The Concept of *pq*-Functions

Mohamed E. Hassani[1]
Institute for Fundamental Research
BP.197, CTR, GHARDAIA 47800, ALGERIA

Abstract: In this article we study the concept of *pq*-functions which should regard as an extension of prior work relating to *pq*-radial functions [1]. Here, our main aim is to generalize this concept to the field of complex numbers. As direct consequences, *new* kind of (partial) differential equations, polynomials, series and integrals are derived, and Joukowski function is generalized.

Keywords: complex analysis, *pq*-functions, *pq*-DEs, *p*-polynomials, *q*-polynomials, *pq*-series, *pq*-integrals

Mathematics Suibject Classification (2010): 30-XX; 31-XX

1. Introduction

In previous work [1], we have heuristically investigated the properties of new class of potential functions results from the concept of *pq*-radial functions with one main real radial variable and three real auxiliary parameters. In the present paper, we wish to generalize this concept to the field of complex numbers and study the resulting properties. Through this paper, we assume that the reader is familiar with [1] and also with the complex analysis.

2. Properties of *pq*-functions:

We begin our investigation on the aforementioned subject by the following specific definition of the concept of *pq*-functions.

2.1. Specific definition: Suppose **U** is an open subset of the complex plan **C**, such that $F_{p,q}$: **U**→**C**, $z \mapsto F_{p,q}(z,s,\rho,\varphi,\theta)$; $s,\rho \in \mathbf{U}$; $p,q,\varphi,\theta \in \mathbf{R}$; denote a continuous and differentiable function; $F_{p,q}$ is said to be *pq*-function if and only if is conceptually expressed in the following form:

$$F_{p,q}(z,s,\rho,\varphi,\theta) = G^{p}(z,s,\varphi)H^{-q}(z,\rho,\theta), \tag{1}$$

where $G(z,s,\varphi)$ and $H(z,\rho,\theta)$ are, respectively, the *weight* function and the *characteristic* function both are defined by

$$G(z,s,\varphi) = z^2 - 2sz\sin\varphi + s^2, \tag{2}$$

$$H(z,\rho,\theta) = z^2 - 2\rho z\cos\theta + \rho^2. \tag{3}$$

In the context of this work, $z = (x+iy)$ with $x,y \in \mathbf{R}$ is the main complex variable and the complex parameters s and ρ are treated as auxiliary variables; the real angular parameters φ and θ stay in general fix. Since the characteristic function $H(z,\rho,\theta)$ plays the role of dominator, thus $F_{p,q}$ is defined for each $z \in \mathbf{U}$ such that

$$z \neq \rho e^{\pm i\theta}. \tag{4}$$

E-mail:hassani641@gmail.com [1]

The *pq*-function (1) is also a fundamental family of solutions of the following *pq*-PDE

$$\frac{\partial}{\partial z}\left[\frac{\partial W}{\partial z}-\left(\frac{p}{G}\frac{\partial G}{\partial z}-\frac{q}{H}\frac{\partial H}{\partial z}\right)W\right]+\frac{\partial}{\partial s}\left[\frac{\partial W}{\partial s}+\left(\frac{q}{G}\frac{\partial G}{\partial s}\right)W\right]+\frac{\partial}{\partial \rho}\left[\frac{\partial W}{\partial \rho}+\left(\frac{q}{H}\frac{\partial H}{\partial \rho}\right)W\right]=0, \quad (5)$$

with

$$W=F_{p,q}, G\neq 0, \quad H\neq 0 .$$

2.2. Specific properties of pq-function

The following properties of $F_{p,q}$ are very similar to those of *pq*-radial functions [1].

2.2.1. Properties of $F_{p,q}\equiv F_{p,q}(z,s,\rho,\varphi,\theta)$ *with respect to* z, s, ρ, φ and θ

1/ $\forall p,q,\varphi,\theta\in\mathbf{R}$ and $\forall z,s,\rho\in\mathbf{U}\setminus\{\rho e^{\pm i\theta}\}$, we have for $z\in\{is\,e^{-i\varphi}, -is\,e^{i\varphi}\}$, $F_{p,q}=0$.

2/ $\forall p,q,\varphi,\theta\in\mathbf{R}$ and $\forall z,s,\rho\in\mathbf{U}\setminus\{\rho e^{\pm i\theta}\}$, we have for $p=0$ and $q=0$, $F_{p,q}=1$.

3/ *Homogeneity of* $F_{p,q}$ *with respect to* $z, s, and\ \rho$
$\forall p,q,\varphi,\theta\in\mathbf{R}$ we have for $\forall\varepsilon\in\mathbf{R}_+\setminus\{0\}$: $F_{p,q}(\varepsilon z,\varepsilon s,\varepsilon\rho,\varphi,\theta)=\varepsilon^{2(p-q)}F_{p,q}(z,s,\rho,\varphi,\theta)$.

4/ *Periodicity* $F_{p,q}$ *with respect to* φ *and* θ
$\forall p,q,\varphi,\theta\in\mathbf{R}$ and $\forall z,s,\rho\in\mathbf{U}\setminus\{\rho e^{\pm i\theta}\}$ we have $\forall k\in\mathbf{Z}$: $F_{p,q}(z,s,\rho,\varphi+2k\pi,\theta+2k\pi)=F_{p,q}(z,s,\rho,\varphi,\theta)$.

Remark: Properties (1) and (2) are very useful particularly for the orthogonality condition of *pq*-functions as we will see, and property (4) means that $F_{p,q}$ is double-periodic.

2.2.2. Properties of $F_{p,q}$ *with respect to the orders* p *and* q

The following series of properties is very important since it shows us how some basic operations performed on *pq*-functions should reduce to the operations performed on their orders. The demonstration of each property should be exclusively based on the compact expression $F_{p,q}=G^p H^{-q}$. Therefore, $\forall p,q,\varphi,\theta\in\mathbf{R}$ and $\forall z,s,\rho\in\mathbf{U}\setminus\{\rho e^{\pm i\theta}\}$, we have the following properties:

1/ $F_{p,q}=F_{p,0}\times F_{p,0}$
2/ $F_{p,q}^{-1}=F_{-p,-q}$
3/ $\forall\ell\in\mathbf{N}^*$: $F_{p,q}^{\ell}=F_{\ell p,\ell q}=F_{\ell(p,q)}$
4/ $\forall\ell,m\in\mathbf{N}^*$: $F_{\ell p,mq}=F_{p/m,q/\ell}^{\ell\cdot m}$
5/ $\forall\ell,m\in\mathbf{N}^*$: $F_{p/\ell,q/m}=F_{mp,\ell q}^{(\ell\cdot m)^{-1}}$
6/ $F_{-p,q}=G^{-2p}F_{p,q}$
7/ $F_{p,-q}=H^{2q}F_{p,q}$
8/ $\forall p,q,p',q'\in\mathbf{R}: F_{p,q}+F_{p',q'}=H^{q'-q}F_{p,q'}+H^{q-q'}F_{p',q}$

9/ $\forall p,q,p',q' \in \mathbf{R} : F_{p,q} - F_{p',q'} = G^{q'-q} F_{p,q'} - G^{q-q'} F_{p',q}$
10/ $\forall p,q,p',q' \in \mathbf{R} : F_{p,q} \,/\, F_{p',q'} = F_{p-p',q-q'}$
11/ $\forall p,q,p',q' \in \mathbf{R} : F_{p,q} \times F_{p',q'} = F_{p+p',q+q'}$

2.2.3. Properties of $F_{p,q}$ with respect to its (partial) derivatives

In this subsection, we study the properties of $F_{p,q}$ with respect to its (partial) derivatives, which are for instance very helpful as new tool to investigate fluid mechanics.

Holmorphicity of $F_{p,q}$: Our main aim here is to show the holomorphicity of $F_{p,q}$. This property is essential because as we know from complex analysis, the holomorphicity of any given function implies its analycity automatically. Since z is the principal independent complex variable of $F_{p,q}$ and s, ρ are only certain auxiliary complex (variable) parameters, hence, this allows us to focus our interest exclusively on the derivatives of $F_{p,q}$ with respect to z. But before all that, let us recall some well-known definitions.

-*Definition.1*: Given a complex-valued function f of a single complex variable, the derivative of f *at* point z_0 in its domain is defined by the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{6}$$

This is the same as the definition of the derivative for real-valued function, except that all the parameters are complex. In particular, the limit is taken as the complex variable z *approaches* z_0, and must have the same value for any sequence of complex values for z approaches z_0 on the complex differentiable at every point z_0 in an open set $\mathbf{S}$, we say that f is complex-differentiable at the point z_0. From all that occurs the following definition.

-*Definition.2*: If f is complex differentiable at every point z_0 in an open set $\mathbf{S}$, say that f is holomorphic on $\mathbf{S}$. We say that f is holomorphic at the point z_0 if it is holomorphic on some *neighborhood* of z_0.

Thus, with the help of definitions (1) and (2), we can prove the holomorphicity of $F_{p,q}$ as follows. Let $z, z_0 \in \mathbf{U} \setminus \left\{\rho e^{\pm i\theta}\right\}$ and $F_{p,q}(z,s,\rho,\varphi,\theta) = G^p(z,s,\varphi) H^{-q}(z,\rho,\theta)$ such that

$$F'_{p,q}(z_\mathbf{0},s,\rho,\varphi,\theta) = \lim_{z \to z_\mathbf{0}} \frac{G^p(z,s,\varphi)H^{-q}(z,\rho,\theta) \,-\, G^p(z_\mathbf{0},s,\varphi)H^{-q}(z_\mathbf{0},\rho,\theta)}{z - z_\mathbf{0}}. \tag{7}$$

Adding and subtracting $G^p(z_\mathbf{0},s,\varphi)H^{-q}(z,\rho,\theta)$ from the numerator of (7), we get

$$F'_{p,q}(z_0,s,\rho,\varphi,\theta) = \lim_{z \to z_0} H^{-q}(z_0,\rho,\theta) \frac{G^p(z,s,\varphi) \,-\, G^p(z_0,s,\varphi)}{z - z_0}$$

$$+\lim_{z\to z_0} G^p(z_0,s,\varphi)\frac{H^{-q}(z,\rho,\theta)\ -\ H^{-q}(z_0,\rho,\theta)}{z-z_0}\ . \tag{8}$$

Finally, we find

$$F'_{p,q}(z_0,s,\rho,\varphi,\theta)=H^{-q}(z_0,\rho,\theta)\left[G^p(z_0,s,\varphi)\right]'+G^p(z_0,s,\varphi)\left[H^{-q}(z_0,\rho,\theta)\right]'. \tag{9}$$

Therefore, it follows that $F_{p,q}$ is holomorphic on $\mathbf{U}\setminus\left\{\rho e^{\pm i\theta}\right\}$ and consequently is analytic.

