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Geometry on Combinatorial Structures

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Contents

1. Introduction

1.1 **A famous proverb---six blind men and an elephant**

touched leg --- pillar

touched tail --- rope

touched trunk --- tree branch

touched **ear --- hand fan**

touched **belly --- wall**

touched **tusk --- solid pipe**

All of you are right!

A wise man explains to them:

why are you telling it differently is because each one of you touched the different part of the elephant.

So, actually the elephant has all those features what you all said.

>>What is the implication of the wise man's saying?

In philosophy, it just means the limitation of one's knowledge, or in another words, one's knowledge on a thing is unilateral.

>>Question: *how to solve the problem of unilateral in one's knowledge?*

What is a thing? Particularly, What is an elephant?

>> *What is an elephant*?

Actually, the elephant has all those features what the blind men said. This means that

An elephant

= {leg} ∪{tail} ∪{trunk} ∪{ear} ∪{belly} ∪{tusk}

={pillar} ∪{rope} ∪{tree branch} ∪{hand fan}∪{wall}∪{solid pipe}

The situation for one realizing the behaviors of natural world is analogous

to the blind men determining what an elephant looks like.

Generally, for an integer $m \geq 2$, let

$$(\Sigma_1; \mathcal{R}_1),\ (\Sigma_2; \mathcal{R}_2),\ \cdots,\ (\Sigma_m; \mathcal{R}_m)$$

be m mathematical system different two by two. A Smarandache multi-

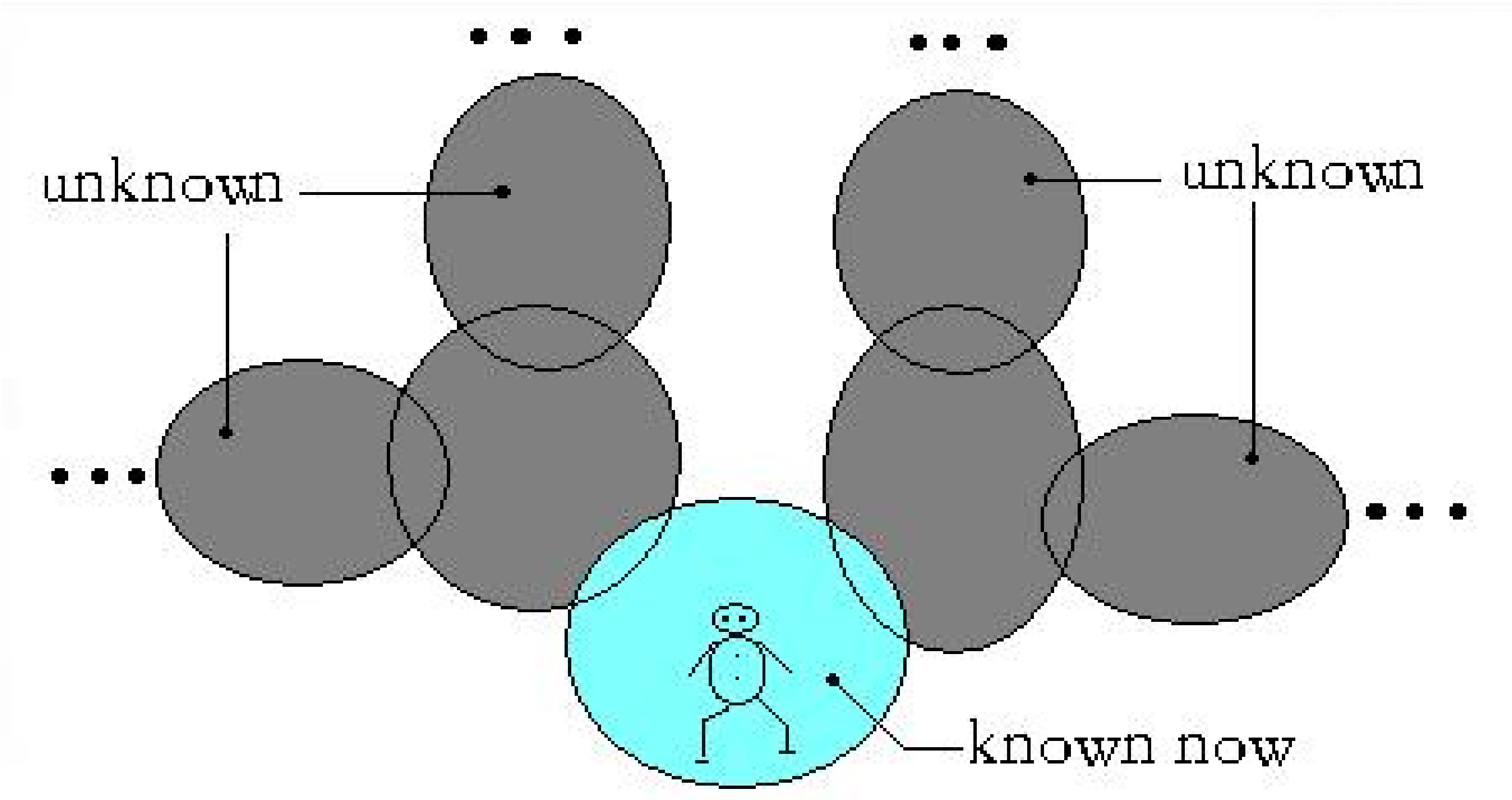
space is a pair $(\widetilde{\Sigma}; \widetilde{\mathcal{R}})$ with $\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i$ and $\widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i$.

>> What is a thing in one's eyes?

Similarly, we can determine what is a thing by

A thing = $\bigcup_{i=1}^{m} \Sigma_i$ where each Σ_i is the feature of the thing. Thus a thing is nothing but a Smarandache multi-space.

- **A depiction of the world by combinatoricians**



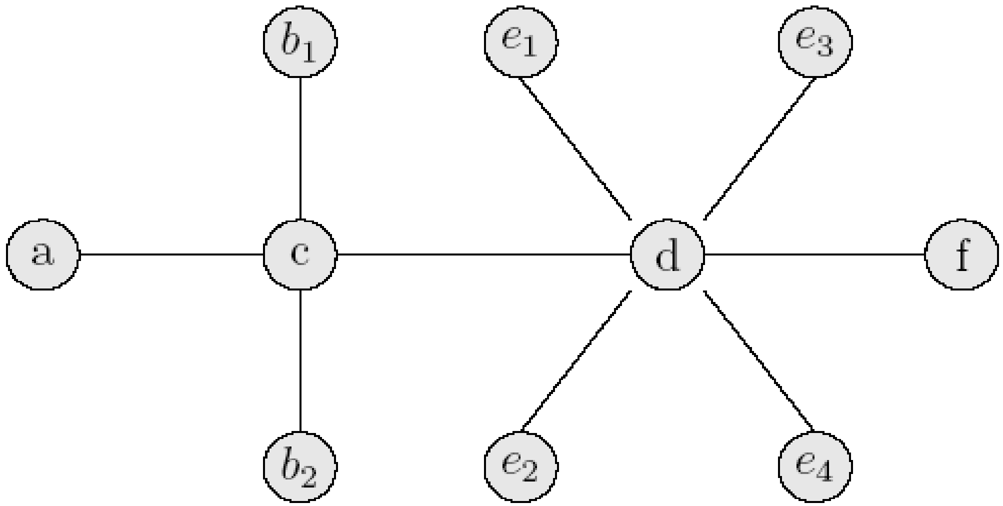
How to characterize it by mathematics? Manifold!

>>the universal relation of things in philosophy implies that the underlying structure in every thing of the universe is nothing but a combinatorial structure.

Then:

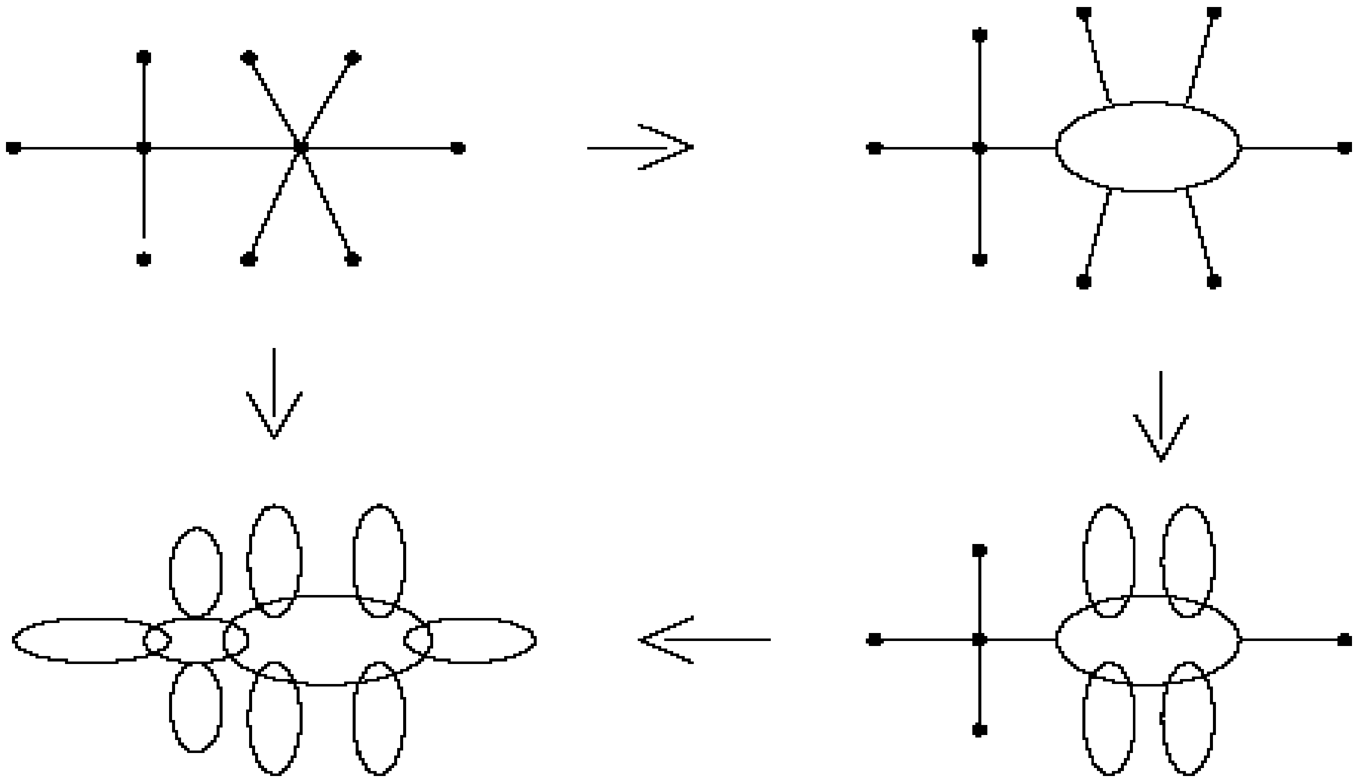
What is the underlying combinatorial structure of an elephant?

