What are the counterparts of Einstein's equations in TGD?

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Abstract

The original motivation of this work was related to Platonic solids. The playing with Einstein's equations and the attempts to interpret them physically forced the return to an old interpretational problem of TGD. TGD allows enormous vacuum degeneracy for Kähler action but the vacuum extremals are not gravitational vacua. Could this mean that TGD forces to modify Einstein's equations? Could space-time surfaces carrying energy and momentum in GRT framework be vacua in TGD context? Of course, also in GRT context cosmological constant means just this and an experimental fact, is that cosmological constant is non-vanishing albeit extremely small.

Trying to understand what is involved led to the realization that the hypothesis that preferred extremals correspond to the solutions of Einstein-Maxwell equations with cosmological constant is too restricted in the case of vacuum extremals and also in the case of standard cosmologies imbedded as vacuum extremals. What one must achieve is the vanishing of the divergence of energy momentum tensor of Kähler action expressing the local conservation of energy momentum currents. The most general analog of Einstein's equations and Equivalence Principle would be just this condition giving in GRT framework rise to the Einstein-Maxwell equations with cosmological constant. The vanishing or llight-likeness of Kähler current guarantees the vanishing of the divergence for the known extremals.

One can however wonder whether it could be possible to find some general ansätze allowing to satisfy this condition. This kind of ansätze can be indeed found and can be written as $kG + \sum \Lambda_i P_i = T$, where Λ_i are cosmological "constants" and P_i are mutually orthogonal projectors such that each projector contribution has a vanishing divergence. One can interpret the projector contribution in terms of topologically condensed matter, whose energy momentum tensor the projectors code in the representation $kG = -\sum \Lambda_i P_i + T$. Therefore Einstein's equations with cosmological constant are generalized. This generalization is not possible in General Relativity, where Einstein's equations follow from a variational principle. This kind of ansätze can be indeed found and involve the analogs of cosmological constant, which are however not genuine constants anymore. Therefore Einstein's equations with cosmological constant are generalized. This generalization is not possible in General Relativity, where Einstein's equations follow from a variational principle.

The suggested quaternionic preferred extremals and preferred extremals involving Hamilton-Jacobi structure could be identified as different families characterized by the little group of particles involved and assignable to time-like/light-like local direction. One should prove that this ansatz works also for all vacuum extremals. This progress - if it really is progress - provides a more refined view about how TGD Universe differs from the Universe according to General Relativity and leads also to a model for how the cosmic honeycomb structure with basic unit cells having size scale 10^8 ly could be modelled in TGD framework.

Contents

1 Introduction

The original motivation of this work was related to Platonic solids. The playing with Einstein's equations and the attempts to interpret them physically forced the return to and old interpretational problem of TGD. TGD allows enormous vacuum degeneracy for Kähler action but the vacuum extremals are not gravitational vacua. Could this mean that TGD forces to modify Einstein's equations? Could space-time surfaces which would carry energy and momentum in GRT frameword be vacua in TGD context?

Trying to understand what is involved led to the realization that the hypothesis that preferred extremals correspond to the solutions of Einstein-Maxwell equations with cosmological constant [K6, K2, K5] is too restricted in the case of vacuum extremals and also in the case of standard cosmologies imbedded as vacuum extremals. What one must achieve is the vanishing of the divergence of energy momentum tensor of Kähler action expressing the local conservation of energy momentum currents. The most general analog of Einstein's equations and Equivalence Principle would be just this condition giving in GRT framework rise to the Einstein-Maxwell equations with cosmological constant.

One can however wonder whether it could be possible to find some general ansätze allowing to satisfy this condition. This kind of ansätze can be indeed found and can be written as $kG+\sum \Lambda_i P_i = T$, where Λ_i are cosmological "constants" and P_i are mutually orthogonal projectors such that each projector contribution has a vanishing divergence. One can interpret the projector contribution in terms of topologically condensed matter, whose energy momentum tensor the projectors code in the representation $kG = -\sum \Lambda_i P_i + T$. Therefore Einstein's equations with cosmological constant are generalized. This generalization is not possible in General Relativity, where Einstein's equations follow from a variational principle. This kind of ansätze can be indeed found and involve the analogs of cosmological constant, which are however not genuine constants anymore. Therefore Einstein's equations with cosmological constant are generalized. This generalization is not possible in General Relativity, where Einstein's equations follow from a variational principl

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The original problem was however not this but the following one. One can decompose Euclidian icosahedron to 20 irregular tetrahedrons emanating from the center of the icosahedron. The ratio of the lengths of the surface edges (edges connecting points at the surface of the sphere) to the radial edges is in good approximation $k = 1 + \epsilon$, $\epsilon = .05$. If one makes the edges of the tetrahedrons equal by shortening of surface edges while preserving the lengths of the radial edges, one obviously obtains a gap [D1].

Is there any manner to make the tedrahedrons regular without creating a gap?