2.2.4. Derivatives of $F_{p,q}$ with respect to z

After we have proved the holomorphicity/analycity of *pq*-functions let us now investigate the properties of $F_{p,q}$ through its first derivative with respect to z: $z\in\mathbf{U}\setminus\left\{\rho e^{\pm i\theta},ise^{-i\theta},-ise^{i\theta}\right\}$. Hence, the first order derivative of $F_{p,q}$ has the form

$$\frac{dF_{p,q}}{dz}=\left(p\,\frac{G'}{G}-q\frac{H'}{H}\right)F_{p,q}\ . \tag{10}$$

We have, according to the property (2) in subsection (2.3) that is $F^{\ell}_{p,q}=F_{\ell p,\ell q}=F_{\ell(p,q)}$, thus after derivation, we get

$$\frac{dF_{\ell(p,q)}}{dz}=\ell\left(p\,\frac{G'}{G}-q\frac{H'}{H}\right)F_{\ell(p,q)}. \tag{11}$$

Introducing the power ℓ in (10) to obtain

$$\left(\frac{dF_{p,q}}{dz}\right)^{\ell}=\left(p\,\frac{G'}{G}-q\frac{H'}{H}\right)^{\ell}F_{\ell(p,q)}. \tag{12}$$

From (11) and (12), we find the relation

$$\frac{dF_{\ell(p,q)}}{dz}=\ell\left(p\,\frac{G'}{G}-q\frac{H'}{H}\right)^{1-\ell}\left(\frac{dF_{p,q}}{dz}\right)^{\ell}, \tag{13}$$

and

$$\left(\frac{dF_{p,q}}{dz}\right)^{\ell}=\frac{1}{\ell}\left(p\,\frac{G'}{G}-q\frac{H'}{H}\right)^{\ell-1}\frac{dF_{\ell(p,q)}}{dz}\ ,\ \forall\ell\in\mathbf{N}^*. \tag{14}$$

2.2.5. First Order Partial Derivatives of $F_{p,q}$ with respect to x and y via z

If we take into account the algebraic form of z, *i.e.*, $z=x+iy$ with $x,y\in\mathbf{R}$. This algebraic form of the principal complex variable implies, among other things, that $F_{p,q}$ is conceptually and implicitly depending on x and y. That's why we can also evaluate the partial derivatives of $F_{p,q}$ with respect to x and y to determine the Wirtinger *pq*-(partial) derivatives in order to establish the link between $F_{p,q}$ and Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\,. \tag{15}$$

Indeed, following Rudin [2], suppose $F_{p,q}$ is defined in an open subset $\mathbf{U} \subset \mathbf{C}$ with $z \neq \rho e^{\pm i\theta}$. Then writing $z = x + iy$ for every $z \in \mathbf{U}$, in this sense, we can also regard $\mathbf{U}$ as an open subset of $\mathbf{R}^2$, and $F_{p,q}$ as a function of two real variables x and y, which maps $\mathbf{U} \subset \mathbf{R}^2$ *to* $\mathbf{C}$. These considerations allow us to say that the existence of the partial derivatives $\partial F_{p,q}/\partial x$ and $\partial F_{p,q}/\partial y$ are in fact a direct consequence of the expressions: $x + iy = z$ and $F_{p,q}(x+iy,s,\rho,\varphi,\theta) = F_{p,q}(z,s,\rho,\varphi,\theta)$, from where we get

$$\frac{\partial F_{p,q}(x+iy,s,\rho,\varphi,\theta)}{\partial x} = \frac{\partial F_{p,q}(z,s,\rho,\varphi,\theta)}{\partial z}\left(\frac{\partial z}{\partial x}\right), \tag{16}$$

and

$$\frac{\partial F_{p,q}(x+iy,s,\rho,\varphi,\theta)}{\partial y} = \frac{\partial F_{p,q}(z,s,\rho,\varphi,\theta)}{\partial z}\left(\frac{\partial z}{\partial y}\right), \tag{17}$$

since $(\partial z/\partial x) = 1$ and $(\partial z/\partial y) = i$, hence (16) and (17) become, respectively

$$\frac{\partial F_{p,q}}{\partial x} = \frac{\partial F_{p,q}}{\partial z}, \tag{18}$$

$$\frac{\partial F_{p,q}}{\partial y} = i\frac{\partial F_{p,q}}{\partial z}. \tag{19}$$

From (18) and (19), we can deduce the following equations just after performing a simple substitution and multiplication

$$\frac{\partial F_{p,q}}{\partial x} = -i\frac{\partial F_{p,q}}{\partial y}, \tag{20}$$

$$\frac{\partial F_{p,q}}{\partial y} = i\frac{\partial F_{p,q}}{\partial x}. \tag{21}$$

Here, Eqs.(20) and (21) play exactly the same role as the Cauchy-Riemann equations for standard complex functions. Now, multiplying the two sides of (19) by $-i$ and adding to (18), to obtain

$$\frac{\partial F_{p,q}}{\partial z} = \frac{1}{2}\left(\frac{\partial F_{p,q}}{\partial x} - i\frac{\partial F_{p,q}}{\partial y}\right). \tag{22}$$

Further, if we replace z with its conjugate $\bar{z} = x - iy$ in (16) and (17), and following exactly the same process as for (22), we get

$$\frac{\partial F_{p,q}}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F_{p,q}}{\partial x} + i\frac{\partial F_{p,q}}{\partial y}\right). \tag{23}$$

Eqs.(22, 23) are the well-known two Wirtinger [3] derivatives, which in the context of the present work, *i.e.*, the formalism of *pq*-functions, we recall them Wirtinger *pq*-(partial) derivatives because, here, they are generalized since $p, q \in \mathbf{R}$. Finally, from (20) or (21) we deduce the equation

$$\left(\frac{\partial F_{p,q}}{\partial x}\right)^2 + \left(\frac{\partial F_{p,q}}{\partial y}\right)^2 = 0. \tag{24}$$

2.2.6. Second Order Partial Derivatives of $F_{p,q}$ with respect to x and y via z

Our main aim in this subsection is firstly to show that $F_{p,q}$ is also a fundamental family of solutions of Laplace equation (15) and secondly establishing the link between Eq.(24) and that to be derived in the form of identity. To this end, we have from (18) and (19)

$$\frac{\partial^2 F_{p,q}}{\partial x^2} = \frac{\partial^2 F_{p,q}}{\partial z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial F_{p,q}}{\partial z}\left(\frac{\partial^2 z}{\partial x^2}\right), \tag{25}$$

and

$$\frac{\partial^2 F_{p,q}}{\partial y^2} = \frac{\partial^2 F_{p,q}}{\partial z^2}\left(\frac{\partial z}{\partial y}\right)^2 + \frac{\partial F_{p,q}}{\partial z}\left(\frac{\partial^2 z}{\partial y^2}\right), \tag{26}$$

sine $(\partial z/\partial x)^2 = 1$, $(\partial^2 z/\partial x^2) = 0$, $(\partial z/\partial y)^2 = -1$ and $(\partial^2 z/\partial y^2) = 0$, thus (25) and (26) reduce to

$$\frac{\partial^2 F_{p,q}}{\partial x^2} = \frac{\partial^2 F_{p,q}}{\partial z^2}, \tag{27}$$

$$\frac{\partial^2 F_{p,q}}{\partial y^2} = -\frac{\partial^2 F_{p,q}}{\partial z^2}. \tag{28}$$

Therefore, from (27) and (28) we obtain the expected equation

$$\frac{\partial^2 F_{p,q}}{\partial x^2} + \frac{\partial^2 F_{p,q}}{\partial y^2} = 0. \tag{29}$$

Eq.(29) means that $F_{p,q}$ is also a fundamental family of solution of Laplace equation (15). Hence, this with result, we can affirm that the *pq*-functions are also a new class of harmonic functions. Finally, from (24) and (29) we get the important identity

$$\left(\frac{\partial F_{p,q}}{\partial x}\right)^2 + \left(\frac{\partial F_{p,q}}{\partial y}\right)^2 = \frac{\partial^2 F_{p,q}}{\partial x^2} + \frac{\partial^2 F_{p,q}}{\partial y^2}. \tag{30}$$

2.3. Orthogonality of pq-functions

We end the study of the properties of *pq*-functions with the determination of orthogonality condition of *pq*-functions on open contour (*C*) of extremities z_1 and z_2 when *pq*-functions are independent of the complex parameters s and ρ. With this aim, it is worthwhile to note that

$$G(z,s,\varphi) = z^2 - 2sz\sin\varphi + s^2 = (z - z_1)(z - z_2), \quad z_1 = -ise^{i\varphi}, \quad z_2 = ise^{-i\varphi}, \tag{31}$$

and

$$H(z,\rho,\theta) = z^2 - 2\rho z\cos\theta + \rho^2 = (z - z_3)(z - z_4), \quad z_3 = \rho e^{i\theta}, \quad z_4 = \rho e^{-i\theta}. \tag{32}$$

Therefore, by substituting (31) and (32) in (1), we get the important expression

$$F_{p,q}(z,s,\rho,\varphi,\theta)=\frac{[(z-z_1)(z-z_2)]^p}{[(z-z_3)(z-z_4)]^q}. \tag{33}$$

It is clear from the expression (33), $z=z_3$ and $z=z_4$ are two poles of $F_{p,q}$. Consequently, since $F_{p,q}$ is supposed independent of the complex parameters s and ρ, thus in such a case Eq.(5) reduces to

$$\frac{d}{dz}\left[W'-\left(p\frac{G'}{G}-q\frac{H'}{H}\right)W\right]=0,\ W=F_{p,q}. \tag{34}$$