Let vertices {trunk}={a}, {ear}={b_1,b_2}, {tusk}={c}, {belly} = {d}, {leg}= {e_1,e_2,e_3,e_4}, {tail}={f} and two vertices adjacent if and only if they touch each other.



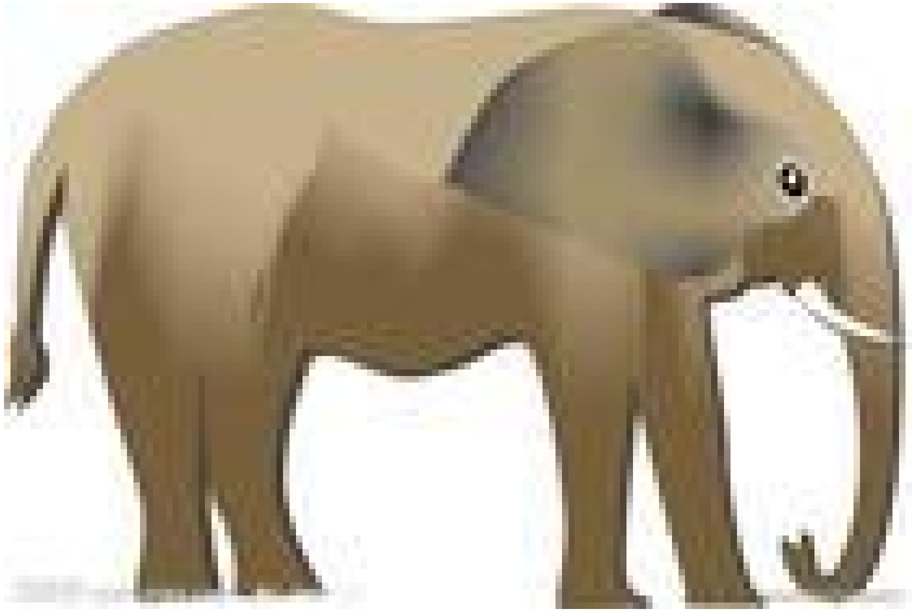
[NOTE] It is quite different from an elephant for drawing just by vertices and edges.

>> How to get a planar elephant from its combinatorial structure?



>> How to get a 3-dimensional elephant from its combinatorial structure?

Blowing up each 2-ball to a 3-ball, we get:



2. Fundamental Groups

Examples of Topological Space:

(1) Real numbers $\mathbb{R}$. Complex numbers $\mathbb{C}$.

(2) Euclidean space $\mathbf{R}^n$, Spheres $\mathbf{S}^n$ for $n \geq 1$;

(3) Product of spaces, such as $\mathbf{S}^2 \times \mathbf{S}^{n-2}$ for $n \geq 4$.

Definition Topological space, Hausdorff space, Open or closed sets, Open neighborhood, Cover, Basis, Compact space, ..., in [1]-[3] following.

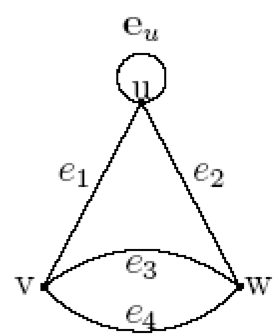
[1] John M.Lee, *Introduction to Topological Manifolds*, Springer-Verlag New York, Inc., 2000.

[2] W.S.Massey, *Algebraic Topology: An Introduction*, Springer-Verlag, New York, etc.(1977).

[3] Munkres J.R., *Topology* (2nd edition), Prentice Hall, Inc, 2000.

Definition *Let S be a topological space and $I=[0,1]\subset\mathbf{R}$. An arc a in S is a continuous mapping $a: I\rightarrow S$ with initial point $a(0)$ and end point $a(1)$, and S is called arcwise connected if every two points in S can be joined by an arc in S. An arc $a: I\rightarrow S$ is a loop based at p if $a(0)=a(1)=p\in S$. A degenerated loop $\mathbf{e}: I\rightarrow x\in S$, i.e., mapping each element in I to a point x, usually called a point loop.*

Example Let G be a planar 2-connected graph on $\mathbf{R}^2$ and S is a topological space consisting of points on each $e\in E(G)$. Then S is a arcwise connected space by definition. For a circuit C in G, we choose any point p on C. Then C is a loop $\mathbf{e}_p$ in S based at p, such as those shown in the following.



Definition *Let a and b be two arcs in a topological space S with $a(1) = b(0)$. A product mapping $a \cdot b$ of a with b is defined by*

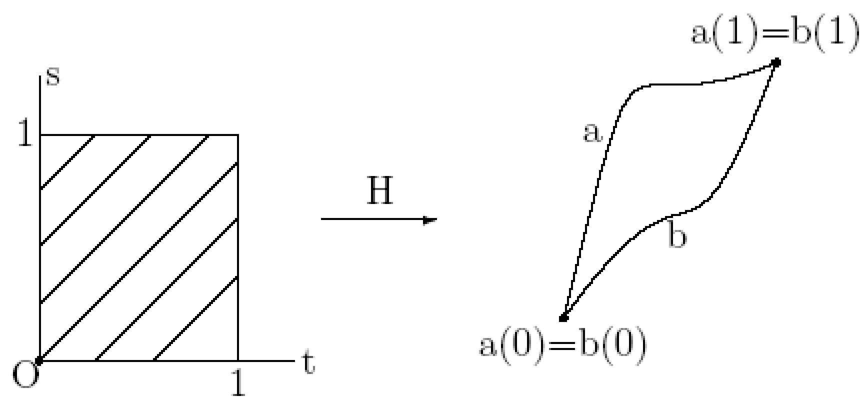
$$a \cdot b(t) = \begin{cases} a(2t), & \text{if} \quad 0 \leq t \leq \frac{1}{2}, \\ b(2t-1), & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

and an inverse mapping $\overline{a} = a(1-t)$ by a.

Definition *Let S be a topological space and $a, b : I \to S$ two arcs with $a(0) = b(0)$ and $a(1) = b(1)$. If there exists a continuous mapping*

$$H : I \times I \to S$$

such that $H(t,0) = a(t)$, $H(t,1) = b(t)$ for $\forall t \in I$, then a and b are said homotopic, denoted by $a \simeq b$ and H a homotopic mapping from a to b.



Theorem *The homotopic $\simeq$ is an equivalent relation, i.e, all arcs homotopic to an arc a is an equivalent arc class, denoted by $[a]$.*

Definition *For a topological space S and $x_0 \in S$, let $\pi_1(S, x_0)$ be a set consisting of equivalent classes of loops based at x_0. Define an operation $\circ$ in $\pi_1(S, x_0)$ by*

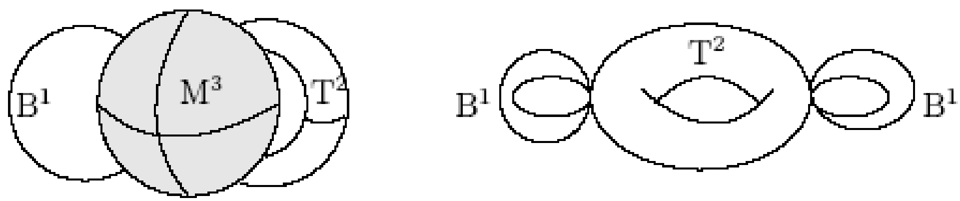
$$[a] \circ [b] = [a \cdot b] \quad \textit{and} \quad [a]^{-1} = [a^{-1}].$$

Theorem *$\pi_1(S, x_0)$ is a group.*

Example: (1) $\pi_1(\mathbf{R}^n, x_0), x_0 \in \mathbf{R}^n$ and $\pi_1(\mathbf{S}^n, y_0), y_0 \in \mathbf{S}^n$ is trivial for $n \geq 2$;
(2) $\pi_1(\mathbf{S}, y_0) \cong Z$ and $\pi_1(T^2, z_0) \cong Z^2$, $z_0 \in T^2$.

