838. Recherches sur les principes mathématiques de la théorie des richesses. Hachette, Paris.

Cournot, A.A., 1927. Researches into the Mathemathical Principles of the Theory of Wealth (translation by N.T. Bacon of Cournot, 1838). Macmillan, New York. Second edition.

Daughety, A.F., 1985. Reconsidering Cournot: The Cournot equilibrium is consistent. Rand Journal of Economics 16, 368–79.

De Finetti, B., 1937a. Problemi di "optimum". Giornale dell'Istituto Italiano degli Attuari 8, 48–67.

De Finetti, B., 1937b. Problemi di "optimum" vincolato. Giornale dell'Istituto Italiano degli Attuari 8, 112–26.

De Finetti, B., 1940. Il problema dei "pieni". Giornale dell'Istituto ltaliano degli Attuari 9, l–88.

Dixit, A.K., 1976. Optimization in Economic Theory. Oxford University Press, Oxford.

Dixit, A.K., 1990. Optimization in Economic Theory. Oxford University Press, Oxford. Second edition.

Edgeworth, F.Y., 1881. Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences. C. Kegan Paul & Co., London.

Fellner, W., 1949. Competition Among the Few. A. Knopf, New York.

Friedman, J.W., 1988. On the strategic importance of prices versus quantities. Rand Journal of Economics 19, 607–22.

Frisch, R., 1933. Monopole–Polypole–La notion de force dans l'économie, "Festskrift till Harald Westergaard". NationalØkonomisk Tidsskrift (Supplement) .

Frisch, R., 1951. Monopoly–Polypoly–The concept of force in the economy (translation by W. Beckerman of Frisch, 1933), in: Peacock et al. (1951). pp. 23–36.

Guilbaud, G.T., 1949. La théorie des yeux: Contributions critiques à la théorie de la valeur. Economie Appliquée 3, 275–319.

Guilbaud, G.T., 1951. The theory of games: Critical contributions to the theory of value (translation by A. L. Minkes of Guilbaud, 1949), in: Peacock et al. (1951). pp. 37–65.

Hathaway, N.J., Rickard, J.A., 1979. Equilibria of price-setting and quantity-setting duopolies. Economics Letters 3, 133–7.

Hildenbrand, W., 1993. Francis Ysidro Edgeworth: Perfect competition and the core. European Economic Review 37, 477–90.

Hsu, J., Wang, X.H., 2005. On welfare under Cournot and Bertrand competition in differentiated oligopolies. Review of Industrial Organization 27, 185–91.

Kamien, M.J., Schwartz, N.L., 1983. Conjectural variations. Canadian Journal of Economics 16, 191–211.

Kaysen, C., 1946–1947. A revolution in economic theory? Review of Economic Studies 14, 1–15.

Kreps, D.M., 1990. A Course in Microeconomic Theory. Harvester Wheatsheaf, London.

Laitner, J., 1980. "Rational" duopoly equilibria. Quarterly Journal of Economics 95, 641–62.

Léonard, D., Van Long, N., 1992. Optimal Control Theory and Static Optimization in Economics. Cambridge University Press, Cambridge, MA.

Luce, R.D., Raiffa, H., 1957. Games and Decisions: Introduction and Critical Survey. John Wiley & Sons, Inc., New York.

Malinvaud, E., 1972. Lectures on Microeconomic Theory. volume 2 of *Advanced Textbooks in Economics*. North Holland Publishing Company, Amsterdam–London.

Markowitz, H., 1952. Portfolio selection. Journal of Finance 6, 77–91.

Mayberry, J.P., Nash, J.F., Shubik, M., 1953. A comparison of treatments of a duopoly situation. Econometrica 21, 141–54.

Nash, J., 1951. Non-cooperative games. Annals of Mathematics 54, 286–95.

Nieuwenhuis, A., 2011. Simultaneous maximisation in economic theory. Translations of De Finetti (1937a,b), available at `http://www.brunodefinetti.it`.

Pareto, V., 1909. Manuel d'Economie Politique. Giard et Brière, Paris.

Peacock, A.T., Houthakker, H.S., Turvey, R., Lutz, F., Henderson, E. (Eds.), 1951. International Economic Papers. volume 1 of *Translations prepared for the International Economic Association*, International Economic Association. Macmillan and Company Ltd and The Macmillan Company, London and New York.

Perry, M.K., 1982. Oligopoly and consistent conjectural variations. The Bell Journal of Economics 13, 197–205.

Salant, D.J., 1984. On the consistency of consistent conjectures. Economics Letters 16, 151–7.

Singh, N., Vives, X., 1984. Price and quantity competition in a differentiated duopoly. Rand Journal of Economics 15, 546–54.

Smale, S., 1974a. Global analysis and economics III: Pareto Optima and price equilibria. Journal of Mathematical Economics 1, 107–17.

Smale, S., 1974b. Global analysis and economics V: Pareto theory with constraints. Journal of Mathematical Economics 1, 213–21.

Smale, S., 1975. Optimizing several functions, in: Manifolds, Tokyo 1973: Proceedings of the International Conference on Manifolds and Related Topics in Topology, University of Tokyo Press, Tokyo. pp. 69–75.

Smale, S., 1976a. Exchange processes with price adjustment. Journal of Mathematical Economics 3, 211–26.

Smale, S., 1976b. Global analysis and economics VI: Geometric analysis of Pareto Optima and price equilibria under classical hypotheses. Journal of Mathematical Economics 3, 1–14.

Stigler, G.J., 1964. A theory of oligopoly. Journal of Political Economy 72, 44–61.

Tanaka, Y., 2001. Evolution to equilibrium in an asymmetric oligopoly with differentiated goods. International Journal of Industrial Organization 19, 1423–40.

Tasnádi, A., 2006. Price vs. quantity in oligopoly games. International Journal of Industrial Organization 24, 541–54.

Theil, H., 1968. Optimal Decision Rules for Government and Industry. volume 1 of *Studies in Mathematical and Managerial Economics*. North Holland Publishing Company / Rand McNally, Amsterdam/Chicago.

Van Damme, E.E.C., 1990. Speltheorie. Economisch Statistische Berichten 75, 1036–42.

Varian, H.R., 1992. Microeconomic Analysis. W. W. Norton & Company, New York and London. Third edition.

Vives, X., 1985. On the efficiency of Bertrand and Cournot equilibria with product differentiation. Journal of Economic Theory 36, 166–75.

Von Neumann, J., Morgenstern, O., 1947. Theory of Games and Economic Behavior. Princeton University Press, Princeton, New Jersey. Second edition.

Von Stackelberg, H., 1934. Marktform und Gleichgewicht. Verlag von Julius Springer, Wien und Berlin.

Zaccagnini, E., 1947. Massimi simultanei in economia pura. Giornale degli Economisti e Annali di Economia 6 (Nuova serie), 258–92.

Zaccagnini, E., 1951. Simultaneous maxima in pure economics (translation by R. Orlando of Zaccagnini, 1947), in: Peacock et al. (1951). pp. 208–44.