1. At sphere S^3 the counterpart of icosahedron consists of 20 counterparts of regular tetrahedrons. One can say that the generation of positive curvature eliminates the gap formed and shortens the surface edges. Deforming Euclidian space to hyperbolic space in turn adds volume. In 2-D case these rules can be visualized. In fact, one can extend the icosahedron to 600-cell denoted by (3,3,5) in Schönflies notation. What the recursive notation (3,3,5) states is that there are 5 tetrahedra (3,3) with common edge and tetrahedra in turn has 3 triangles with common vertex. The 4-D 600-cell has as boundary 600 tetrahedra assignable to a surface of 3-sphere in a manner completely analogous to the Platonic solids regarded as 2-D surfaces bounding 3-D cells.

2. One is however interested in possibly existing Euclidian variants of tetrahedral Penrose tiling and QC, and perhaps also in tetrahedral dense packing (still poorly understood), which must be however distinguished from tetrahedral QC which can quite well contain intersecting tetrahedrons. The problem is that it is difficult to imagine a unique Euclidianization by somehow mapping S^3 to E^3. Here different topologies pose the basic problem: one cannot avoid local deformations and the presence of singular 2-D surface. The best that one can hope of achieving are clusters of 600 tetrahedra.

The TGD inspired idea to be discussed is that sub-manifold gravity could help to achieve a unique map from S^3 to E^3 and also the counterparts the isosahedral Penrose tiling with icosahedrons consisting of regular tetrahedrons.

1. This would be achieved by a local deformation of the E^3 metric obtained by deforming canonically imbedded E^3 in CP_2 directions to make the 20 tetrahedra in the decomposition of the icosahedron regular and space-filling. This deformation would be just a piece of S^3, say the upper hemisphere. Entire S^3 would require two-sheeted 3-surface and is also possible in TGD. The Euler angles (α, θ, ϕ) for S^3 would correspond to the spherical coordinates of (r, θ, ϕ) via the formula: $\alpha = arcsin(r/R)$, where R is the radius of S^3.

2. Although the S^3 tetrahedra in the induced metric are regular, they do not look so in E^3 metric, and an interesting question is whether the irregularity of the tetrahedral structures seen in 3-space usually identified as E^3 could correspond to regularity in S^3. The correspondence between coordinates allows to predict precisely the E^3 coordinates of tetrahedral vertices in the decomposition of S^3 icosahedron so that the hypothesis is testable. The physics of water provides an especially interesting test bench for the idea.

3. It is also possible to imagine a construction crystal and quasicrystal (QC) like structures consisting of tetrahedrons by gluing together pieces of S^3 realized as static surfaces in $M^4 \times CP_2$ along their boundaries just as one glues together cubes along their faces to build cubic crystals. Note that this proposal for tetrahedral QC differs from the earlier proposal for twisted QC [**?**] in which each supercell (icosahedron for Penrose tiling, and icosahedron, dodecahedron, or icosidodecahedron for icosahedral QC) contains single tetrahedron twisted so that it does not intersect the tetrahedra of neighboring and possibly intersecting supercells.

Deformations of pieces of E^3 to pieces of S^3 is not the only possibility. Also deformations respecting the topology of E^3 are possible.

1. One can consider a more general deformation of the metric of E^3 to $ds^2 = k^2(r)^2 + r^2 d\Omega^2$. One obtains an infinite family of functions $k(r)$ satisfying the condition that an icosahedron with center at North pole $(r, \theta, \phi) = (0, 0, 0)$ of S^3 consists of regular tetrahedrons since the only condition is $\int_0^R k(r)dr = s$, where s is the length of the surface edge and integral defines the length of the radial edge. $k = 1 + \delta = constant$ option however fails as shown in Appendix C. In Appendix B it is shown that the 1+3 decomposition $kG + \sum_{i=1,2} \Lambda_i P_i$ allow to deduce differential equation for $k(r)$ but still one has a large number of solutions guaranteeing the regularity of the icosahedral tetrahedra.

2. If one accepts also deformations with more general functions $k(r)$ deduced in Appendix B, quite large number of quasi-lattice like structures consisting of regular tetrahedra becomes possible. One can ask whether these geometries could define possible tetrahedral quasi-lattice structures for water clusters having interpretation as some kind of geometric coding of information so that the apparent randomness would reflect hidden geometric order.

In GRT framework the deformation of E^3 to S^3 or a non-compact manifold for more general $k(r)$ and icosahedral Penrose tiling [A1] with icosahedrons consisting of 20 *regular* tetrahedrons are not plausible in condensed matter length scales for the simple reason that the gravitational deformation of the metric is so weak. If one accepts the cautious proposal for TGD variant of Equivalence Principle, it might be possible to realize these tetrahedral dreams. The projector contributions $-\Lambda_i Pi$ would represent average density of topologically condensed whereas T would be vanishing for vacuum extremals.