3. Combinatorial Manifolds

An *n-dimensional manifold* is a second countable Hausdorff space such that each point has an open neighborhood homomorphic to a Euclidean space $\mathbf{R}^n$ of dimension n, abbreviated to *n-manifold*.



(a) (b)

Definition *A combinatorial Euclidean space is a combinatorial system $\mathscr{C}_G$ of Euclidean spaces R^{n_1}, R^{n_2}, $\cdots$, R^{n_m} underlying a connected graph G defined by*

$$V(G) = \{\mathrm{R}^{n_1}, \mathrm{R}^{n_2}, \cdots, \mathrm{R}^{n_m}\},$$

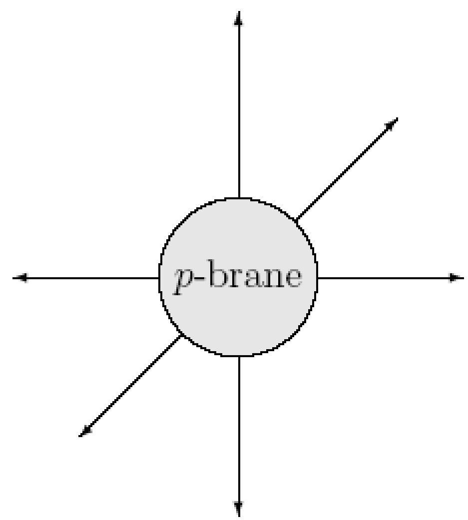
$$E(G) = \{\ (\mathrm{R}^{n_i}, \mathrm{R}^{n_j}) \mid \mathrm{R}^{n_i} \bigcap \mathrm{R}^{n_j} \neq \emptyset, 1 \leq i, j \leq m\},$$

denoted by $\mathscr{E}_G(n_1, \cdots, n_m)$ and abbreviated to $\mathscr{E}_G(r)$ if $n_1 = \cdots = n_m = r$, which enables us to view an Euclidean space $\mathbf{R}^n$ for $n \geq 4$.

Definition *A combinatorial fan-space* $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ *is the combinatorial Euclidean space* $\mathscr{E}_{K_m}(n_1, \cdots, n_m)$ *of* $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ *such that*

$$\mathbf{R}^{n_i} \bigcap \mathbf{R}^{n_j} = \bigcap_{k=1}^{m} \mathbf{R}^{n_k}.$$

for any integers $i, j,\ 1 \leq i \neq j \leq m$.



Definition *For a given integer sequence* $0 < n_1 < n_2 < \cdots < n_m$, $m \geq 1$, *a combinatorial manifold* $\widetilde{M}$ *is a Hausdorff space such that for any point* $p \in \widetilde{M}$, *there is a local chart* (U_p, φ_p) *of* p, *i.e., an open neighborhood* U_p *of* p *in* $\widetilde{M}$ *and a homoeomorphism* $\varphi_p : U_p \to \widetilde{\mathbf{R}}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$, *a combinatorial fan-space with*

$$\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\},$$
$$\bigcup_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\},$$

denoted by $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *or* $\widetilde{M}$ *on the context and*

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$.

A combinatorial manifold $\widetilde{M}$ *is finite if it is just combined by finite manifolds with an underlying combinatorial structure* G *without one manifold contained in the union of others.*

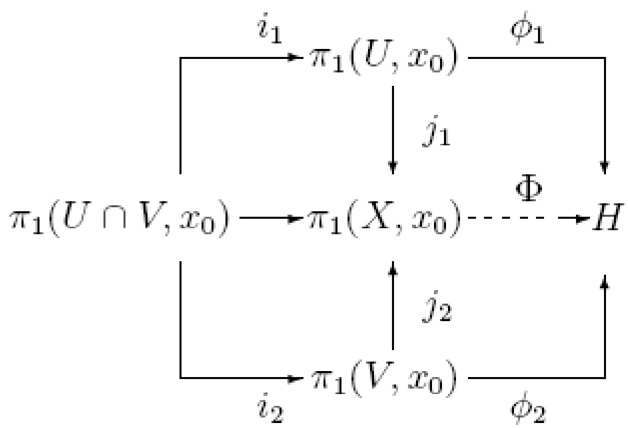
[4] Linfan Mao, Geometrical theory on combinatorial manifolds, JP J.Geometry and Topology, Vol.7, No.1(2007),65-114.

4. Classical Seifert-Van Kampen Theorem with Applications

Theorem 4.1(Seifert and Van-Kampen) *Let $X = U \cup V$ with U, V open subsets and let X, U, V, $U \cap V$ be non-empty arcwise-connected with $x_0 \in U \cap V$ and H a group. If there are homomorphisms*

$$\phi_1 : \pi_1(U, x_0) \to H \quad \text{and} \quad \phi_2 : \pi_1(V, x_0) \to H$$

and



with $\phi_1 \cdot i_1 = \phi_2 \cdot i_2$, where $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$, $i_2 : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$, $j_1 : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_2 : \pi_1(V, x_0) \to \pi_1(X, x_0)$ are homomorphisms induced by inclusion mappings, then there exists a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \cdot j_1 = \phi_1$ and $\Phi \cdot j_2 = \phi_2$.

Theorem 4.2(Seifert and Van-Kampen theorem, classical version) *Let spaces* X, U, V *and* x_0 *be in Theorem* 2.1. *If*

$$j : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$$

is an extension homomorphism of j_1 *and* j_2*, then* j *is an epimorphism with kernel* $\mathrm{Ker} j$ *generated by* $i_1^{-1}(g)i_2(g),\ g \in \pi_1(U \cap V, x_0)$*, i.e.,*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{\left\langle i_1^{-1}(g) \cdot i_2(g) |\ g \in \pi_1(U \cap V, x_0) \right\rangle^N}.$$

Corollary 4.1 *Let spaces* X, U, V *and* x_0 *be in Theorem* 2.1. *If* $U \cap V$ *is simply connected, then*

$$\pi_1(X) = \pi_1(U, x_0) * \pi_1(V, x_0).$$

Application: Let $B_n = \bigcup_{i=1}^{n} S_i^1$ be a bouquet shown in Fig.2.1 with $v_i \in S_i^1$, $W_i = S_i^1 - \{v_i\}$ for $1 \le i \le n$ and

$$U = S_1^1 \bigcup W_2 \bigcup \cdots \bigcup W_n \ \text{ and } \ V = W_1 \bigcup S_2^1 \bigcup \cdots \bigcup S_n^1.$$

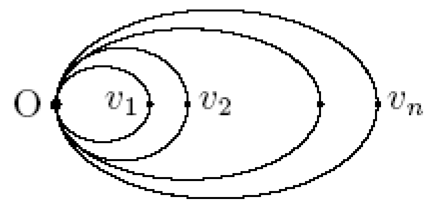


Fig.2.1

Then $U \bigcap V = S_{1.n}$, an arcwise-connected star. Whence,

$$\pi_1(B_n, O) = \pi_1(U, O) * \pi_1(V, O) \cong \pi_1(B_{n-1}, O) * \left\langle S_1^1 \right\rangle.$$

By induction, we easily get that

$$\pi_1(B_n, O) = \left\langle S_i^1, \ 1 \le i \le n \right\rangle.$$

5. Dimensional Graphs

Definition 5.1 *A topological graph $\mathscr{T}[G]$ is a pair (X, X^0) of a Hausdorff space X with its a subset X^0 such that*

(1) *X^0 is discrete, closed subspaces of X;*

(2) *$X - X^0$ is a disjoint union of open subsets $e_1, e_2, \cdots, e_m$, each of which is homeomorphic to an open interval $(0,1)$;*

(3) *the boundary $\overline{e}_i - e_i$ of e_i consists of one or two points. If $\overline{e}_i - e_i$ consists of two points, then $(\overline{e}_i, e_i)$ is homeomorphic to the pair $([0,1],(0,1))$; if $\overline{e}_i - e_i$ consists of one point, then $(\overline{e}_i, e_i)$ is homeomorphic to the pair $(S^1, S^1 - \{1\})$;*

(4) *a subset $A \subset \mathscr{T}[G]$ is open if and only if $A \cap \overline{e}_i$ is open for $1 \leq i \leq m$.*

Theorem 5.1([2]) *Any tree is contractible.*

Theorem 5.2([2]) *Let T_{span} be a spanning tree in the topological graph $\mathscr{T}[G]$, $\{e_\lambda : \lambda \in \Lambda\}$ the set of edges of $\mathscr{T}[G]$ not in T_{span} and $\alpha_\lambda = A_\lambda e_\lambda B_\lambda \in \pi(\mathscr{T}[G], v_0)$ a loop associated with $e_\lambda = a_\lambda b_\lambda$ for $\forall \lambda \in \Lambda$, where $v_0 \in \mathscr{T}[G]$ and A_λ, B_λ are unique paths from v_0 to a_λ or from b_λ to v_0 in T_{span}. Then*

$$\pi(\mathscr{T}[G], v_0) = \langle \alpha_\lambda | \lambda \in \Lambda \rangle .$$

