This a translation of “Световой луч круговой поляризации несет удвоенный момент импульса” http://www.mai.ru/science/trudy/published.php?ID=34422 (2003). JMO made an inquiry about the translation.

Circularly polarized light beam carries the double angular momentum (2003)

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A calculation is presented of absorption of a circularly polarized electromagnetic beam without an azimuth phase structure in a dielectric. The calculation shows that such a beam carries the double angular momentum as compared with predictions of the modern electrodynamics, which are used by experimenters. This angular momentum is consists of two equal parts, orbital and spin. A classical spin tensor of electromagnetic waves is used in the paper.

1 Introduction

In the last years an explosion of interest is observed to measuring of forces acting on optically trapped microparticles. It may concern contents of living cells [1] or characteristics of suspensions. If a circularly or elliptically polarized beam is used, the trapped particle experiences a torque and may be driven to a rotation [2 – 8]. In this case, it is important to know angular momentum which is carried by the beam

Today all experts hold the opinion [9 –13] that total angular momentum J of a light beam without an azimuth phase structure is given by the formula

$$J = U/\omega, \tag{1}$$

where U is energy of the beam, and this angular momentum is calculated according

$$\mathbf{J} = \int \mathbf{r}\times(\mathbf{E}\times\mathbf{H})dV\,. \tag{2}$$

Heitler wrote [9, p.401]:

“A plane wave traveling in the z-direction and with infinite extension in the xy-directions can have no angular momentum about the z-axis, because $\mathbf{E}\times\mathbf{H}$ is in the z-direction and $[\mathbf{r}\times(\mathbf{E}\times\mathbf{H})]_z = 0$. However, this is no longer the case for a wave with finite extension in the xy-plane. Consider a cylindrical wave with its axis in the z-direction and traveling in this direction. At the **wall** of the cylinder $r = R$, say, we let the amplitude drop to zero. It can be shown that the wall of such a wave packet gives a finite contribution to J_z”.

Ohanian wrote [10, p.502]:

“In a wave of finite transverse extent, the $\mathbf{E}$ and $\mathbf{B}$ fields have a component parallel to the wave vector (the field lines are closed loops) and the energy flow has components perpendicular to the wave vector… The circulating energy flow in the wave implies the existence of angular momentum, whose direction is along the direction of propagation. This angular momentum is the spin of the wave”.

Simmonds and Guttman [11, p.227] wrote:

``The electric and magnetic field can have a nonzero z-component only within the **skin** region of the wave. Having z-component within this region implies the possibility of a nonzero z-component of angular momentum within this region. Since the wave is identically zero outside the **skin** and constant inside the **skin** region, the skin region is the only one in which the z-component of angular momentum does not vanish”

Jackson [12] used an expression

$$\breve{\mathbf{E}} = \exp[i(kz-\omega t)][\mathbf{x}+i\mathbf{y}+\mathbf{z}\frac{1}{k}(i\partial_x-\partial_y)]E_0(x,y), \quad \breve{\mathbf{B}} = -ik\breve{\mathbf{E}}/\omega, \qquad k^2 = \varepsilon\omega^2. \tag{3}$$

for the circularly polarized beam. Here $E_0(x, y) = \mathtt{Const}$ inside the beam, and $E_0(x, y) = 0$ outside the beam, ε is the permittivity. We set the light speed $c = 1$ and mark complex vectors and numbers by sign breve.

Equation (2) gives

$$J = \int E_0^2 dV / \omega$$

for vacuum $(\varepsilon = 1)$. And it is obvious that

$$U = \int E_0^2 dV .$$

Thus, the ratio $U / J = \omega$ is equal to $U / S = \omega,$ i.e. to the ratio energy/spin for a photon.