Eq.(34) will be henceforth be called '*pq*-differential equation' or shortly *pq*-DE, which here should play a central role as follows. Let $W_1=F_{p_1,q_1}$ and $W_2=F_{p_2,q_2}$ be two fundamental families of solutions of the following *pq*-DEs:

$$\frac{d}{dz}\left[W_1'-\left(p_1\frac{G'}{G}-q_1\frac{H'}{H}\right)W_1\right]=0, \tag{35}$$

and

$$\frac{d}{dz}\left[W_2'-\left(p_2\frac{G'}{G}-q_2\frac{H'}{H}\right)W_2\right]=0, \tag{36}$$

with $(p_1,q_1)\neq(p_2,q_2)$ and $p_1,q_1,p_2,q_2\in\mathbf{R}$. Integrating Eqs.(35) and (36), to get

$$W_1'-\left(p_1\frac{G'}{G}-q_1\frac{H'}{H}\right)W_1=c_1, \tag{37}$$

$$W_2'-\left(p_2\frac{G'}{G}-q_2\frac{H'}{H}\right)W_2=c_2. \tag{38}$$

Multiplying (37) by GW_2 and (38) by HW_1, to find

$$GW_2(W_1'-c_1)=\left(p_1G'-q_1\frac{H'G}{H}\right)W_1W_2, \tag{39}$$

and

$$GW_1(W_2'-c_2)=\left(p_2G'-q_1\frac{H'G}{H}\right)W_1W_2. \tag{40}$$

Subtracting (40) from (39) to obtain, after omitting the integration constants

$$G(W_1'W_2-W_2'W_1)=(p_2-p_1)(q_2-q_1)\left(\frac{G}{H}\frac{H'}{p_2-p}-\frac{G'}{q_2-q_1}\right)W_1W_2. \tag{41}$$

Integrating from $z=z_1$ to $z=z_2$ to get

$$\int_{z_1}^{z_2}(W_1'W_2-W_2'W_1)Gdz=(p_2-p_1)(q_2-q_1)\int_{z_1}^{z_2}\left(\frac{G}{H}\frac{H'}{p_2-p}-\frac{G'}{q_2-q_1}\right)W_1W_2dz. \tag{42}$$

If we take into account the property (1) in *Sub-subsection 2.2.1* and the expression (31), the left hand side of (42) should equal to zero, therefore, we have

$$(p_2 - p_1)(q_2 - q_1)\int_{z_1}^{z_2}\left(\frac{G}{H}\frac{H'}{p_2 - p} - \frac{G'}{q_2 - q_1}\right)W_1 W_2 dz = 0\,. \tag{43}$$

Since $(p_1, q_1) \neq (p_2, q_2)$, thus (43) becomes

$$\int_{z_1}^{z_2}\left(\frac{G}{H}\frac{H'}{p_2 - p} - \frac{G'}{q_2 - q_1}\right)W_1 W_2 dz = 0\,. \tag{44}$$

Furthermore, according to property (10) in *Sub-subsection 2.2.2,* we have $W_1 W_2 = F_{p_1 + p_2,\, q_1 + q_2}$, hence the relation (44) becomes after substitution

$$\int_{z_1}^{z_2}\left(\frac{G}{H}\frac{H'}{p_2 - p} - \frac{G'}{q_2 - q_1}\right)F_{p_1 + p_2,\, q_1 + q_2} dz = 0\,. \tag{45}$$

The relation (45) is exactly the very expected orthogonality condition of *pq*-functions.

3. Consequences of *pq*-functions

In this section we will show the existence of two new types of polynomials called in the context of this work, *p*-polynomials and *q*-polynomials as a direct consequence of *pq*-functions. The study of some properties of these polynomials revealing that the well-known the Legendre polynomials are in fact special case of *q*-polynomials.

3.1. p-Polynomials

In order to establish the existence of *p*-polynomials, we must return to *pq*-function (4) and write it in its explicit form

$$F_{p,q}(z,s,\rho,\varphi,\theta) = \frac{\left(z^2 - 2sz\sin\varphi + s^2\right)^p}{\left(z^2 - 2\rho z\cos\theta + \rho^2\right)^q}\,. \tag{46}$$

The property (1) in *Sub-subsection 2.2.2* allow us to write $F_{p,q} = F_{p,0} \times F_{0,q}$. First, focusing our attention on $F_{p,0}$ for the case $|z| < 1$, that is

$$F_{p,0}(z,s,\rho,\varphi,\theta) = \left(z^2 - 2sz\sin\varphi + s^2\right)^p\,. \tag{47}$$

As we can remark it, the *pq*-function (47) for $q = 0$ is explicitly independent of the parameters ρ and θ, hence it may be written as follows

$$F_{p,0}(z,s,\rho,\varphi,\theta) = s^{2p}\left(1 - 2\xi\sin\varphi + \xi^2\right)^p \text{, with } \xi = z/s \text{ and } |\xi| < 1\,. \tag{48}$$

We have according to the Newton’s generalized binomial (theorem) formula

$$(1+\chi)^\alpha = 1 + \frac{\alpha}{1!}\chi + \frac{\alpha(\alpha-1)}{2!}\chi^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\chi^3 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}\chi^n + \cdots\,, \tag{49}$$

with $\chi < 1$ and $\alpha \in \mathbf{R}$.

By putting $\chi = \left(\xi^2 - 2\xi \sin\varphi\right)$ and $\alpha = p$ in (49), and after rearranging and collecting terms in powers of ξ, we find

$$\left[1 + \left(\xi^2 - 2\xi\sin\varphi\right)\right]^p = 1 - \left(\frac{2p}{1!}\sin\varphi\right)\xi + \left(\frac{4p(p-1)}{2!}\sin^2\varphi + \frac{p}{1!}\right)\xi^2 -$$
$$\left(\frac{8p(p-1)(p-2)}{3!}\sin^3\varphi + \frac{4p(p-1)}{2!}\sin\varphi\right)\xi^3 +$$
$$\left(\frac{16p(p-1)(p-2)(p-3)}{4!}\sin^4\varphi + \frac{12p(p-1)(p-2)}{3!}\sin^2\varphi + \frac{p(p-1)}{2!}\right)\xi^4 -$$
$$\left(\frac{32p(p-1)(p-2)(p-3)(p-4)}{5!}\sin^5\varphi + \frac{32p(p-1)(p-2)(p-3)}{4!}\sin^3\varphi + \frac{6p(q-1)(p-2)}{3!}\sin\varphi\right)\xi^5 + \cdots \qquad (50)$$

Therefore, the coefficients of ξ should take the explicit expressions

$$A_0(\sin\varphi, p) = 1;$$
$$A_1(\sin\varphi, p) = -\left(\frac{2p}{1!}\sin\varphi\right);$$
$$A_2(\sin\varphi, p) = \left(\frac{4p(p-1)}{2!}\sin^2\varphi + \frac{p}{1!}\right);$$
$$A_3(\sin\varphi, p) = -\left(\frac{8p(p-1)(p-2)}{3!}\sin^3\varphi + \frac{4p(p-1)}{2!}\sin\varphi\right);$$
$$A_4(\sin\varphi, p) = \left(\frac{16p(p-1)(p-2)(p-3)}{4!}\sin^4\varphi + \frac{12p(p-1)(p-2)}{3!}\sin^2\varphi + \frac{p(p-1)}{2!}\right)$$
$$A_5(\sin\varphi, p) = -\left(\frac{32p(p-1)(p-2)(p-3)(p-4)}{5!}\sin^5\varphi + \frac{32p(p-1)(p-2)(p-3)}{4!}\sin^3\varphi + \right.$$
$$\left. + \frac{6p(p-1)(p-2)}{3!}\sin\varphi\right) \qquad (51)$$

…

The coefficients $A_n(\sin\varphi, p)$ are exactly the very expected *p*-polynomials. Thus (50) may be written as

$$\left(1 - 2\xi\sin\varphi + \xi^2\right)^p = \sum_{n=0}^{\infty} A_n(\sin\varphi,\, p)\xi^n, \quad |\xi| < 1. \qquad (52)$$

Result: for the case when $q = 0$ and $|z| < 1$, the *pq*-function (46) may be written in the form of *p*-series as follows

$$F_{p,0}(z, s, \rho, \varphi, \theta) = \sum_{n=0}^{\infty} s^{2p-n} A_n(\sin\varphi,\, p) z^n, \quad |z| < 1. \qquad (53)$$

3.2. Properties of p-Polynomials
3.2.1. Expression of p-polynomials for $\varphi = \pm\,\pi/2$

Many important properties of *p*-polynomials can be obtained from (52). Here, we derive immediately a few ones as follows. Let $\varphi = \pi/2$ in (52), and then the left-hand side is

$$(1-\xi)^{2p} = 1 - \frac{2p}{1!}\xi + \frac{2p(2p-1)}{2!}\xi^2 - \frac{2p(2p-1)(2p-2)}{3!}\xi^3 + \cdots + (-1)^n \frac{2p(2p-1)\ldots(2p-n+1)}{n!}\xi^n + \cdots$$

The right-hand side is

$$A_0(1,p) + A_1(1,p)\xi + A_2(1,p)\xi^2 + A_3(1,p)\xi^3 + \cdots + A_n(1,p)\xi^n + \cdots$$

Comparing the coefficients of ξ^n on both sides we get

$$A_n(1,p) = (-1)^n \frac{2p(2p-1)\ldots(2p-n+1)}{n!}. \tag{54}$$

And when we substitute $\varphi = -\pi/2$ in (52), we obtain

$$A_n(-1,p) = \frac{2p(2p-1)\ldots(2p-n+1)}{n!}. \tag{55}$$

3.2.2. Recurrence Relation for p-Polynomials

To obtain the recurrence relation, first we put $t = \sin\varphi$ in (52) to get

$$\left(1 - 2\xi t + \xi^2\right)^p = \sum_{n=0}^{\infty} A_n(t,p)\xi^n. \tag{56}$$

Differentiating (56) with respect to ξ on both sides and rearranging to obtain

$$2p(\xi - t)\left(1 - 2\xi t + \xi^2\right)^p = \left(1 - 2\xi t + \xi^2\right)\sum_{n=0}^{\infty} n A_n(t,p)\xi^{n-1}, \tag{57}$$

or equivalently

$$2p(\xi - t)\sum_{n=0}^{\infty} A_n(t,p)\xi^n = \left(1 - 2\xi t + \xi^2\right)\sum_{n=0}^{\infty} n A_n(t,p)\xi^{n-1}. \tag{58}$$

Equating the coefficients of powers of ξ^n to get the very expected recurrence relation

$$(n+1)A_{n+1}(t,p) = 2(n-p)t\,A_n(t,p) - (n-2p-1)A_{n-1}(t,p), \tag{59}$$

with

$$A_0(t,p) = 1,\ \ A_1(t,p) = -2pt \ \text{ and } \ t = \sin\varphi.$$

The recurrence relation (59), along with the first two *p*-polynomials $A_0(t,p)$ and $A_1(t,p)$, allows the *p*-polynomials to be explicitly expressed.