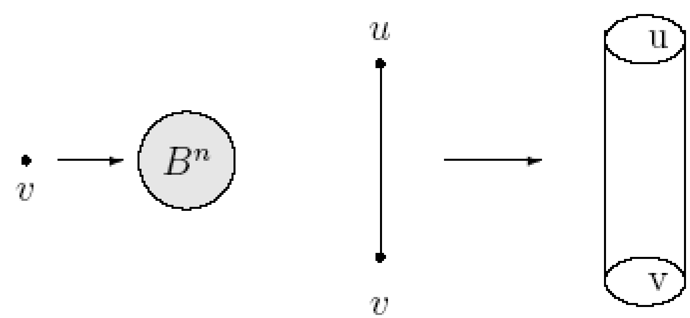
Definition 5.2 *An n-dimensional graph $\widetilde{M}^n[G]$ is a combinatorial Euclidean space $\mathscr{E}_G(n)$ of $\mathbf{R}^n_\mu, \mu \in \Lambda$ underlying a combinatorial structure G such that*

(1) $V(G)$ is discrete consisting of B^n, i.e., $\forall v \in V(G)$ is an open ball B^n_v;

(2) $\widetilde{M}^n[G] \setminus V(\widetilde{M}^n[G])$ is a disjoint union of open subsets $e_1, e_2, \cdots, e_m$, each of which is homeomorphic to an open ball B^n;

(3) the boundary $\overline{e}_i - e_i$ of e_i consists of one or two B^n and each pair $(\overline{e}_i, e_i)$ is homeomorphic to the pair $(\overline{B}^n, B^n)$;

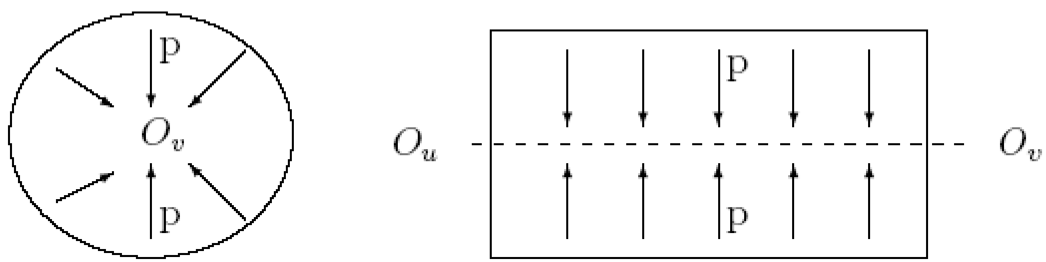
(4) a subset $A \subset \widetilde{M}^n[G]$ is open if and only if $A \cap \overline{e}_i$ is open for $1 \leq i \leq m$.



Theorem 5.3 *For any integer $n \geq 1$, $\mathscr{T}_0[G]$ is a deformation retract of $\widetilde{M}^n[G]$.*

Sketch of Proof If $n = 1$, then $\widetilde{M}^n[G] = \mathscr{T}_0[G]$ is itself a topological graph. So we assume $n \geq 2$.

For $n \geq 2$, let$f(\overline{x}, t) = (1 - t)\overline{x} + t\overline{x}_0$ be a mapping $f : \widetilde{M}^n[G] \times I \to \widetilde{M}^n[G]$ for $\forall \overline{x} \in \widetilde{M}^n[G]_1, t \in I$, where $\overline{x}_0 = O_v$ if $\overline{x} \in B_v^n$, and $\overline{x}_0 = p(\overline{x})$ if $\overline{x} \in e_i$, where $p : uv \to e_{uv}$ a projection for $1 \leq i \leq m$, such as those shown in Fig.2.3.



B_v^n uv

Fig.2.3

Then f is such a deformation retract. □

6. Generalized Seifert-Van Kampen Theorem

[5] LinfanMao, Graph structure of manifolds with listing, International J.Contemp. Math. Sciences, Vol.5, 2011, No.2,71-85.

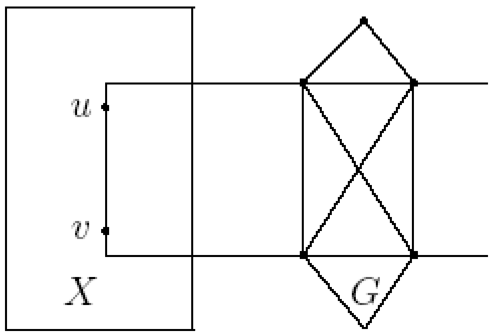
[6] Linfan Mao, A generalization of Seifert-Van Kampen theorem for fundamental groups, Far East Journal of Mathematical Sciences Vol.61 No.2 (2012), 141-160.

Definition 6.1 *A topological space X attached with a graph G is a space $X \odot G$ such that*

$$X \bigcap G \neq \emptyset, \quad G \not\subset X$$

and there are semi-edges $e^+ \in (X \bigcap G) \setminus G$, $e^+ \in G \setminus X$.

An example for $X \odot G$ can be found in Fig.4.1.



$X \odot G$

Theorem 6.1 *Let X be arc-connected space, G a graph and H the subgraph $X \cap G$ in $X \odot G$. Then for $x_0 \in X \cap G$,*

$$\pi_1(X \odot G, x_0) \cong \frac{\pi_1(X, x_0) * \pi_1(G, x_0)}{\left\langle i_1^{-1}(\alpha_{e_\lambda}) i_2(\alpha_{e_\lambda}) | \ e_\lambda \in E(H) \setminus T_{span} \right\rangle^N},$$

where $i_1 : \pi_1(H, x_0) \to X$, $i_2 : \pi_1(H, x_0) \to G$ are homomorphisms induced by inclusion mappings, T_{span} is a spanning tree in H, $\alpha_\lambda = A_\lambda e_\lambda B_\lambda$ is a loop associated with an edge $e_\lambda = a_\lambda b_\lambda \in H \setminus T_{span}$, $x_0 \in G$ and A_λ, B_λ are unique paths from x_0 to a_λ or from b_λ to x_0 in T_{span}.

Sketch of Proof Let $U = X$ and $V = G$. Applying the Seifert-Van Kampen theorem, we get that

$$\pi_1(X \odot G, x_0) \cong \frac{\pi_1(X, x_0) * \pi_1(G, x_0)}{\left\langle i_1^{-1}(g) i_2(g) | \ g \in \pi_1(X \cap G, x_0) \right\rangle},$$

Applying Theorem 3.2, We finally get the following conclusion,

$$\pi_1(X \odot G, x_0) \cong \frac{\pi_1(X, x_0) * \pi_1(G, x_0)}{\left\langle i_1^{-1}(\alpha_{e_\lambda}) i_2(\alpha_{e_\lambda}) | \ e_\lambda \in E(H) \setminus T_{span} \right\rangle^N}$$

□

Corollary 6.1 *Let X be arc-connected space, G a graph. If $X \cap G$ in $X \odot G$ is a tree, then*

$$\pi_1(X \odot G, x_0) \cong \pi_1(X, x_0) * \pi_1(G, x_0).$$

Particularly, if G is graphs shown in the following

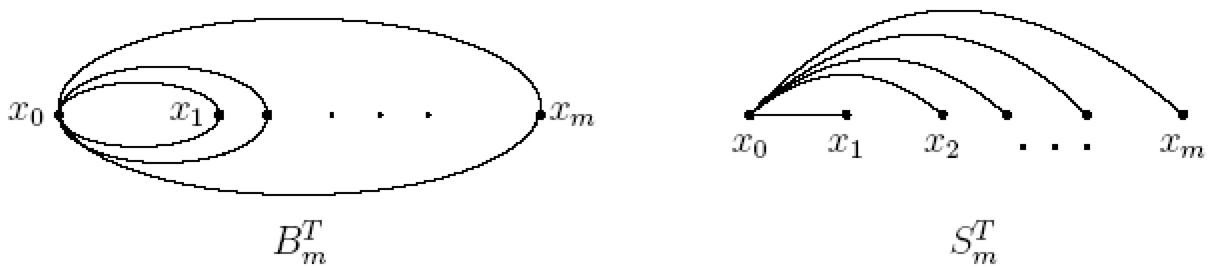


Fig.4.2

and $X \cap G = K_{1,m}$, Then

$$\pi_1(X \odot B_m^T, x_0) \cong \pi_1(X, x_0) * \langle L_i | 1 \leq i \leq m \rangle ,$$

where L_i is the loop of parallel edges (x_0, x_i) in B_m^T for $1 \leq i \leq m-1$ and

$$\pi_1(X \odot S_m^T, x_0) \cong \pi_1(X, x_0).$$

Theorem 6.2 *Let $\mathscr{X}_m \odot G$ be a topological space consisting of m arcwise-connected spaces $X_1, X_2, \cdots, X_m$, $X_i \cap X_j = \emptyset$ for $1 \leq i, j \leq m$ attached with a graph G, $V(G) = \{x_0, x_1, \cdots, x_{l-1}\}$, $m \leq l$ such that $X_i \cap G = \{x_i\}$ for $0 \leq i \leq l-1$. Then*

$$\begin{aligned}\pi_1(\mathscr{X}_m \odot G, x_0) &\cong \left(\prod_{i=1}^{m} \pi_1(X_i^*, x_0)\right) * \pi_1(G, x_0) \\ &\cong \left(\prod_{i=1}^{m} \pi_1(X_i, x_i)\right) * \pi_1(G, x_0),\end{aligned}$$

where $X_i^* = X_i \bigcup (x_0, x_i)$ with $X_i \cap (x_0, x_i) = \{x_i\}$ for $(x_0, x_i) \in E(G)$, integers $1 \leq i \leq m$.