We criticized this concept [14 – 27]. Here we calculate the total angular momentum of Jackson's beam (3) by considering absorption in dielectric. We show that, in reality, the total angular momentum of the beam is twice as much as U / ω, i.e. $J = 2U / \omega.$

2. Cylindrical coordinates

Because the cylindrical symmetry of the beam, the cylindrical coordinates $r, \varphi, z,$

$$x = r\cos\varphi, \quad y = r\sin\varphi,$$

with metrics

$$dl^2 = dr^2 + r^2 d\varphi^2 + dz^2, \quad g_{rr} = 1, \quad g_{\varphi\varphi} = r^2, \quad g_{zz} = 1, \quad \sqrt{g}_{\wedge} = r, \quad g^{\varphi\varphi} = 1/r^2.$$

are in use. The root of determinant of metric tensor is a scalar density of weight +1; the sign "wedge" at the level of lower indices marks this. A volume element is the density of weight -1 and is marked by wedge at the level of top indices, $dV^{\wedge} = dr d\varphi dz,$ as well, as absolute antisymmetric density $e^{\wedge}_{r\varphi z} = \pm 1, 0$. (We ignore the fact that $e^{\wedge}_{r\varphi z}$ is, in reality, a pseudo density).

The transformation of *covariant* components of vectors $\mathbf{E}, \mathbf{B}$ in (3) gives

$$\underset{\to}{\breve{E}} = \exp[i(kz - \omega t + \varphi)](\underset{\to}{r} + ir\underset{\to}{\varphi} + \underset{\to}{z}\frac{i}{k}\partial_r)E_0(r), \qquad \underset{\to}{\breve{B}} = -ik\,\underset{\to}{\breve{E}}/\,\omega. \tag{4}$$

Here the arrow placed under a character means covariant vectors or covariant coordinate vectors. $E_0 \approx Const$ if $r < R, \quad E_0 \approx 0$ if $r > R$.

3. Dielectric

Electric field polarizes dielectric when an electromagnetic wave passes through it. The polarization vector $\mathbf{P}$, its time derivative, i.e. the displacement current $\mathbf{j}$, and density of the Lorentz force $\mathbf{f}$ are given by equations

$$\mathbf{P} = (\varepsilon - 1)\mathbf{E}, \quad \mathbf{j} = \partial_t \mathbf{P}, \quad \mathbf{f} = \mathbf{j} \times \mathbf{B}. \tag{5}$$

Besides, the circular polarization gives rise to volume density of a torque [28]

$$\mathbf{M} = \mathbf{P} \times \mathbf{E}. \tag{6}$$

At the beginning, let us consider quantity (5), or rather the z-component of the cross product, which is a torque density, $d\underset{f}{\tau}_z / dV = (\mathbf{r} \times \mathbf{f})_{\mathbf{z}}$. We interpret the torque $\underset{f}{\tau}_z$, which is provoked by the force $\mathbf{f}$ and acts within the **skin** region, as an orbital torque. It can be obtained by integration of quantity

$$d\underset{f}{\tau}_z = rf_{zr}\sqrt{g}_{\wedge} dV^{\wedge} \tag{7}$$

over all volume of the dielectric ($z > 0$) with time averaging (here $f_{zr} = 2j_{[z}B_{r]}$).

We must insert in (7) the complex values

$$\breve{E}_r = \exp[i(\breve{k}z - \omega t + \varphi)]E_0(r), \qquad \breve{E}_z = \exp[i(\breve{k}z - \omega t + \varphi)]i\partial_r E_0(r)/k,$$

$$\breve{B}_r = -i\breve{k}\breve{E}_r / \omega, \qquad \breve{B}_z = -i\breve{k}\breve{E}_z / \omega,$$

and take into account that permittivity and wave number are complex numbers as well:

$$\breve{\varepsilon} = \varepsilon' + i\varepsilon'', \qquad \breve{k} = \sqrt{\breve{\varepsilon}}\,\omega = k' + ik''.$$

Time averaging and integrating over φ, z, give

$$\underset{f}{\tau}_z = \pi\int r^2 \Re[(\breve{\varepsilon}-1)(\partial_t \breve{E}_z \bar{B}_r - \partial_t \breve{E}_r \bar{B}_z)]drdz = \pi\int r^2 \Re[(\breve{\varepsilon}-1)(\bar{k}/\breve{k}+1)]\partial_r(E_0^2/2)dr/2k''.$$

The line bar marks complex numbers.