3.2.3. Associated p-Functions

Our purpose here is to show the existence of *p*-functions. As we will see, this *kind of functions* is in fact a direct consequence of *p*-polynomials. First, it is important and easy to show that the special case when $p=-1/2$, the *p*-polynomials become a fundamental solution of the second order homogeneous ODE:

$$\frac{1}{cos\varphi}\frac{d}{d\varphi}\left[cos\varphi\,\frac{d\mathscr{L}}{d\theta}\right]+n(n+1)\mathscr{L}=0\,, \tag{60}$$

or by substituting $t=\sin\varphi$, we find

$$\frac{d}{dt}\left[\left(1-t^2\right)\frac{d\mathscr{L}}{dt}\right]+n(n+1)\mathscr{L}=0\,,\quad \mathscr{L}=A_n(t,-1/2). \tag{61}$$

Accordingly, for the general case that is $\forall\, p\in\mathbf{R}$, the *p*-polynomials $A_n(sin\,\varphi,p)\equiv A_n(t,p)$ should be also a fundamental solution of the following second order non-homogenous ODE:

$$\frac{d}{dt}\left[\left(1-t^2\right)\frac{d\mathscr{K}}{dt}\right]+n(n+1)\mathscr{K}=f_n(t,p)\,,\quad \mathscr{K}=A_n(t,\,p). \tag{62}$$

It is worthwhile to note that Eq.(62) should reduce to Eq.(61) when $p=-1/2$, this implies

$$f_n(t,-1/2)=0\,,\quad \forall\, n\in\mathbf{N}\,. \tag{63}$$

Result: It follows from all that the *p*-functions $f_n(t,p)$ are associated to *p*-polynomials $A_n(t,\,p)$ *through* Eq.(62) that 's why are called 'associated *p*-functions'. To illustrate this association, the Table 1 below gives us the first few *p*-polynomials and their associated *p*-functions.

p-polynomial	associated *p*-function
$A_1(t,p)$	$f_1(t,p)=0$
$A_2(t,p)$	$f_2(t,p)=2p(2p+1)$
$A_3(t,p)$	$f_3(t,p)=-4p(p-1)(2p+1)t$
$A_4(t,p)$	$f_4(t,p)=4p\,(p-1)(p-2)(2p+1)t^2+2p(p-1)(2p+1)$
$A_5(t,p)$	$f_5(t,p)=-\frac{8}{3}p(p-1)(p-2)(p-3)(2p+1)t^3-4p(p-1)(p-2)(2q+1)t$

Table 1: Expressions of the associated *p*-functions $f_n(t,p)$, $n=1,2\ldots5$

3.2.4. Orthogonality of p-Polynomials

We have already seen the orthogonality of *pq*-functions, now we will show the orthogonality of *p*-polynomials on the interval $(-1,1)$. With this aim, let $g = A_m(t,p)$ and $h = A_n(t,p)$ then by Eq.(62), we have

$$\frac{d}{dt}\left[\left(1-t^2\right)g'\right] + k_m\, g = f_m\,, \tag{64}$$

and

$$\frac{d}{dt}\left[\left(1-t^2\right)h'\right] + k_n\, h = f_n\,, \tag{65}$$

with

$$f_m \equiv f_m(t,p);\; f_n \equiv f_n(t,p);\; k_m = m(m+1);\; k_n = n(n+1) \text{ and } m \neq n\,.$$

Multiplying (64) by h and integrating from $t=-1$ to $t=1$ to obtain

$$\int_{-1}^{1}\frac{d}{dt}\left[\left(1-t^2\right)g'\right]h\,dt + k_m\int_{-1}^{1} g\,h\,dt = \int_{-1}^{1} h\, f_m dt\,.$$

Integrating the first integral by parts we get

$$\left[\left(1-t^2\right)g'h\right]_{-1}^{1} - \int_{-1}^{1}\left(1-t^2\right)g'h'\,dt + k_m\int_{-1}^{1} g\,h\,dt = \int_{-1}^{1} h\, f_m dt\,.$$

But since $\left(1-t^2\right)$ is zero both at $t=-1$ and $t=1$ this becomes

$$-\int_{-1}^{1}\left(1-t^2\right)g'h'\,dt + k_m\int_{-1}^{1} g\,h\,dt = \int_{-1}^{1} h\, f_m dt\,. \tag{66}$$

In exactly the same way we can multiply (65) by g and integrating from $t=-1$ to $t=1$ to get

$$-\int_{-1}^{1}\left(1-t^2\right)g'h'\,dt + k_n\int_{-1}^{1} g\,h\,dt = \int_{-1}^{1} g\; f_n dt\,. \tag{67}$$

Subtracting (67) from (66), we find

$$\left(k_m - k_n\right)\int_{-1}^{1} g\,h\,dt = \int_{-1}^{1} h\, f_m dt - \int_{-1}^{1} g\; f_n dt\,.$$

Or since $g = A_m(t,p)$; $h = A_n(t,p)$; $f_m \equiv f_m(t,p)$ and $f_n \equiv f_n(t,p)$, hence we obtain after substitution

$$\left(k_m - k_n\right)\int_{-1}^{1} A_m(t,p)A_n(t,p)dt = \int_{-1}^{1}\left[A_n(t,p)\,f_m(t,p) - A_m(t,p)\,f_n(t,p)\right]dt\,,$$

this gives us the following expected orthogonality condition

$$\int_{-1}^{1} A_m(t,p)A_n(t,p)\left[1-\frac{1}{k_m-k_n}\left(\frac{f_m(t,p)}{A_m(t,p)}-\frac{f_n(t,p)}{A_n(t,p)}\right)\right]dt=0,\ \ m\neq n\,. \tag{68}$$

According to (63), we should have $f_m(t,-1/2)\equiv f_n(t,-1/2)=0$, thus as a special case the orthogonality condition (68) reduces to

$$\int_{-1}^{1} A_m(t,-1/2)\,A_n(t,-1/2)dt=0,\ \ m\neq n\,. \tag{69}$$

Besides the important property (68), there is another, namely $\int_{-1}^{1} A_n^2(t,p)dt$, which may be determined as follows: first putting $t=\sin\varphi$ in (52), squaring and integrating from $t=-1$ to $t=1$. Due to orthogonality only the integrals of terms having $A_n^2(t,p)$ survive on the right-hand side. So we have

$$\int_{-1}^{1}\left(1-2\xi t+\xi^2\right)^{2p} dt=\sum_{n=0}^{\infty}\xi^{2n}\int_{-1}^{1}A_n^2(t,p)dt\ . \tag{70}$$

For the special case when $p=-1/2$, we have from (70)

$$\frac{1}{\xi}\ln\left(\frac{1+\xi}{1-\xi}\right)=\sum_{n=0}^{\infty}\frac{2\xi^{2n}}{2n+1}=\sum_{n=0}^{\infty}\xi^{2n}\int_{-1}^{1}A_n^2(t,-1/2)dt\ . \tag{71}$$

Comparing the coefficient of ξ^{2n} we get the important relation

$$\int_{-1}^{1}A_n^2(t,-1/2)dt=\frac{2}{2n+1}\,. \tag{72}$$

Hence, what we need for the general case is only to put

$$K_n(p)=\int_{-1}^{1}A_n^2(t,p)dt\,,\quad p\in\mathbf{R}\,. \tag{73}$$

The formula (73) defines us the polynomials $K_n(p)$ that exclusively depend on the real parameter p. As we will see, $K_n(p)$ are characterized by the following properties:

$$K_0(p)=2\,,\quad \forall\, p\in\mathbf{R}\,, \tag{74}$$

and

$$K_n(0)=0\,,\quad \forall n\in\mathbf{N}\,,\ n\neq 0\ . \tag{75}$$

Expressions of $K_n(p)$ for $n=0,1,2,3$:

$$K_0(p)=\int_{-1}^{1}A_0^2(t,p)dt=2\,;\ \ K_1(p)=\int_{-1}^{1}A_1^2(t,p)dt=\frac{8}{3}p^2\,;$$

$$K_2(p)=\int_{-1}^{1}A_2^2(t,p)dt=\frac{8}{5}p^2(p-1)^2+\frac{8}{3}p^2(p-1)+2p^2\,;$$

$$K_3(p) = \int_{-1}^{1} A_3^2(t,p)\,dt = \frac{32}{63}p^2(p-1)^2(p-2)^2 + \frac{32}{15}p^2(p-1)^2(p-2) + \frac{8}{3}p^2(p-1)^2 .$$