Sketch of Proof The proof is by induction on m with Theorem 4.1 and the Seifert-Van Kampen theorem. □

Corollary 6.2 *Let G be the graph B_m^T or a star S_m^T. Then*

$$\pi_1(\mathscr{X}_m \odot B_m^T, x_0) \cong \left(\prod_{i=1}^{m} \pi_1(X_i^*, x_0)\right) * \pi_1(B_m^T, x_0)$$

$$\cong \left(\prod_{i=1}^{m} \pi_1(X_i, x_{i-1})\right) * \langle L_i | 1 \leq i \leq m \rangle,$$

where L_i is the loop of parallel edges (x_0, x_i) in B_m^T for integers $1 \leq i \leq m$ and

$$\pi_1(\mathscr{X}_m \odot S_m^T, x_0) \cong \prod_{i=1}^{m} \pi_1(X_i^*, x_0).$$

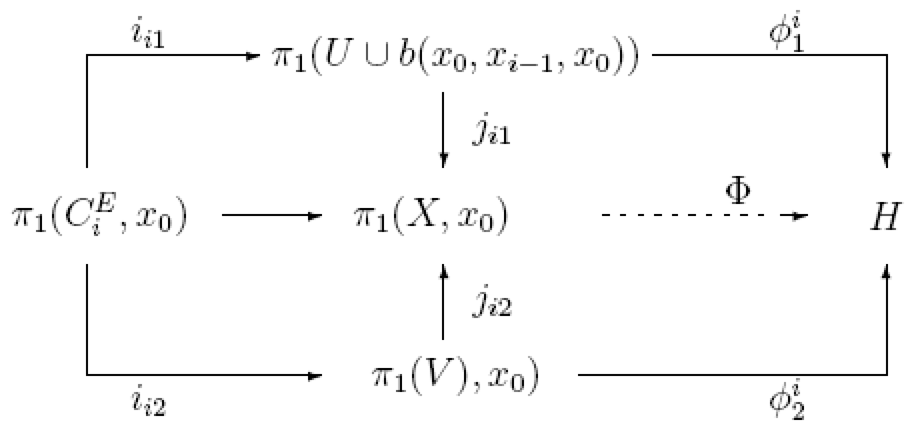
Corollary 6.3 Let $X = \mathscr{X}_m \odot G$ be a topological space with a simply-connected space X_i for any integer i, $1 \leq i \leq m$ and $x_0 \in X \cap G$. Then we know that

$$\pi_1(X, x_0) \cong \pi_1(G, x_0).$$

Theorem 6.3 *Let $X = U \cup V$, $U, V \subset X$ be open subsets and X, U, V arcwise-connected and let $C_1, C_2, \cdots, C_m$ be arcwise-connected components in $U \cap V$ for an integer $m \geq 1$, $x_{i-1} \in C_i$, $b(x_0, x_{i-1}) \subset V$ an arc $: I \to X$ with $b(0) = x_0, b(1) = x_{i-1}$ and $b(x_0, x_{i-1}) \cap U = \{x_0, x_{i-1}\}$, $C_i^E = C_i \bigcup b(x_0, x_{i-1})$ for any integer i, $1 \leq i \leq m$, H a group and there are homomorphisms*

$$\phi_1^i : \pi_1(U \bigcup b(x_0, x_{i-1}), x_0) \to H, \quad \phi_2^i : \pi_1(V, x_0) \to H$$

such that



with $\phi_1^i \cdot i_{i1} = \phi_2^i \cdot i_{i2}$, where $i_{i1} : \pi_1(C_i^E, x_0) \to \pi_1(U \cup b(x_0, x_{i-1}), x_0)$, i_{i2} : $\pi_1(C_i^E, x_0) \to \pi_1(V, x_0)$ and $j_{i1} : \pi_1(U \cup b(x_0, x_{i-1}, x_0)) \to \pi_1(X, x_0)$, $j_{i2} : \pi_1(V, x_0)) \to \pi_1(X, x_0)$ are homomorphisms induced by inclusion mappings, then there exists a unique homomorphism $\Phi :\ \pi_1(X, x_0) \to H$ such that $\Phi \cdot j_{i1} = \phi_1^i$ and $\Phi \cdot j_{i2} = \phi_2^i$ for integers $1 \leq i \leq m$.

Sketch of Proof Define $U^E = U \bigcup \{ b(x_0, x_i) \mid 1 \leq i \leq m-1\}$. Then we get that $X = U^E \cup V$, $U^E, V \subset X$ are still opened with an arcwise-connected intersection $U^E \cap V = \mathscr{X}_m \odot S_m^T$, where S_m^T is a graph formed by arcs $b(x_0, x_{i-1})$, $1 \leq i \leq m$.

Notice that $\mathscr{X}_m \odot Sm^T = \bigcup\limits_{i=1}^{m} C_i^E$ and $C_i^E \bigcap C_j^E = \{x_0\}$. Therefore, we get that

$$\pi_1(\mathscr{X}_m \odot S_m^T, x_0) = \bigotimes_{i=1}^{m} \pi_1(C_i^E, x_0).$$

This fact enables us knowing that there is a unique m-tuple $(h_1, h_2, \cdots, h_m)$, $h_i \in \pi_1(C_i^E, x_{i-1})$, $1 \leq i \leq m$ such that

$$\mathscr{I} = \prod_{i=1}^{m} h_i$$

for $\forall \mathscr{I} \in \pi_1(\mathscr{X}_m \odot S_m^T, x_0)$ and inclusion maps

$$i_1^E : \pi_1(\mathscr{X}_m \odot S_m^T, x_0) \to \pi_1(U^E, x_0),$$

$$i_2^E : \pi_1(\mathscr{X}_m \odot S_m^T, x_0) \to \pi_1(V, x_0),$$

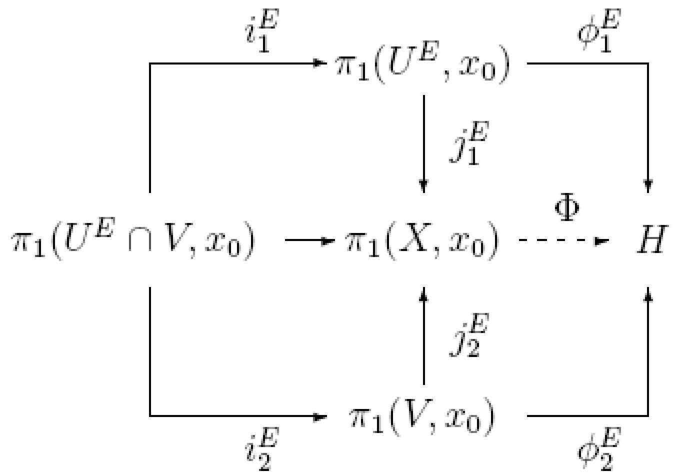
$$j_1^E : \pi_1(U^E, x_0) \to \pi_1(X, x_0), \quad j_2^E : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

with $i_1^E|_{\pi_1(C_i^E, x_0)} = i_{i1}$, $i_2^E|_{\pi_1(C_i^E, x_0)} = i_{i2}$, $j_1^E|_{\pi_1(U \cup b(x_0, x_{i-1}, x_0))} = j_{i1}$ and $j_2^E|_{\pi_1(V, x_0)} = j_{i2}$ for integers $1 \leq i \leq m$.

Define ϕ_1^E and ϕ_2^E by

$$\phi_1^E(\mathscr{I}) = \prod_{i=1}^{m} \phi_1^i(i_{i1}(h_i)), \quad \phi_2^E(\mathscr{I}) = \prod_{i=1}^{m} \phi_2^i(i_{i2}(h_i)).$$

Then the following diagram



is commutative. Applying Theorem 2.1, we get the conclusion. □

Theorem 6.4 *Let* X, U, V, C_i^E, $b(x_0, x_{i-1})$ *be arcwise-connected spaces for any integer* i, $1 \leq i \leq m$ *as in Theorem* 3.1, $U^E = U \bigcup \{\ b(x_0, x_i) \mid 1 \leq i \leq m-1\}$ *and* B_m^T *a graph formed by arcs* $a(x_0, x_{i-1})$, $b(x_0, x_{i-1})$, $1 \leq i \leq m$, *where* $a(x_0, x_{i-1}) \subset U$ *is an arc* : $I \to X$ *with* $a(0) = x_0, a(1) = x_{i-1}$ *and* $a(x_0, x_{i-1}) \cap V = \{x_0, x_{i-1}\}$. *Then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(B_m^T, x_0)}{\left\langle (i_1^E)^{-1}(g) \cdot i_2(g) |\ g \in \prod_{i=1}^{m} \pi_1(C_i^E, x_0) \ \right\rangle^N},$$

where $i_1^E : \pi_1(U^E \cap V, x_0) \to \pi_1(U^E, x_0)$ *and* $i_2^E : \pi_1(U^E \cap V, x_0) \to \pi_1(V, x_0)$ *are homomorphisms induced by inclusion mappings.*

Theorem 6.5 *Let* X, U, V, $C_1, C_2, \cdots, C_m$ *be arcwise-connected spaces,* $b(x_0, x_{i-1})$ *arcs for any integer* i, $1 \leq i \leq m$ *as in Theorem* 3.1, $U^E = U \bigcup \{\ b(x_0, x_{i-1}) \mid 1 \leq i \leq m\}$ *and* B_m^T *a graph formed by arcs* $a(x_0, x_{i-1})$, $b(x_0, x_{i-1})$, $1 \leq i \leq m$. *Then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(B_m^T, x_0)}{\left\langle (i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{i=1}^{m} \pi_1(C_i^E, x_0) \right\rangle^N},$$

where $i_1^E : \pi_1(U^E \cap V, x_0) \to \pi_1(U^E, x_0)$ *and* $i_2^E : \pi_1(U^E \cap V, x_0) \to \pi_1(V, x_0)$ *are homomorphisms induced by inclusion mappings.*

Corollary 6.4 *Let $X = U \cup V$, $U, V \subset X$ be open subsets and X, U, V and $U \cap V$ arcwise-connected. Then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{\left\langle i_1^{-1}(g) \cdot i_2(g) |\ g \in \pi_1(U \cap V, x_0) \right\rangle^N},$$

where $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $i_2 : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$ are homomorphisms induced by inclusion mappings.