The integrand is nonzero only within the **skin** region near $r = R$ because of the derivative $\partial_r E_0$. Therefore integrating by parts gives the orbital torque acting on the dielectric:

$$\underset{f}{\tau}_z = -\pi R^2 E_0^2 \Re[i(\breve{\varepsilon}-1)(\bar{k}/\breve{k}+1)]/4k'' = \pi R^2 E_0^2 k'(k^2+\omega^2)/(2\omega^2 k^2). \tag{8}$$

Now let us integrate density (6)

$$\tau^z = \int \Re(\breve{P}_r \bar{E}_\varphi - \breve{P}_\varphi \bar{E}_r)\underset{\wedge}{e}^{r\varphi z}(drd\varphi dz)^\wedge/2 = \pi R^2 E_0^2 k'/\omega^2. \tag{9}$$

We interpret it as spin absorbed by the dielectric.

4. Space in front of the dielectric and the total torque

There is torque besides (8) and (9).

Beam (4) propagates in the dielectric where $z > 0$ because there are incident and outgoing beams where $z < 0$:

$$\underset{\to}{\breve{E}_1} = \exp[i(\omega z - \omega t + \varphi)](\breve{k}+1)(\underset{\to}{r} + ir\underset{\to}{\varphi} + \underset{\to}{z}\frac{i}{\omega}\partial_r)E_0(r)/2, \qquad \underset{\to}{\breve{B}_1} = -i\underset{\to}{\breve{E}_1}. \tag{10}$$

$$\underset{\to}{\breve{E}_2} = \exp[i(-\omega z - \omega t + \varphi)](\breve{k}-1)(-\underset{\to}{r} - ir\underset{\to}{\varphi} + \underset{\to}{z}\frac{i}{\omega}\partial_r)E_0(r)/2, \qquad \underset{\to}{\breve{B}_2} = i\underset{\to}{\breve{E}_2}. \tag{11}$$

At the boundary $z = 0$, all components of **B** and tangential components of **E** are continuous, and the normal component E_z is multiplied by factor $1/\breve{\varepsilon}$. This means charge presence with surface density

$$\breve{\sigma} = [\breve{E}_z - \breve{E}_{1z} - \breve{E}_{2z}]_{z=0} = -i\exp[i(-\omega t + \varphi)]\partial_r E_0(\breve{\varepsilon}-1)/\breve{k}.$$

This charge experiences tangential forces σE_φ from the electric field. Moment of the forces is an orbital torque $\underset{\sigma}{\tau}$ acting on the surface of the dielectric. Integrating gives:

$$\underset{\sigma}{\tau} = \int r\Re(\breve{\sigma}\bar{E}_\varphi)drd\varphi/2 = -\pi\int r^2\partial_r(E_0^2/2)\Re[(\breve{\varepsilon}-1)/\breve{k}]dr = \pi R^2 E_0^2 k'(k^2-\omega^2)/(2\omega^2 k^2). \tag{12}$$

It is remarkable that the sum of skin orbital torque (8) and surface orbital torque (12) is equal to spin torque (9),

$$\underset{f}{\tau} + \underset{\sigma}{\tau} = \tau = \pi R^2 E_0^2 k'/\omega^2. \tag{13}$$

So the total torque is equal to the double quantity:

$$\underset{tot}{\tau} = \underset{f}{\tau} + \tau + \underset{\sigma}{\tau} = 2\pi R^2 E_0^2 k'/\omega^2. \tag{14}$$

5. The flows of torque and energy in space in front of the dielectric

Torque (14) is provided by flux of angular momentum in the space. Let us, first, calculate flux of moment of momentum by the use of Maxwell tensor density

$$\underset{\wedge}{T}^{\varphi z} = -\Re[(\breve{E}_{1\varphi}+\breve{E}_{2\varphi})(\bar{E}_{1z}+\bar{E}_{2z}) + (\breve{B}_{1\varphi}+\breve{B}_{2\varphi})(\bar{B}_{1z}+\bar{B}_{2z})]g^{\varphi\varphi}\underset{\wedge}{\sqrt{g}}/2 = -k'\partial_r E_0^2/2\omega^2.$$