3.2.5. Series of p-Polynomials

As a direct consequence of the existence of *p*-polynomials we can refer to the series of *p*-polynomials; that is to say any continuous function $f(t)$ such that $-1<t<1$, may be expanded in series of *p*-polynomials. More precisely, let us prove that if

$$f(t) = \sum_{k=0}^{\infty} c_k A_k(t,p)\,,\quad -1<t<1,\ \ \forall\, p \in \mathbf{R}\,, \tag{76}$$

this implies

$$c_k = K_k^{-1}(p)\int_{-1}^{1} A_k(t,p)\, f(t)\,dt\,. \tag{77}$$

To this end, multiplying the series (76) by $A_n(t,p)$ and integrating from $t=-1$ to $t=1$, and taking into account the previous result, namely formula (73), we get

$$\int_{-1}^{1} A_n(t,p)\, f(t)\,dt = \sum_{k=0}^{\infty} c_k \int_{-1}^{1} A_n(t,p) A_k(t,p)\, f(t)\,dt\,,$$

for the case when $n=k$, we have

$$\int_{-1}^{1} A_n(t,p)\, f(t)\,dt = c_n \int_{-1}^{1} A_n^2(t,p)\,dt = c_n\, K_n(p)\,,$$

from where we obtain the very expected formula (77). Furthermore, if we consider the important special case that is when $p=-1/2$, we get according to (72), (76) and (77)

$$f(t) = \sum_{k=0}^{\infty} c_k A_k(t,-1/2)\,,\quad -1<t<1\,, \tag{78}$$

and

$$c_k = \frac{2k+1}{2}\int_{-1}^{1} A_k(t,-1/2)\, f(t)\,dt\,. \tag{79}$$

3.3. q-Polynomials

After we have established the existence and studied the properties of *p*-polynomials which are a direct consequence of *pq*-functions, at present, we would derive the other polynomials, namely *q*-polynomials. For this purpose, we must follow exactly the same previous process that led to *p*-polynomials. Thus let us return to the expression (46) and consider the second case that is when $p=0$, $q \neq 0$ and $|z|<1$ to obtain

$$F_{0,q}(z,s,\rho,\varphi,\theta) = \left(z^2 - 2\rho\, z\cos\theta + \rho^2\right)^{-q}\,,\quad |z|<1\,. \tag{80}$$

As we can remark it, the *pq*-function (80) for $p=0$ is explicitly independent of the parameters s and φ, hence it may be written as follows

$$F_{0,q}(z,s,\rho,\varphi,\theta) = \rho^{-2q}\left(1 - 2\zeta\cos\theta + \zeta^2\right)^{-q}, \quad \text{with } \zeta = \frac{z}{\rho} \text{ and } |\zeta| < 1. \tag{81}$$

By putting $\chi = \left(\zeta^2 - 2\zeta\cos\theta\right)$ and $\alpha = -q$ in (49), and after rearranging and collecting terms in powers of ζ, we find

$$\begin{aligned}
&\left[1 + \left(\zeta^2 - 2\zeta\cos\theta\right)\right]^{-q} = 1 + \left(\frac{2q}{1!}\cos\theta\right)\zeta + \left(\frac{4q(q+1)}{2!}\cos^2\theta - \frac{q}{1!}\right)\zeta^2 + \\
&\left(\frac{8q(q+1)(q+2)}{3!}\cos^3\theta - \frac{4q(q+1)}{2!}\cos\theta\right)\zeta^3 + \\
&\left(\frac{16q(q+1)(q+2)(q+3)}{4!}\cos^4\theta - \frac{12q(q+1)(q+2)}{3!}\cos^2\theta + \frac{q(q+1)}{2!}\right)\zeta^4 + \\
&\left(\frac{32q(q+1)(q+2)(q+3)(q+4)}{5!}\cos^5\theta - \frac{32q(q+1)(q+2)(q+3)}{4!}\cos^3\theta + \frac{6q(q+1)(q+2)}{3!}\cos\theta\right)\zeta^5 + \cdots
\end{aligned} \tag{82}$$

Therefore the coefficients of ζ should take the explicit expressions

$$\begin{aligned}
&B_0(\cos\theta, q) = 1;\ B_1(\cos\theta, q) = \frac{2q}{1!}\cos\theta;\ B_2(\cos\theta, q) = \frac{4q(q+1)}{2!}\cos^2\theta - \frac{q}{1!}; \\
&B_3(\cos\theta, q) = \frac{8q(q+1)(q+2)}{3!}\cos^3\theta - \frac{4q(q+1)}{2!}\cos\theta; \\
&B_4(\cos\theta, q) = \frac{16q(q+1)(q+2)(q+3)}{4!}\cos^4\theta - \frac{12q(q+1)(q+2)}{3!}\cos^2\theta + \frac{q(q+1)}{2!} \\
&B_5(\cos\theta, q) = \frac{32q(q+1)(q+2)(q+3)(q+4)}{5!}\cos^5\theta - \frac{32q(q+1)(q+2)(q+3)}{4!}\cos^3\theta + \frac{6q(q+1)(q+2)}{3!}\cos\theta \\
&\ldots
\end{aligned} \tag{83}$$

The coefficients $B_n(\cos\theta, q)$ are exactly the expected q-polynomials. Further, it is clear that when $q = 1/2$, the q-polynomials (83) reduce to those of Legendre, that is

$$B_n(\cos\theta, 1/2) = P_n(\cos\theta). \tag{84}$$

This implies, among other things, that the Legendre polynomials $P_n(\cos\theta)$ are in fact a special case of q-polynomials $B_n(\cos\theta, q)$ for the case $q = 1/2$. Therefore, expression (83) may be written as

$$\left(1 - 2\zeta\cos\theta + \zeta^2\right)^{-q} = \sum_{n=0}^{\infty} B_n(\cos\theta, q)\zeta^n, \quad |\zeta| < 1, \tag{85}$$

Result: for the case when $p = 0$ and $|z| < 1$, the pq-function (46) may be written in the form of q-series:

$$F_{0,q}(z,s,\rho,\varphi,\theta) = \sum_{n=0}^{\infty} \rho^{-(2q+n)} B_n(\cos\theta, q) z^n, \quad |z| < 1. \tag{86}$$

Recall that since the beginning our main interest is essentially focused on the investigation of structure, properties and consequences of pq-functions as an extension of pq-radial functions [1] that's why, here, we are not particularly concerned with the study of the Legendre polynomials because they are well-known since their introduction in 1784 by the French mathematician A. M. Legendre [4]. Also, the q-polynomials have already been studied in [1] but in the present work $B_n(\cos\theta, q)$ are direct consequences of *pq*-functions. However, the reader who is interested in *q*-polynomials and their properties can refer to [1]. Nevertheless, it seems that the determination of the recurrence relation for *q*-polynomials is necessary because, as we shall see, $A_n(\sin\varphi, q)$ and $B_n(\cos\theta, q)$ are essential for *pq*-series.

3.3.1. *Recurrence Relation for q-Polynomials*

The recurrence relation for *q*-polynomials is so important, although it was already determined in the previous work [1], here we are obliged to drive it again in the context of *pq*-functions since with the aid of this relation and the first two *q*-polynomials $B_0(\cos\theta, q)$ and $B_1(\cos\theta, q)$ we can explicitly express the *q*-polynomials of any degree. To this aim, putting $\tau = \cos\varphi$ in (85), we get

$$\left(1-2\zeta\tau+\zeta^2\right)^{-q} = \sum_{n=0}^{\infty} B_n(\tau, q)\zeta^n \ . \qquad (87)$$

Differentiating (87) with respect to ζ on both sides and rearranged to obtain

$$\frac{2q(\tau-\zeta)}{\left(1-2\zeta\tau+\zeta^2\right)^q} = \left(1-2\zeta\tau+\zeta^2\right)\sum_{n=0}^{\infty} n\, B_n(\tau, q)\zeta^{n-1}, \qquad (88)$$

or equivalently

$$2q(\tau-\zeta)\sum_{n=0}^{\infty} B_n(\tau, q)\zeta^n = \left(1-2\zeta\tau+\zeta^2\right)\sum_{n=0}^{\infty} n\, B_n(\tau, q)\zeta^{n-1}. \qquad (89)$$

Replacing the dominator with its definition (87), and equating the coefficients of powers of ζ^n in the resulting expansion gives the expected Recurrence Relation for *q*-polynomials

$$(n+1)B_{n+1}(\tau, q) = 2(n+q)\tau B_n(\tau, q) - (n+2q-1)B_{n-1}(\tau, q), \qquad (90)$$

with

$$B_0(\tau, q) = 1 \ \text{ and } \ B_1(\tau, q) = 2q\tau$$

This relation, along with the first two polynomials $B_0(\tau, q)$ and $B_1(\tau, q)$, allows the Legendre Polynomials to be generalized recursively. Furthermore, it may be worth noting that the *p*-polynomials and *q*-polynomials have almost the same general properties, for instance, $A_n(t, p)$ and $B_n(\tau, q)$ both have the same periodicity with respect to their angular parameters since $t = \sin\theta$ and $\tau = \cos\varphi$. Also, there is another interesting particular case that is when $(\theta, \varphi) = (\pi/4, \pi/4)$ and $p = -q$, we get $A_n\left(\sqrt{2}/2, -q\right) \equiv B_n\left(\sqrt{2}/2, q\right)$.

3.4. pq-Series

The existence of *p*-series (53) and *q*-series (86), as a direct consequence of *pq*-functions, allows us to introduce the notion of *pq*-series that may be considered as a very important useful tool particularly for expanding any *pq*-function with $p, q \in \mathbf{R}$ and $|z| < 1$, also may be used for evaluating certain '*new*' type of integrals as we will see.