Corollary 6.5 *Let X, U, V, C_i, $a(x_0, x_i)$, $b(x_0, x_i)$ for integers i, $1 \leq i \leq m$ be as in Theorem 3.1. If each C_i is simply-connected, then*

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(B_m^T, x_0).$$

Corollary 6.6 *Let X, U, V, C_i, $a(x_0, x_i)$, $b(x_0, x_i)$ for integers i, $1 \leq i \leq m$ be as in Theorem 3.1. If V is simply-connected, then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(B_m^T, x_0)}{\left\langle (i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{i=1}^{m} \pi_1(C_i^E, x_0) \right\rangle^N},$$

where $i_1^E : \pi_1(U^E \cap V, x_0) \to \pi_1(U^E, x_0)$ and $i_2^E : \pi_1(U^E \cap V, x_0) \to \pi_1(V, x_0)$ are homomorphisms induced by inclusion mappings.

7. Fundamental Groups of Spaces

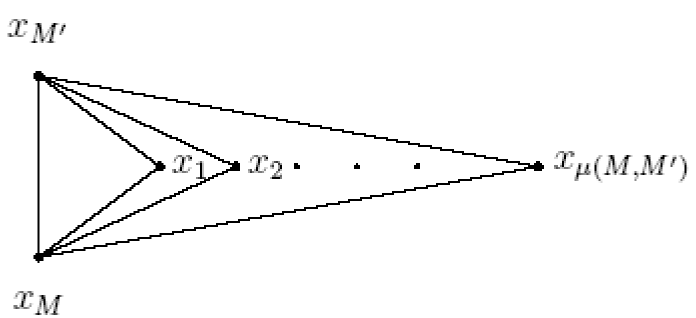
Definition 7.1 *Let $\widetilde{M}$ be a combinatorial manifold underlying a graph $G[\widetilde{M}]$. An edge-induced graph $G^\theta[\widetilde{M}]$ is defined by*

$$V(G^\theta[\widetilde{M}]) = \{x_M, x_{M'}, x_1, x_2, \cdots, x_{\mu(M,M')} | \text{ for } \forall (M, M') \in E(G[\widetilde{M}])\},$$

$$E(G^\theta[\widetilde{M}]) = \{(x_M, x_{M'}), (x_M, x_i), (x_{M'}, x_i) | \ 1 \leq i \leq \mu(M, M')\},$$

where $\mu(M, M')$ is called the edge-index of (M, M') with $\mu(M, M') + 1$ equal to the number of arcwise-connected components in $M \cap M'$.

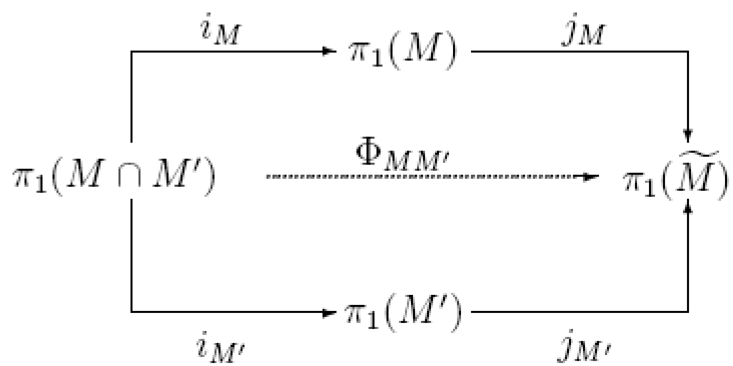
By definition, $G^\theta[\widetilde{M}]$ of a combinatorial manifold $\widetilde{M}$ is gotten by replacing each edge (M, M') in $G[\widetilde{M}]$ by a subgraph $TB^T_{\mu(M,M')}$ shown in the following with $x_M = M$ and $x_{M'} = M'$.



Theorem 7.1 *Let $\widetilde{M}$ be a finitely combinatorial manifold. Then*

$$\pi_1(\widetilde{M}) \cong \frac{\left(\prod\limits_{M\in V(G[\widetilde{M}])} \pi_1(M)\right) * \pi_1(G^\theta[\widetilde{M}])}{\left\langle (i_1^E)^{-1}(g)\cdot i_2^E(g) \mid g \in \prod\limits_{(M_1,M_2)\in E(G[\widetilde{M}])} \pi_1(M_1 \bigcap M_2) \right\rangle^N},$$

where i_1^E and i_2^E are homomorphisms induced by inclusion mappings $i_M : \pi_1(M \cap M') \to \pi_1(M)$, $i_{M'} : \pi_1(M \cap M') \to \pi_1(M')$ such as those shown in the following diagram:



for $\forall (M, M') \in E(G[\widetilde{M}])$.

Corollary 7.1 *Let $\widetilde{M}$ be a finitely combinatorial manifold. If for $\forall(M_1, M_2) \in E(G^L[\widetilde{M}])$, $M_1 \cap M_2$ is simply connected, then*

$$\pi_1(\widetilde{M}) \cong \left(\bigotimes_{M \in V(G[\widetilde{M}])} \pi_1(M) \right) \bigotimes \pi_1(G[\widetilde{M}]).$$

Theorem 7.2 *Let M be a compact n-manifold with charts $\{(U_\lambda, \varphi_\lambda) | \varphi_\lambda : U_\lambda \to \mathbf{R}^n, \lambda \in \Lambda)\}$. Then*

$$\pi_1(M) \cong \frac{\pi_1(G^\theta[M])}{\left\langle (i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{(U_\mu, U_\nu) \in E(G[M])} \pi_1(U_\mu \cap U_\nu) \right\rangle^N},$$

where i_1^E and i_2^E are homomorphisms induced by inclusion mappings $i_{U_\mu} : \pi_1(U_\mu \cap U_\nu) \to \pi_1(U_\mu)$, $i_{U_\nu} : \pi_1(U_\mu \cap U_\nu) \to \pi_1(U_\nu)$, $\mu, \nu \in \Lambda$.

Corollary 7.2 *Let M be a simply connected manifold with charts $\{(U_\lambda, \varphi_\lambda) | \varphi_\lambda : U_\lambda \to \mathbf{R}^n, \lambda \in \Lambda)\}$. Then $G^\theta[M] = G[M]$ is a tree.*

Corollary 7.3 *Let M be a compact n-manifold with charts $\{(U_\lambda, \varphi_\lambda) | \varphi_\lambda : U_\lambda \to \mathbf{R}^n, \lambda \in \Lambda)\}$. If $U_\mu \cap U_\nu$ is simply connected for $\forall \mu, \nu \in \Lambda$, then*

$$\pi_1(M) \cong \pi_1(G[M]).$$

Theorem 7.2 *Let M be a compact n-manifold with charts $\{(U_\lambda, \varphi_\lambda)|\ \varphi_\lambda : U_\lambda \to \mathbf{R}^n, \lambda \in \Lambda)\}$. Then*

$$\pi_1(M) \cong \frac{\pi_1(G^\theta[M])}{\left\langle (i_1^E)^{-1}(g) \cdot i_2^E(g)|\ g \in \prod_{(U_\mu, U_\nu) \in E(G[M])} \pi_1(U_\mu \cap U_\nu) \right\rangle^N},$$

where i_1^E and i_2^E are homomorphisms induced by inclusion mappings $i_{U_\mu} : \pi_1(U_\mu \cap U_\nu) \to \pi_1(U_\mu)$, $i_{U_\nu} : \pi_1(U_\mu \cap U_\nu) \to \pi_1(U_\nu)$, $\mu, \nu \in \Lambda$.

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$$\pi_1(M) \cong \pi_1(G[M]).$$

8. Combinatorially Differential Theory

Differential n-manifolds

An *differential n-manifold* $(M^n, \mathcal{A})$ is an n-manifold M^n, $M^n = \bigcup_{i \in I} U_i$, endowed with a C^r differential structure $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$ on M^n for an integer r with following conditions hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of M^n;

(2) For $\forall \alpha, \beta \in I$, atlases $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the *overlap maps*

$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\beta(U_\beta) \quad \text{and} \quad \varphi_\beta \varphi_\alpha^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\alpha(U_\alpha)$$

are C^r;

(3) $\mathcal{A}$ is maximal, i.e., if (U, φ) is an atlas of M^n equivalent with one atlas in $\mathcal{A}$, then $(U, \varphi) \in \mathcal{A}$.