The torque (2) is

$$\underset{L}{\tau}_z = \int r\underset{\wedge}{T}^{\varphi z} da^{\wedge}_z \underset{\wedge}{e}_{r\varphi z}\underset{\wedge}{\sqrt{g}} = \int r^2 \underset{\wedge}{T}^{\varphi z} drd\varphi = \pi R^2 E_0^2 k'/\omega^2 \tag{15}$$

We interpret this torque as orbital torque. Space orbital torque (15) coincides with orbital torque acting on the dielectric $\underset{f}{\tau} + \underset{\sigma}{\tau}$, (8) + (12).

It is important that Maxwell tensor in the space cannot provide spin torque (9). To provide this torque a spin flux must be in the space. We calculate the spin flux using a component of spin tensor [16 - 20]

$$\Upsilon_{r\varphi z} = A_{[r}\partial_{|z|}A_{\varphi]} + \Pi_{[r}\partial_{|z|}\Pi_{\varphi]}, \tag{16}$$

where A_i, Π_i are the magnetic and electric vector potentials.

In this case the potentials are complex:

$$\underset{\to}{\breve{A}} = -\int(\underset{\to}{\breve{E}_1} + \underset{\to}{\breve{E}_2})dt = -i(\underset{\to}{\breve{E}_1} + \underset{\to}{\breve{E}_2})/\omega, \tag{17}$$

$$\underset{\to}{\breve{\Pi}} = \int(\underset{\to}{\breve{B}_1} + \underset{\to}{\breve{B}_2})dt = i(\underset{\to}{\breve{B}_1} + \underset{\to}{\breve{B}_2})/\omega, \tag{18}$$

The spin flux, i.e. spin torque is calculated by integrating over a cross-section of the beam in vacuum:

$$\underset{S}{\tau_z} = \int \Upsilon^{r\varphi z}_{\wedge} da^{\wedge}_{z} e^{\wedge}_{r\varphi z}\sqrt{g}_{\wedge},$$

where $\Upsilon^{r\varphi z}_{\wedge} = \Upsilon_{r\varphi z} g^{\varphi\varphi}\sqrt{g}_{\wedge}$, so

$$\underset{S}{\tau_z} = \int \Upsilon_{r\varphi z} dr d\varphi.$$

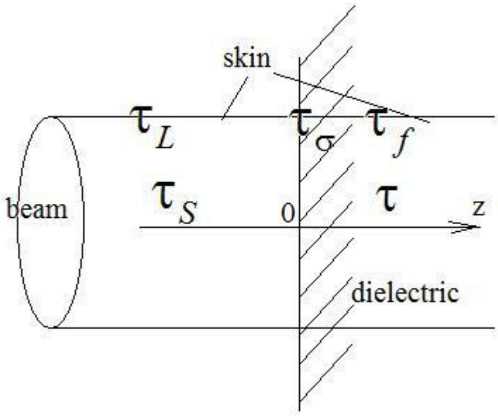
Inserting the complex expressions (17), (18) and time averaging gives the quantity (9)

$$\underset{S}{\tau_z} = \pi R^2 E_0^2 k'/\omega^2. \tag{19}$$

Thus the total flux of angular momentum in space is equal to the total torque acting on the dielectric:

$$\underset{L}{\tau_z} + \underset{S}{\tau_z} = \underset{f}{\tau_z} + \underset{\sigma}{\tau_z} + \tau = 2\pi R^2 E_0^2 k'/\omega^2, \tag{20}$$

and orbital and spin parts are equal to each other pairwise. The quantity of equation (20) are depicted in Figure.



We can apply our calculating to energy.