Since *p*-series and *q*-series are in fact power series, therefore, to arrive at the explicit expression of *pq*-series, it suffices to multiply side to side the above mentioned series (53) and (86) to obtain:

$$F_{p,q}(z,s,\rho,\varphi,\theta) = \sum_{n=0}^{\infty}\left(\frac{s^{2p-n}}{\rho^{2q+n}}\right)A_n(\sin\varphi,p)\,B_n(\cos\theta,q)z^{2n}\ , \quad |z|<1, \tag{91}$$

with

$$s,\rho\in\mathbf{U}\subset\mathbf{C},\ \ (s,\rho)\neq(0,0)\quad\text{and}\quad p,q,\varphi,\theta\in\mathbf{R}.$$

3.4.1. Properties of pq-Series

i) Let $F_{p,q}\equiv F_{p,q}(z,s,\rho,\varphi,\theta)$ be an expandable *pq*-function in *pq*-series for any $z\in\mathbf{U}\setminus\{\rho e^{\pm i\theta}\}$ and $|z|<1$; let $-z\in\mathbf{U}$ such that if $z=-z\ \Rightarrow\ F_{p,q}(-z,s,\rho,\varphi,\theta) = F_{p,q}(z,s,\rho,\varphi,\theta)$. This means there is *parity* between $-z\mapsto F_{p,q}$ and $z\mapsto F_{p,q}$ via the *pq*-series.

ii) Let $p,q\in\mathbf{R}$ and $k\in\mathbf{N}$. If $F_{k(p,q)}$ is an expandable *pq*-function in *pq*-series for any $z\in\mathbf{U}\setminus\{\rho e^{\pm i\theta}\}$ and $|z|<1$: we should have

$$F_{k(p,q)}(z,s,\rho,\varphi,\theta) = \sum_{n=0}^{\infty}\left(\frac{s^{2kp-n}}{\rho^{2kq+n}}\right)A_n(\sin\varphi,kp)\,B_n(\cos\theta,kq)z^{2n}\ , \quad |z|<1, \quad k\in\mathbf{N}. \tag{92}$$

3.4.2. Application of pq-Series

By the present pedagogical example, we would show how a given standard complex function may be expanded in *pq*-series. With this aim, let ϑ be a standard complex functions defined as follows. $\vartheta:\mathbf{U}\setminus\{-1\}\to\mathbf{C},\ z\mapsto\vartheta(z)=(z-1)^{1/2}(z+1)^{-2/3}$ such that ϑ is continuous and differentiable for all $z\mid z\in\mathbf{U}\setminus\{-1\}$. Our goal is to expand $\vartheta(z)$ in *pq*-series with $|z|<1$. For this reason, we must rewrite ϑ in the form of *pq*-function (46), for the particular case when $(p,q)=(1/4,1/3)$, $(s,\rho)=(1,1)$, $(\varphi,\theta)=(\pi/2,\pi)$. After substitution, we get

$$F_{1/4,1/3}(z,1,1,\pi/2,\pi)=\frac{(z-1)^{1/2}}{(z+1)^{2/3}}.$$

Now, with this expression, we can expand ϑ in *pq*-series (91) *through* $F_{1/4,1/3}(z,1,1,\pi/2,\pi)$ and we find

$$F_{1/4,1/3}(z,1,1,\pi/2,\pi) = \sum_{n=0}^{\infty} A_n(1/2,1/4)\,B_n(-1,1/3)z^{2n}\ , \quad |z|<1$$

The coefficients $A_n(1/2,1/4)$ and $B_n(-1,1/3)$ may be easily determined from the recurrence relations (59) and (90), respectively.

4. *pq*-Integrals

As it was already mentioned, at the present we are dealing with the application of *p*-series, *q*-series and *pq*-series for the purpose of evaluating some '*new*' kind of integrals, the *pq*-integrals of general form: $\int_D F_{p,q}dz$ which, according to the explicit expression (33) of *pq*-function, it is not easy task to evaluate the integral of (33) even with the help of the usual methods. Indeed, the notion of *pq*-integrals is a natural consequence of *pq*-functions that may be defined as follows. Let $F_{p,q}\equiv F_{p,q}(z,s,\rho,\varphi,\theta)$ be a holomorphic *pq*-function on the complex plane $\mathbf{U} \subset \mathbf{C}$ with $z \neq \rho e^{\pm i\theta}$ and in the unit disc $D = \left\{ z \in \mathbf{U} \setminus \left\{ \rho e^{\pm i\theta} \right\} : |z| < 1 \right\}$; we call *pq*-integral any integral of the general form:

$$\int_D F_{p,q} dz\,, \quad |z| < 1. \tag{93}$$

There are two special cases that may be derived from the *pq*-integral (93), namely the *p*-integral for the case when $p \neq 0$ and $q = 0$:

$$\int_D F_{p,0} dz\,, \quad |z| < 1. \tag{94}$$

And the *q*-integral for the opposite case, that is when $p = 0$ and $q \neq 0$:

$$\int_D F_{0,q} dz\,, \quad |z| < 1. \tag{95}$$

The integrals (93), (94) and (95) should be evaluated by using the *pq*-series, *p*-series and *q*-series, respectively.

4.1. Properties of pq-Integrals

Generally, all the properties of *pq*-functions with respect to the orders *p* and *q* are transposable to *pq*-integrals. Here, we are particularly concerned with four properties:

i) $\forall\, p, q \in \mathbf{R}$:
$$\int_D F_{p,q} dz = \int_D F_{p,0} \times F_{0,q} dz\,, \quad |z| < 1, \tag{96}$$

ii) $\forall\, p, q \in \mathbf{R}$:
$$\int_D F_{p,q}^{-1} dz = \int_D F_{-p,-q} dz\,, \quad |z| < 1, \tag{97}$$

iii) $\forall\, p, q \in \mathbf{R}$:
$$\int_D F_{p,q}^{\ell} dz = \int_D F_{\ell(p,q)} dz\,, \quad |z| < 1, \tag{98}$$

iv) $\forall\, p, q \in \mathbf{R}$:
$$\int_D F_{p,q} \times F_{p',q'} dz = \int_D F_{p+p',q+q'}\, dz\,, \quad |z| < 1. \tag{99}$$

To illustrate the practical importance of *pq*-series and its strong link with *pq*-integrals, let us examine the following pedagogical example: Let $F_{p,q} \equiv F_{p,q}(z, s, \rho, \varphi, \theta)$ be a holomorphic *pq*-function in the unit disc $D = \left\{ z \in \mathbf{U} \setminus \left\{ \rho e^{\pm i\theta} \right\} : |z| < 1 \right\}$, here, our aim is to determine the explicit expression of another *pq*-function $f_{p,q}(z, z_0)$, $z, z_0 \in D$ such that

$$f_{p,q}(z, z_0) = \int_{z_0}^{z} F_{p,q} dz\,, \quad |z| < 1. \tag{100}$$

Since $F_{p,q}$ is holomorphic in D and $|z| < 1$, thus $F_{p,q}$ is expandable in *pq*-series (91). Therefore, for the purpose of finding this *pq*-integral, substituting (91) in (100) to get, after integration from z to z_0:

$$f_{p,q}(z,z_0) = \sum_{n=0}^{\infty}\left(\frac{s^{2p-n}}{\rho^{2q+n}}\right)A_n(\sin\varphi,p)B_n(\cos\theta,q)\,\frac{z^{2n+1}-z_0^{2n+1}}{2n+1}, \quad |z|<1. \tag{101}$$

As we can remark it, by using the *pq*-series, we have easily and rapidly determined the explicit expression (101) of *pq*-function $f_{p,q}(z,z_0)$ by evaluating the *pq*-integral (100). However if, for example, we want to evaluate the same *pq*-integral without using *pq*-series, in such situation we must, first, rewrite (100) according to (33) as follows:

$$f_{p,q}(z,z_0)=\int_{z_0}^{z}F_{p,q}\,dz=\int_{z_0}^{z}\frac{[(z-z_1)(z-z_2)]^p}{[(z-z_3)(z-z_4)]^q}\,dz.$$

With this above expression and under the condition 'do not use the *pq*-series', the advanced student or even the professional mathematician should have a great difficulty and hard task to arrive at the expected explicit expression of $f_{p,q}(z,z_0)$ by using the usual methods only. Even with the help of the repetitive usage of the usual processes of evaluation, there is always some residual *pq*-integral to evaluate again! Hence, the *pq*-series is essential for *pq*-integrals.

In what follows it is a systematic application of the concept of *pq*-functions to standard complex functions in order to *rewrite* their integrals in the form of *pq*-integral with the pedagogical aim to show how for example *pq*-integral apply to evaluate some standard integrals.

4.2. Application of pq-Integral

Let *u* be a standard complex functions defined as follows. $u:\mathbf{U}\setminus\{\pm i\}\to\mathbf{C},\ z\mapsto u(z)=(z^2+1)^{-2}$ and *u* is holomorphic in unit disc $D_u=\{\,z\in\mathbf{U}\setminus\{\pm i\}:\ |z|<1\,\}$. Our goal is to evaluate the integral of function $u(z)$ from z to z_0 with $z,z_0\in D_u$. This is equivalent to find new standard complex function defined by $\bar{u}(z,z_0)=\int_{z_0}^{z}u(z)dz$, $z,z_0\in D_u$, $|z|<1$. First, we look at the function $u(z)$, which has singularities at $z=i$ and $z=-i$ but since $u(z)$ is holomorphic in the unit disc $D_u=\{\,z\in\mathbf{U}\setminus\{\pm i\}:\ |z|<1\,\}$ this allows us to rewrite it in the form of *pq*-function that is $u(z)$ should be a special case of *pq*-function (46). Note that there are, in fact, several manners to rewrite $u(z)$ in the form of *pq*-function by, of course, selecting the adequate values for the parameters and orders. For example, in addition to its initial standard form, $u(z)$ may be rewritten in the form $u(z)=(z^2+1)^{-2}$ this means (z^2+1) may be interpreted as a weight function or a characteristic function. This remark allows us to choose according to the expression (46): $(p,\,q)=(-1,1)$; $(s,\,\rho)=(1,1)$ and $(\varphi,\,\theta)=(\pi,\pi/2)$, and we get after substitution in (46)

$$F_{-1,1}(z,1,1,\pi,\pi/2) = \frac{(z^2+1)^{-1}}{(z^2+1)^{1}} = \frac{1}{(z^2+1)^2} = (z^2+1)^{-2}. \tag{1.a}$$

After this process, we can say: we have rewritten the standard complex function $u(z)$ in the form of complex *pq*-function or we have *transformed* the standard complex function $u(z)$ into complex *pq*-function or simply we write: $u(z)\to F_{-1,1}(z,1,1,\pi,\pi/2)$ and its integral becomes *via* this transformation $\bar{u}(z,z_0)\equiv f_{-1,1}(z,z_0)$ or more explicitly

$$\bar{u}(z,z_0) \equiv f_{-1,1}(z,z_0) = \int_{z_0}^{z} F_{-1,1}\, dz \,. \tag{1.b}$$