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(2) For $\forall \alpha, \beta \in I$, atlases $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are *equivalent*, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the *overlap maps*

$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\beta(U_\beta) \quad \text{and} \quad \varphi_\beta \varphi_\alpha^{-1} : \varphi_\beta(U_{\alpha \bigcap U_\beta}) \to \varphi_\alpha(U_\alpha)$$

are C^r;

(3) $\mathcal{A}$ is maximal, i.e., if (U, φ) is an atlas of M^n equivalent with one atlas in $\mathcal{A}$, then $(U, \varphi) \in \mathcal{A}$.

Differentiable Combinatorial Manifold

Definition 3.1 *For a given integer sequence $1 \leq n_1 < n_2 < \cdots < n_m$, a combinatorial C^h-differential manifold $(\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ is a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$, $\widetilde{M}(n_1, n_2, \cdots, n_m) = \bigcup_{i \in I} U_i$, endowed with a atlas $\widetilde{\mathcal{A}} = \{(U_\alpha; \varphi_\alpha) | \alpha \in I\}$ on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ for an integer $h, h \geq 1$ with conditions following hold.*

(1) *$\{U_\alpha; \alpha \in I\}$ is an open covering of $\widetilde{M}(n_1, n_2, \cdots, n_m)$.*

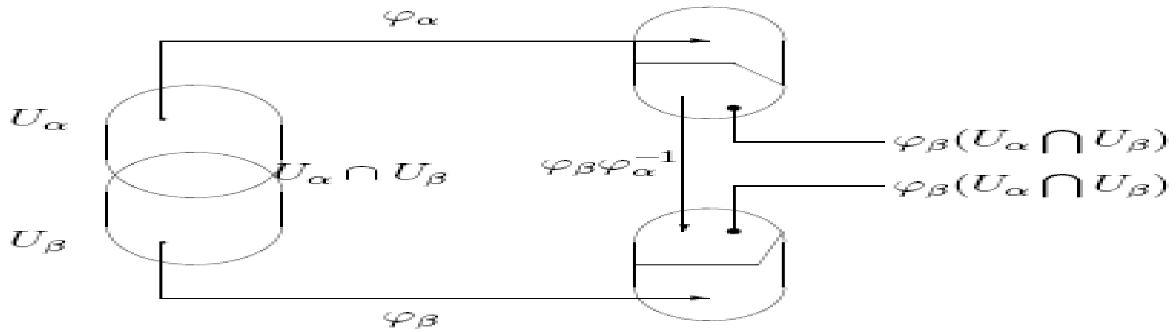
(2) *For $\forall \alpha, \beta \in I$, local charts $(U_\alpha; \varphi_\alpha)$ and $(U_\beta; \varphi_\beta)$ are equivalent, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the overlap maps*

$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \bigcap U_\beta) \rightarrow \varphi_\beta(U_\beta) \text{ and } \varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \bigcap U_\beta) \rightarrow \varphi_\alpha(U_\alpha)$$

are C^h-mappings.

(3) *$\widetilde{\mathcal{A}}$ is maximal, i.e., if $(U; \varphi)$ is a local chart of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ equivalent with one of local charts in $\widetilde{\mathcal{A}}$, then $(U; \varphi) \in \widetilde{\mathcal{A}}$.*

Explains for condition (2)



Extence Theorem *Let* $\widetilde{M}(n_1, n_2, \cdots, n_m)$ *be a finitely combinatorial manifold and* $d, 1 \leq d \leq n_1$ *an integer. If* $\forall M \in V(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)])$ *is* C^h*-differential and* $\forall (M_1, M_2) \in E(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)])$ *there exist atlas*

$$\mathcal{A}_1 = \{(V_x; \varphi_x) | \forall x \in M_1\} \quad \mathcal{A}_2 = \{(W_y; \psi_y) | \forall y \in M_2\}$$

such that $\varphi_x|_{V_x \bigcap W_y} = \psi_y|_{V_x \bigcap W_y}$ *for* $\forall x \in M_1, y \in M_2$*, then there is a differential structures*

$$\widetilde{\mathcal{A}} = \{(U_p; [\varpi_p]) | \forall p \in \widetilde{M}(n_1, n_2, \cdots, n_m)\}$$

such that $(\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ *is a combinatorial* C^h*-differential manifold.*

Local Properties of Combinatorial Manifolds

Denote by $\mathscr{X}_p$ all these C^∞-functions at a point $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$.

Definition 3.2 *Let $(\widetilde{M}(n_1, n_2, \cdots, n_m), \widetilde{\mathcal{A}})$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. A tangent vector $\overline{v}$ at p is a mapping $\overline{v} : \mathscr{X}_p \to \mathbf{R}$ with conditions following hold.*

(1) $\forall g, h \in \mathscr{X}_p, \forall \lambda \in \mathbf{R},\ \overline{v}(h + \lambda h) = \overline{v}(g) + \lambda \overline{v}(h)$;

(2) $\forall g, h \in \mathscr{X}_p, \overline{v}(gh) = \overline{v}(g)h(p) + g(p)\overline{v}(h)$.

Theorem 3.2 *For any point $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ is*

$$\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p))$$

with a basis matrix $[\frac{\partial}{\partial \overline{x}}]_{s(p)\times n_{s(p)}} =$

$$\begin{bmatrix} \frac{1}{s(p)}\frac{\partial}{\partial x^{11}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{1\widehat{s}(p)}} & \frac{\partial}{\partial x^{1(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{1n_1}} & \cdots & 0 \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{21}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{2\widehat{s}(p)}} & \frac{\partial}{\partial x^{2(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{2n_2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)1}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)\widehat{s}(p)}} & \frac{\partial}{\partial x^{s(p)(\widehat{s}(p)+1)}} & \cdots & \cdots & \frac{\partial}{\partial x^{s(p)(n_{s(p)}-1)}} & \frac{\partial}{\partial x^{s(p)n_{s(p)}}} \end{bmatrix}$$

where $x^{il} = x^{jl}$ for $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$.

Tensor Field

Definition 3.3 *Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. A tensor of type (r, s) at the point p on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is an $(r+s)$-multilinear function τ,*

$$\tau : \underbrace{T_p^*\widetilde{M} \times \cdots \times T_p^*\widetilde{M}}_{r} \times \underbrace{T_p\widetilde{M} \times \cdots \times T_p\widetilde{M}}_{s} \to \mathbf{R},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$.

Theorem 3.3 *Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. Then*

$$T_s^r(p, \widetilde{M}) = \underbrace{T_p\widetilde{M} \otimes \cdots \otimes T_p\widetilde{M}}_{r} \otimes \underbrace{T_p^*\widetilde{M} \otimes \cdots \otimes T_p^*\widetilde{M}}_{s},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$, particularly,

$$\dim T_s^r(p, \widetilde{M}) = (\widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p)))^{r+s}.$$

Curvature Tensor

Definition 3.4 *Let* $\widetilde{M}$ *be a smoothly combinatorial manifold. A connection on tensors of* $\widetilde{M}$ *is a mapping* $\widetilde{D} : \mathscr{X}(\widetilde{M}) \times T_s^r\widetilde{M} \to T_s^r\widetilde{M}$ *with* $\widetilde{D}_X\tau = \widetilde{D}(X,\tau)$ *such that for* $\forall X, Y \in \mathscr{X}\widetilde{M}$, $\tau, \pi \in T_s^r(\widetilde{M})$,$\lambda \in \mathbf{R}$ *and* $f \in C^\infty(\widetilde{M})$,

(1) $\widetilde{D}_{X+fY}\tau = \widetilde{D}_X\tau + f\widetilde{D}_Y\tau$; and $\widetilde{D}_X(\tau + \lambda\pi) = \widetilde{D}_X\tau + \lambda\widetilde{D}_X\pi$;

(2) $\widetilde{D}_X(\tau \otimes \pi) = \widetilde{D}_X\tau \otimes \pi + \sigma \otimes \widetilde{D}_X\pi$;

(3) *for any contraction* C *on* $T_s^r(\widetilde{M})$, $\widetilde{D}_X(C(\tau)) = C(\widetilde{D}_X\tau)$.

A *combinatorial connection space* is a 2-tuple $(\widetilde{M}, \widetilde{D})$ consisting of a smoothly combinatorial manifold $\widetilde{M}$ with a connection $\widetilde{D}$ on its tensors.

For $\forall X, Y \in \mathscr{X}(\widetilde{M})$, *a combinatorial curvature operator*

$$\widetilde{\mathcal{R}}(X,Y) : \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$$

is defined by

$$\widetilde{\mathcal{R}}(X,Y)Z = \widetilde{D}_X\widetilde{D}_YZ - \widetilde{D}_Y\widetilde{D}_XZ - \widetilde{D}_{[X,Y]}Z$$

for $\forall Z \in \mathscr{X}(\widetilde{M})$.