The energy flux of the beam in space consists of two parts spatially separated, as well as the angular momentum flux. The Poynting vector is directed along z-axis in the bulk of the beam. This part of energy flux is accompanied by spin flux, which is distributed uniformly, just as the part of energy flux. The Poynting vector of the **skin** region has an angular component which causes the orbital angular momentum. However, the mass-energy circulating in the skin region is smallish and does not contribute to the beam power

The beam power *W* is obtained by integrating of the Poynting vector over a cross-section of the beam in vacuum as well as in (19):

$$W = \int \Re[(\breve{E}_{1r} + \breve{E}_{2r})(\overline{B}_{1\varphi} + \overline{B}_{2\varphi}) - (\breve{E}_{1\varphi} + \breve{E}_{2\varphi})(\overline{B}_{1r} + \overline{B}_{2r})]d\varphi dr/2 = \pi R^2 E_0^2 k'/\omega^2. \tag{21}$$

Thereby the equation $W/\tau = \omega$, which is characteristic of photons, holds. However, besides spin (19), the beam contains orbital angular momentum (15), which is equal to spin (19), but is not connected with beam power (21).

6. Conclusion

Thereby the total angular momentum of a circularly polarized beam is twice as much as the Maxwell tensor predicts. And angular momentum (2), which is recognized by physicists and is equal to U/ω, is orbital angular momentum rather than spin, i.e. we must write $L = U/\omega$. But L does not associated with *U,* beam's energy. The flow of beam's energy U is accompanied by the flow of spin $S = U/\omega$, and the density of the spin flux is given by formula (16), which is absent in the modern electrodynamics. The orbital angular momentum of the beam, L, is simply equal to the spin S. As a result, total angular momentum of a circularly polarized beam without an azimuth phase structure is

$$J = L + S = 2U/\omega.$$

We introduce a classical spin tensor into the modern electrodynamics. Physicists overlooked the spin tensor because of the negation of a local sense of energy-momentum tensors. The significance of the Lagrange formalism is exaggerated.

7. History

Material of the paper was rejected or ignored by following journals (dates of submissions are in brackets).

Phys. Rev. D (25.09.01, 22.09.02),
American J. of Physics (15.09.99, 10.09.01, 03.06.02, 12.03.03),
Foundation of Physics (28.05.01, 05.05.02, 08.08.02, 23.10.02, 16.04.03, 17.04.03),
Acta Physica Polonica B (28.01.02, 09.05.02, 02.06.02, 12.03.03, 13.03.03),
Physics Lett. A (22.07.02, 28.08.02, 14.11.02, 14.12.02, 12.03.03, 21.04.03),
Optics Com. (22.09.02, 16.06.03),
Europhysics Letters (08.10.02, 12.05.03, 15.04.03,18.07.03)
J. of Physics A (23.06.02, 02.03.03, 12.03.03)
J. of Math Phys. (28.11.02, 13.03.03,28.03.03),
Apeiron (28.04.03, 30.04.03),
Optics Letters (29.07.03),
New J. of Physics (27.06.03, 15.07.03),
J. of Optics A (30.11.03)
arXiv (21.01.02, 18.02.02, 02.06.02, 13.06.02, 15.05.03).
JETP Letters (14.05.98, 17.06.02, 05.08.02, 20.11.02, 02.10.03),
JETP (27.01.99, 25.02.99, 13.04.00, 25.05.00, 16.0501, 26.11.01, 05.07.02, 19.12.02, 02.10.03),
TMP (29.04.99, 17.02.00, 29.05.00, 18.10.00, 05.07.02),
UFN (25.02.99, 12.01.00, 31.05.00, 26.06.02, 31.07.02),
RPJ (18.05.99, 15.10.99, 01.03.00, 25.05.00, 31.05.01, 24.11.01, 05.07.02, 19.12.03, 22.10.03).
Optics and Spectroscopy (01.09.03).

Editors-in-Chief of JETP Letters, JETP, TMP, RPJ, V.F. Gantmakher, A.F. Andreev, A.A. Logunov, V.N. Detinko ignored my complaints about lack of reviews or mistaken reviews.

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