Since $F_{-1,1}$ is holomorphic in $D_{u\to F_{-1,1}}$ and $|z|<1$, thus $F_{-1,1}$ is expandable in *pq*-series (91)

$$F_{-1,1}(z,1,1,\pi,\pi/2) = \sum_{n=0}^{\infty} A_n(0,-1)\, B_n(0,1) z^{2n} \,. \tag{1.c}$$

The coefficients $A_n(0,-1)$ and $B_n(0,1)$ may be easily determined from the expressions (59) and (90), respectively. Therefore substituting (1.c) in (1.b), to find, after a direct evaluation

$$\bar{u}(z,z_0) \equiv f_{-1,1}(z,z_0) = \sum_{k=0}^{\infty} A_n(0,-1)\, B_n(0,1) \frac{z^{2n+1} - z_0^{2n+1}}{2n+1}, \quad |z|<1 \tag{1.d}$$

5. Physical Interpretation of *pq*-Functions

In this section, we are particularly interesting in the importance of *pq*-functions in physics especially when the phenomena depend at the same time on the main complex variable z, the auxiliary complex parameters (s,ρ) and the real angular parameters (φ,θ). For instance, in the framework of fluid mechanics, $F_{p,q}$ may be interpreted under, of course, some experimental and/or theoretical conditions as a power law of complex potential flow. This power law should be defined by the expression (33). The *pq*-DE that governing such a complex potential is, according to (33) and (34), of the form

$$\frac{d}{dz}\left[W' - 2\left(p\, \frac{z-\alpha}{(z-z_1)(z-z_2)} - q\, \frac{z-\beta}{(z-z_3)(z-z_4)} \right) W \right] = 0 \,, \tag{102}$$

with

$$\alpha = \frac{z_1 + z_2}{2}\,, \quad \beta = \frac{z_3 + z_4}{2}\,.$$

As illustration, let us show that the well-known Joukowski function (JF) is a special case of *pq*-function. This property should, among other things, extend the field of application of *pq*-functions. Historically, the Russian scientist N. E. Joukowski (1847-1921) who first studied the properties of the function

$$J(z) = \frac{1}{2}\left(z + \frac{1}{z} \right), \quad z \neq 0\,, \tag{103}$$

in the early 20th century. He showed that the image of a circle passing through $z_1 = 1$ and $z_2 = -1$ is mapped onto a curve shaped like the cross section of an airplane wing. We call this curve the Joukowski airfoil as shown in Figs. 1 and 2.

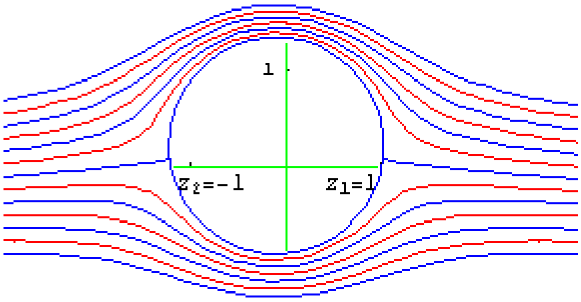

Fig.1: Image of a fluid flow under the Joukowski function

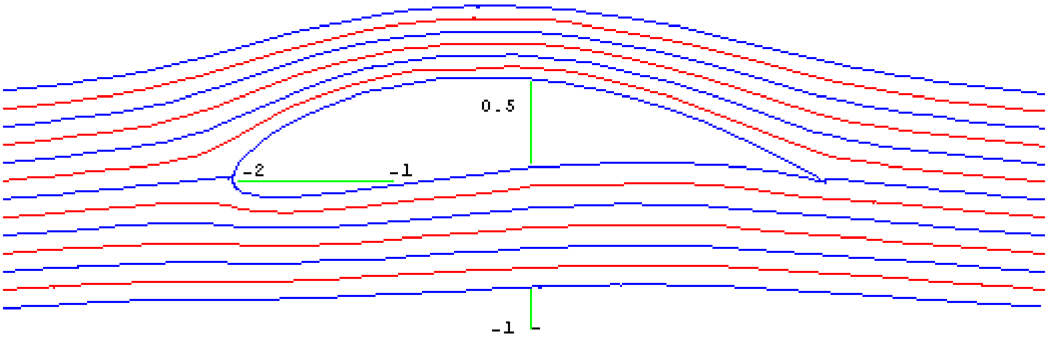

Fig.2: Illustration of Joukowski airfoil

JF is very important for its applications in fluid mechanics. For example, if the streamlines for a flow around the circle are known, then their images under the mapping $w = J(z)$ will be streamlines for a flow around Joukowski airfoil. However, technically, the Joukowski airfoil/profile suffers from a cusp at trailing edge. This implies that if, for instance, one had to build wings with such a profile, and one should obtain a very thin, hence fragile rear part of the wing. For this reason more general profiles having a singularity with distinct tangents at the trailing edges have been introduced (Karman-Trefftz profile).

Another generalization of Joukowski profile goes in the direction of enlarging the number of parameters (Von Mises profile). Now, returning to the explicit expression (46) and showing that with the help of property (3) in *Sub-subsection 2.2.1* regarding the homogeneity of $F_{p,q}$ with respect to $z, s, and\ \rho$, and by an appropriate choice of some values for the real angular parameters (φ,θ), we can generalize JF. For this purpose, let us begin by rewriting (46)

$$F_{p,q}(z,s,\rho,\varphi,\theta) = \frac{\left(z^2 - 2sz\sin\varphi + s^2\right)^p}{\left(z^2 - 2\rho z\cos\theta + \rho^2\right)^q}.$$

Thus, according to the above mentioned property (3), we have $\forall \varepsilon \in \mathbf{R}_+ \setminus \{0\}$:

$$F_{p,q}\left(\frac{z}{\varepsilon},\frac{s}{\varepsilon},\frac{\rho}{\varepsilon},\varphi,\theta\right) = \frac{1}{\varepsilon^{2(p-q)}}\frac{\left(z^2 - 2sz\sin\varphi + s^2\right)^p}{\left(z^2 - 2\rho z\cos\theta + \rho^2\right)^q}. \qquad (104)$$

So let us deduce the expected generalized JF for the case when $(p,q)=(p,p/2)$, $(s,\rho)=(s,0)$ and $(\varphi,\theta)=(\pi,\theta)$:

$$F_{p,\frac{p}{2}}\left(\frac{z}{\varepsilon},\frac{s}{\varepsilon},0,\pi,\pi\right)=\frac{1}{\varepsilon^{p}}\left(z+\frac{s^{2}}{z}\right)^{p}. \tag{105}$$

Here, the generalization is done in power form with the presence of two parameters ε and s. Furthermore, it is worthwhile to note that (105) reduces to the usual JF (103) for the special case when $p=1$, $\varepsilon=2$ and $s=1$. Therefore, the study of properties and behavior of (105) *via* its graphical representations should depend on the appropriate choice of the numerical values for p, ε and s, respectively.

6. Structural Properties of *pq*-DE

In this section, we would focus our attention exclusively on the structural properties of *pq*-DE (34), which as we know is derived from *pq*-PDE (5) when the *pq*-function is supposed independent of the complex auxiliary parameters s and ρ. But first, let us rewrite (34) in its more explicit form, namely:

$$W''-\left(p\frac{G'}{G}-q\frac{H'}{H}\right)W'-\left[p\left(\frac{G''}{G}-\frac{G'^{2}}{G^{2}}\right)-q\left(\frac{H''}{H}-\frac{H'^{2}}{H^{2}}\right)\right]W=0\,,\ W=F_{p,q}. \tag{106}$$

We remark from Eq.(106) that the orders, the weight function, the characteristic function and their derivatives all are essential elements that entering in the structure of this equation. This allows us to say that the study of structural properties of Eq.(106) is completely depending on those mentioned elements as we shall see.

6.1. Relationship between pq-DE and Fuchs' class

Our aim, here, is to prove that under some conditions relative to very interesting particular cases, Eq.(106) belongs to Fuchs' class. For this purpose, considering the following cases.

Case.1: when $p\neq 0$ and $q=0$, Eq.(106) takes the dorm

$$W''-p\left(\frac{G'}{G}\right)W'-p\left(\frac{G''}{G}-\frac{G'^{2}}{G^{2}}\right)W=0. \tag{107}$$

Anyone well familiarized with the equations of Fuchs' class can immediately affirm that Eq.(107) is really belonging to Fuchsian class since its variable coefficients satisfying Fuchs' condition, and according to the explicit expression of the weight function (31), Eq.(107) has two regular singular points: $z=z_1$ and $z=z_2$.

Case.2: when $p=0$ and $q\neq 0$, Eq.(106) takes the dorm

$$W''+q\left(\frac{H'}{H}\right)W'+q\left(\frac{H''}{H}-\frac{H'^{2}}{H^{2}}\right)W=0. \tag{108}$$

Also, the variable coefficients of Eq.(108) satisfying Fuchs' condition, and the explicit expression of the characteristic function (32) implies that Eq.(108) has two regular singular points: $z=z_3$ and $z=z_4$.