Definition 3.5 *Let $\widetilde{M}$ be a smoothly combinatorial manifold and $g \in A^2(\widetilde{M}) = \bigcup_{p\in\widetilde{M}} T_2^0(p, \widetilde{M})$. If g is symmetrical and positive, then $\widetilde{M}$ is called a combinatorial Riemannian manifold, denoted by $(\widetilde{M}, g)$. In this case, if there is a connection $\widetilde{D}$ on $(\widetilde{M}, g)$ with equality following hold*

$$Z(g(X,Y)) = g(\widetilde{D}_Z, Y) + g(X, \widetilde{D}_Z Y)$$

then $\widetilde{M}$ is called a combinatorial Riemannian geometry, denoted by $(\widetilde{M}, g, \widetilde{D})$.

In this case, $\widetilde{R} = \widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} dx^{\sigma\varsigma} \otimes dx^{\eta\theta} \otimes dx^{\mu\nu} \otimes dx^{\kappa\lambda}$ with

$$\begin{aligned} \widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} &= \frac{1}{2}\left(\frac{\partial^2 g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}\partial x^{\eta\theta}} + \frac{\partial^2 g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\mu\nu\nu}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\mu\nu)(\eta\theta)}}{\partial x^{\kappa\lambda}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\kappa\lambda)(\sigma\varsigma)}}{\partial x^{\mu\nu}\partial x^{\eta\theta}}\right) \\ &+ \Gamma^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)}\Gamma^{\xi o}_{(\kappa\lambda)(\eta\theta)} g_{(\xi o)(\vartheta\iota)} - \Gamma^{\xi o}_{(\mu\nu)(\eta\theta)}\Gamma_{(\kappa\lambda)(\sigma\varsigma)^{\vartheta\iota}} g_{(\xi o)(\vartheta\iota)}, \end{aligned}$$

where $g_{(\mu\nu)(\kappa\lambda)} = g(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}})$.

9. Application to Gravitational Field Theory

[7] LinfanMao, Relativity in combinatorial gravitational fields, Progress in Physics,Vol.3(2010), 39-50.

Principle 3.1 *These gravitational forces and inertial forces acting on a particle in a gravitational field are equivalent and indistinguishable from each other.*

Principle 3.2 *An equation describing a law of physics should have the same form in all reference frame.*

Principle 3.3 *A physics law in a combinatorial field is invariant under a projection on its a field.*

Projective Principle *Let $(\widetilde{M}, g, \widetilde{D})$ be a combinatorial Riemannian manifold and $\mathscr{F} \in T_s^r(\widetilde{M})$ with a local form*

$$\mathscr{F}_{(\mu_1\nu_1)\cdots(\mu_s\nu_s)}^{(\kappa_1\lambda_1)\cdots(\kappa_r\lambda_r)} e_{\kappa_1\lambda_1} \otimes \cdots \otimes e_{\kappa_r\lambda_r} \omega^{\mu_1\nu_1} \otimes \cdots \otimes \omega^{\mu_s\nu_s}$$

in $(U_p, [\varphi_p])$. If

$$\mathscr{F}_{(\mu_1\nu_1)\cdots(\mu_s\nu_s)}^{(\kappa_1\lambda_1)\cdots(\kappa_r\lambda_r)} = 0$$

for integers $1 \leq \mu_i \leq s(p), 1 \leq \nu_i \leq n_{\mu_i}$ with $1 \leq i \leq s$ and $1 \leq \kappa_j \leq s(p), 1 \leq \lambda_j \leq n_{\kappa_j}$ with $1 \leq j \leq r$, then for any integer $\mu, 1 \leq \mu \leq s(p)$, there must be

$$\mathscr{F}_{(\mu\nu_1)\cdots(\mu\nu_s)}^{(\mu\lambda_1)\cdots(\mu\lambda_r)} = 0$$

for integers ν_i, $1 \leq \nu_i \leq n_\mu$ with $1 \leq i \leq s$.

Let $\mathscr{L}_{G^L[\widetilde{M}]}$ be the Lagrange density of a combinatorial spacetime $(\mathscr{C}_G|\overline{t})$. Then we know equations of the combinatorial gravitational field $(\mathscr{C}_G|\overline{t})$ to be

$$\partial_\mu \frac{\partial \mathscr{L}_{G^L[\widetilde{M}]}}{\partial \partial_\mu \phi_{\widetilde{M}}} - \frac{\partial \mathscr{L}_{G^L[\widetilde{M}]}}{\partial \phi_{\widetilde{M}}} = 0, \quad (3-1)$$

by the Euler-Lagrange equation, where $\phi_{\widetilde{M}}$ is the wave function of $(\mathscr{C}_G|\overline{t})$. Choose its Lagrange density $\mathscr{L}_{G^L[\widetilde{M}]}$ to be

$$\mathscr{L}_{G^L[\widetilde{M}]} = \widetilde{R} - 2\kappa \mathscr{L}_F,$$

where $\kappa = -8\pi G$ and $\mathscr{L}_F$ the Lagrange density for all other fields with

$$\widetilde{R} = g^{(\mu\nu)(\kappa\lambda)} \widetilde{R}_{(\mu\nu)(\kappa\lambda)}, \ \widetilde{R}_{(\mu\nu)(\kappa\lambda)} = \widetilde{R}^{\sigma\varsigma}_{(\mu\nu)(\sigma\varsigma)(\kappa\lambda)}.$$

Applying the Euler-Lagrange equation we get the equation of combinatorial gravitational field following

$$\widetilde{\mathcal{R}}_{(\mu\nu)(\kappa\lambda)} - \frac{1}{2}\widetilde{R} g_{(\mu\nu)(\kappa\lambda)} = \kappa \mathscr{E}_{(\mu\nu)(\kappa\lambda)}, \quad (3-2)$$

where $\mathscr{E}_{(\mu\nu)(\kappa\lambda)}$ is the energy-momentum tensor.

Let $(\mathscr{C}_G|\bar{t})$ be a gravitational field. We know its Schwarzschild metric, i.e., a spherically symmetric solution of Einstein's gravitational equations in vacuum is

$$ds^2 = \left(1 - \frac{r_s}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - -r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (3-4)$$

where $r_s = 2Gm/c^2$. Now we generalize it to combinatorial gravitational fields to find the solutions of equations

$$R_{(\mu\nu)(\sigma\tau)} - \frac{1}{2}g_{(\mu\nu)(\sigma\tau)}R = -8\pi G\mathscr{E}_{(\mu\nu)(\sigma\tau)}$$

in vacuum, i.e., $\mathscr{E}_{(\mu\nu)(\sigma\tau)} = 0$. Notice that the underlying graph of combinatorial field consisting of m gravitational fields is a complete graph K_m. For such a objective, we only consider the homogenous combinatorial Euclidean spaces $\widetilde{M} = \bigcup_{i=1}^{m} \mathbf{R}^{n_i}$, i.e., for any point $p \in \widetilde{M}$,

$$[\varphi_p] = \begin{bmatrix} x^{11} & \cdots & x^{1n_1} & \cdots & 0 \\ x^{21} & \cdots & x^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x^{m1} & \cdots & \cdots & \cdots & x^{mn_m} \end{bmatrix}$$

with $\widehat{m} = \dim(\bigcap_{i=1}^{m} \mathbf{R}^{n_i})$ a constant for $\forall p \in \bigcap_{i=1}^{m} \mathbf{R}^{n_i}$ and $x^{il} = \frac{x^l}{m}$ for $1 \le i \le m, 1 \le l \le \widehat{m}$.

Let $(\mathscr{C}_G|\overline{t})$ be a combinatorial field of gravitational fields $M_1, \cdots, M_m$ with masses $m_1, \cdots, m_m$ respectively. For usually undergoing, we consider the case of $n_\mu = 4$ for $1 \le \mu \le m$ since line elements have been found concretely in classical gravitational field in these cases. Now establish m spherical coordinate subframe $(t_\mu; r_\mu, \theta_\mu, \phi_\mu)$ with its originality at the center of such a mass space. Then we have known its a spherically symmetric solution by $(3-4)$ to be

$$\begin{aligned} ds_\mu^2 &= (1-\frac{r_{\mu s}}{r_\mu})dt_\mu^2 - (1-\frac{r_{\mu s}}{r_\mu})^{-1}dr_\mu^2 - \\ &\quad -r_\mu^2(d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2). \end{aligned}$$

for $1 \leq \mu \leq m$ with $r_{\mu s} = 2Gm_\mu/c^2$. By Theorem 3.1, we know that

$$ds^2 = {}_1ds^2 + {}_2ds^2 + \cdots + {}_mds^2,$$

where ${}_\mu ds^2 = ds_\mu^2$ by the projective principle on combinatorial fields. Notice that $1 \leq \widehat{m} \leq 4$. We therefore get the geometrical of $(\mathscr{C}_G|\overline{t})$ dependent on $\widehat{m}$ following.

For example, if $\widehat{m} = 1$, *i.e.*, $t_\mu = t$ *for* $1 \leq \mu \leq m$, then **the combinatorial metric *ds* is**

$$ds^2 = \sum_{\mu=1}^{m}\left(1 - \frac{2Gm_\mu}{c^2 r_\mu}\right)dt^2 - \sum_{\mu=1}^{m}(1 - \frac{2Gm_\mu}{c^2 r_\mu})^{-1}dr_\mu^2 - \sum_{\mu=1}^{m} r_\mu^2(d\theta_\mu^2 + \sin^2\theta_\mu d\phi_\mu^2).$$

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