6.2. Relationship between pq-DE and DE of Sturm-Liouville form

After we have proven that *pq*-DE (106) belongs to Fuchs' class under some well-established conditions, at present we shall show that the same equation may be written in classical form of Sturm-Liouville DE, particularly, when its spectral (eigenvalue) $\lambda = 1$, and when the orders $(p,q) = (-1,1)$ for Eq.(106). First, let us write the classical form of Sturm-Liouville DE:

$$\frac{dR}{dz}\left[\alpha(z)R'\right] + \left[\lambda\,\beta(z) - \gamma(z)\right]R = 0\,, \quad R \equiv R(z), \;\; \alpha(z) \neq 0\,. \qquad (109)$$

Considering the very important case when $\lambda = 1$ and R is supposed holomorphic in its domain. Hence, after substitution, differentiation and rearrangement, we get

$$R'' + \frac{\alpha'(z)}{\alpha(z)}R' + \frac{1}{\alpha(z)}\left[\beta(z) - \gamma(z)\right]R = 0. \qquad (110)$$

Concerning Eq.(106), we have for the case $(p,q) = (-1,1)$:

$$W'' + \left(\frac{G'}{G} + \frac{H'}{H}\right)W' + \left[\left(\frac{G''}{G} - \frac{G'^2}{G^2}\right) + \left(\frac{H''}{H} - \frac{H'^2}{H^2}\right)\right]W = 0\,. \qquad (111)$$

Or equivalently

$$W'' + \left(\frac{G'H + H'G}{GH}\right)W' + \frac{1}{GH}\left[\left(G''H + H''G\right) - \left(\frac{G'^2 H}{G} + \frac{H'^2 G}{H}\right)\right]W = 0\,, \quad GH \neq 0\,. \qquad (112)$$

As we can remark it, the expression of Eq.(112) is comparable to that of Eq.(110), consequently we can rewrite it in the following form

$$\frac{d}{dz}\left[(GH)W'\right] + \left[\left(G''H + H''G\right) - \left(\frac{G'^2 H}{G} + \frac{H'^2 G}{H}\right)\right]W = 0\,. \qquad (113)$$

Eq.(113) is exactly the expected classical form of Sturm-Liouville DE for the case when $\lambda = 1$. However, if we take into account the previous result we find that the variable coefficients of Eq.(113) do not justify Fuchs' condition, therefore, Eq.(113) does not belong to Fuchsian class, in this sense we call it " *pq*-DE in Sturm-Liouville form for $\lambda = 1$ and $(p,q) = (-1,1)$ ".

6.3. Question

From all that we arrive at the central question that arises in the context of *pq*-DEs: Is there some relationship between the DEs of Fuchsian class and the DEs of Sturm-Liouville form in spite of their quite distinct structures?

From previous result concerning the structure of *pq*-DEs that are belonging to Fuchsian class and Eq.(112), we begin to answer this question as follows. The above mentioned relationship may be really exist through pq-DEs if and only if $(p,q) = (-1,0)$ or $(p,q) = (0,1)$ when $\lambda = 1$. Indeed, for the case $(p,q) = (-1,0)$, Eq.(106) reduces to

$$W'' + \left(\frac{G'}{G}\right)W' + \left(\frac{G''}{G} - \frac{G'^2}{G^2}\right)W = 0\,,\;\; W = F_{-1,0}\,. \tag{114}$$

It is clear from the expression of Eq.(114), which is also an important special case of Eq.(107) when $p = -1$, therefore it follows that the variable coefficients of Eq.(114) satisfying Fuchs' condition and consequently the equation has two regular singular points similar to those of Eq.(107). Furthermore, the structure of Eq.(114) allows us to write in Sturm-Liouville form for the case $\lambda = 1$:

$$\frac{d}{dz}\left(GW'\right) + \left(G'' - \frac{G'^2}{G}\right)W = 0\,. \tag{115}$$

Eq.(115) is precisely the first answer to our question -relating to the relationship between the DEs of Fuchsian class and the DEs of Sturm-Liouville form. The second answer comes from the case when $(p,q) = (0,1)$, thus Eq.(106) reduces to

$$W'' + \left(\frac{H'}{H}\right)W' + \left(\frac{H''}{H} - \frac{H'^2}{H^2}\right)W = 0\,,\;\; W = F_{0,1}\,. \tag{116}$$

Eq.(116) is also an interesting special case of Eq.(108) when $q = 1$. Hence, it follows that the variable coefficients of (116) satisfying Fuchs' condition therefore the equation has two regular singular points similar to those of Eq.(108). Moreover, the structure of Eq.(116) permits us to write in Sturm-Liouville form for the case $\lambda = 1$:

$$\frac{d}{dz}\left(HW'\right) + \left(H'' - \frac{H'^2}{H}\right)W = 0\,. \tag{117}$$

Eq.(117) is exactly the second answer to our central question. That is to say, in the context of *pq*-DEs, there is really a certain relationship between the DEs of Fuchsian class and the DEs of Sturm-Liouville form.

6.4. Reciprocal characteristic properties

The main purpose of this subsection is to show the existence of some reciprocal properties that characterize at the same time the structure of *pq*-function and its *pq*-DE. Hence, we must return to (106), which has in reality three independent families of solutions, namely:

$$W_1 = F_{p,q}\,;\;\; W_2 = W_1 \int W_1^{-1} dz\,;\;\; W_3 = c_1 W_1 + c_2 W_2\,;\;\; c_1, c_2 \in \mathbf{C}\,. \tag{118}$$

In order to make the understanding of this investigation more easy let us, first, begin with the following theorem.

Theorem: -The Wronskian of two fundamental families of solutions of pq-DE (106) is itself a fundamental family of solutions of the same equation.
Proof of theorem: -Let $W_1 = F_{p,q}$ and $W_2 = W_1 \int W_1^{-1} dz$ two fundamental families of solutions of *pq*-DE (106), thus their Wronskian is $\mathscr{W}\left(W_1, W_2\right) = W_1 W_2' - W_2 W_1'$.
We have for the derivative of W_2, the expression $W_2' = W_1' \int W_1^{-1} dz + 1$. Hence, after substitution in the Wronskian, we get $\mathscr{W}\left(W_1, W_2\right) = W_1\left(W_1' \int W_1^{-1} dz + 1\right) - W_1\left(W_1' \int W_1^{-1} dz\right) = W_1$.

Secondly, we would show that the solutions (118) and their *pq*-DE (106) are in fact special case. With this aim, let $\ell \in \mathbf{Z} \setminus \{0\}$, hence the property (2) in *Sub-subsection 2.2.2* allows us to write

$$w_1 = F_{\ell(p,q)}; \quad w_2 = w_1 \int w_1^{-1} dz\,; \quad w_3 = c_1 w_1 + c_2 w_2\,. \tag{119}$$

Note that since the solutions (118) are special case of (119) when $\ell = 1$, this implies that the solutions (119) *themselves* should be families of solutions of the following *pq*-DE:

$$\frac{d}{dz}\left[w' - \ell \left(p\,\frac{G'}{G} - q\,\frac{H'}{H} \right) w \right] = 0\,. \tag{120}$$

The *mutual* presence of the parameter ℓ in the solutions (119) and their *pq*-DE (120) defines us, in this sense, the *reciprocal characteristic properties* of *pq*-DE and its solutions. Indeed, like its solutions, *pq*-DE (120) reduces to (106) when $\ell = 1$. Moreover, if presently we suppose $\ell \in \mathbf{N} \setminus \{0\}$ such that ℓ is not fixed, thus in such a case w is not simply a fundamental family of solutions, but it should be a system of fundamental families of solutions defined by finite summation:

$$w_n = \sum_{\ell=1}^{n} F_{\ell(p,q)}\,. \tag{121}$$

Therefore, *pq*-DE (120) becomes

$$\frac{d}{dz}\left[w_n' - \ell \left(p\,\frac{G'}{G} - q\,\frac{H'}{H} \right) w_n \right] = 0\,, \tag{122}$$

or equivalently

$$\frac{d}{dz}\left\{ \sum_{\ell=1}^{n} \left[F'_{\ell(p,q)} - \ell \left(p\,\frac{G'}{G} - q\,\frac{H'}{H} \right) F_{\ell(p,q)} \right] \right\} = 0\,. \tag{123}$$

Recall that until now the orders (p,q) are always considered as fixed real numbers, however, if hereafter they are supposed to be non fixed positive integers, that is $p,q \in \mathbf{N}$; in such case we can distinguish two systems (of fundamental families) of solutions defined as a finite summation.

Case 1: $\ell \in \mathbf{N} \setminus \{0\}$; $p,q \in \mathbf{N}$ and $p > q$ such that

$$w_n = \sum_{\ell=1}^{n} \sum_{p>q}^{n} F_{\ell(p,q)} = F_{1(1,0)} + F_{2(2,1)} + F_{3(3,2)} + \cdots + F_{n(n,n-1)}\,, \tag{124}$$

this justifies the following system of *pq*-DEs

$$\frac{d}{dz}\left\{ \sum_{\ell=1}^{n} \sum_{p>q}^{n} \left[F'_{\ell(p,q)} - \ell \left(p\,\frac{G'}{G} - q\,\frac{H'}{H} \right) F_{\ell(p,q)} \right] \right\} = 0\,. \tag{125}$$

Case 2: $\ell \in \mathbf{N} \setminus \{0\}$; $p,q \in \mathbf{N}$ and $p < q$ such that

$$w_n = \sum_{\ell=1}^{n} \sum_{p<q}^{n} F_{\ell(p,q)} = F_{1(0,1)} + F_{2(1,2)} + F_{3(2,3)} + \cdots + F_{n(n-1,n)}\,, \tag{126}$$

The corresponding system of *pq*-DEs takes the form

$$\frac{d}{dz}\left\{\sum_{\ell=1}^{n}\sum_{p<q}^{n}\left[F'_{\ell(p,q)} - \ell\left(p\,\frac{G'}{G} - q\,\frac{H'}{H}\right)F_{\ell(p,q)}\right]\right\} = 0\,. \qquad (127)$$

Hence, Eqs.(125) and (127) define us two systems of *pq*-DEs when p and q are non fixed positive integers and w_n is defined by (124) and/or (126). Furthermore, as it was previously mentioned, the different structural properties of Eq.(108) as a system of *pq*-DEs depend exclusively on the expressions of *pq*-function and *vice versa*.

7. Conclusion

In this paper, we have developed a theory based exclusively on the concept of pq-functions which should regard as an extension of previous work. We have studied the specific properties of *pq*-functions and the structural properties of *pq*-(P) DE and their consequences which, to our knowledge, have not previously been reported in the literature.

References

[1] M. E. Hassani, Heuristic Study of the Concept of *pq*-Radial Functions as a New Class of Potentials, *viXra:* 1304.0151 (2013)

[2] W. Rudin, Real and Complex analysis, (3rd ed.), Mc Graw Hill (1987)

[3] W. Wirtinger, Zur formalen Theorie der Funktionen von mehr komplexen Veränderlichen, *Mathematische Annalen* **97**, 357-375 (1926)

[4] M. Le Gendre, Recherches sur l'attraction des sphéroïdes homogènes, Mémoires de Mathématiques et de Physique, présentés à l'Académie Royale des Sciences, par divers savants, et lus dans ses Assemblées, Tome X, pp. 411-435 (Paris, 1785)