It becomes also possible to build a model for cosmic honeycombs and quasicrystal like structures consisting of units with size of order 10^8 ly and having galaxies at the boundaries of otherwise almost empty regions known as cosmic voids [E1]. The basic unit would be either a piece of hyperbolic space, of Euclidian space, or of 3-sphere. In hyperbolic case there is infinite number of tesselations. In GRT framework these pieces could carry a constant mass density (sub-critical, critical, or over-critical) but in TGD framework they would be vacua and galactic mass would be associated with their boundaries and idealizable as being due to the discontinuity of the normal component g_{nn} of the induced metric at the 3-D facets along which the super-cells are glued together. The time evolution of critical and super-critical options is unique part from the duration of the nonsingular period, and leads to TGD counterpart of blackhole having Euclidian induced metric. Note that cosmic honeycomb would provide a rather concrete realization for the notion of space-time foam usually assigned with Planck length scale.

It deserves to be mentioned that cosmic honeycombs and their possible counterparts for water clusters modeled as consisting of icosahedral pieces of S^3 bring in mind foams. Soap film foam is perhaps the most familiar example about foam. Plateau's laws govern the structure of many foams. Mean curvature is constant for each film and physically derives from area minimization assuming constant pressure difference over the film. 3 films meet at angle of 120 degrees along a line known as Plateau border and 4 Plateau borders meet at each vertex at tetrahedral angle of $arcos(-1/3) \simeq 109.47$ degrees (tetrahedral angle is defined as the angle between radii drawn from the center of tetrahedron to its vertices). This suggests spherical tetrahedron as a basic building brick in a model as a honeycomb built from pieces of S^3. Plateau's laws can be derived mathematically for foams, for which films are minimal surfaces (pressure difference vanishes).

2 Does the $SO(3)$ symmetry preserving deformation of the metric of E^3 regularize the icosahedral tetrahedra?

In GRT framework quantum gravitational effects are extremely small in everyday length scales - say in condensed matter physics. In TGD the situation can be different due to sub-manifold gravity predicting new physical effects. Sub-manifold gravity and the notion of many-sheeted space-time indeed challenges the flatness of 3-space as approximation broken only by very weak gravitational effects. The field equations demand that Kähler energy momentum tensor has vanishing divergence. This can be guaranteed if Einstein-Maxwell equations with cosmological term are satisfied: both are in principle predictions of the theory and depend on the preferred extremal [K6].

This is however not the most general option if one is ready to accept TGD and allow the decomposition of the cosmological constant term to a sum of terms proportional to projectors to orthogonal subspaces multiplied with cosmological "constants", which are not constant anymore. The interpretation of the sum of these terms in $kG = -\sum_i \Lambda_i P_i + T$ is in terms of topologically condensed matter representing topological inhomogenuities smaller than the length scale resolution used (for a more detailed discussion see Appendix A). Therefore it might be also physically possible to modify the 3-metric without giving up $SO(3)$ symmetry and the topology of E^3, and there would be no need to map S^3 to E^3 in the hope of obtaining what might be called tetrahedral Penrose tiling. In fact, also a deformation of a piece of E^3 to that of S^3 exists and defines this kind of map uniquely.

An obvious deformation of Euclidian metric is obtained by the scaling of the radial component of the E^3 metric

$$ds^2 = dr^2 + r^2 d\Omega^2 \rightarrow k^2(r)dr^2 + r^2 d\Omega^2 \ \ , \ \ k = 1 + \delta \ . \tag{2.1}$$

$k(r)$ could be fixed by the condition that the radial edges of the 20 icosahedral tetrahedra have the same length as the surface edges defined as geodesic lines in the deformed metric. This poses a condition on $k(r)$ but it is not at all obvious whether any solutions to the condition exist.

1. $SO(3)$ symmetry alone allows $k(r)$ to be an arbitrary function of the radial coordinate. The original guess was $k = 1+\delta(r)$ with $\delta(r) \geq 0$ non-vanishing only near the center of the icosahedron so that in the region containing surface edges the metric would be strictly Euclidian. The deviation from E^3 metric near the center of the icosahedron could be due to the presence of a particle. $\delta = constant$ option is excluded as shown in Appendix C but in Appendix B good arguments supporting $\delta(r) \rightarrow 0$ option are developed. The regularity of the tetrahedral decomposition of the icosahedron with center at $(r, \theta, \phi) = (0, 0, 0)$ follows from the assumption that radial and surface edges have same length:

$$\int_0^R k(r)dr = s \ \ , \tag{2.2}$$

   where s is the length of the surface edge of the icosahedron identified as geodesic line.

2. The local deformation of E^3 metric to S^3 metric obtained by allowing CP_2 coordinates to depend on r having $k(r) = 1/(1-(r/R)^2)$ certainly gives rise to a decomposition of S^3 icosahedron to 20 regular S^3 tetrahedrons realized as as a piece of S^3 - say around North pole so that one has radial and surface edges of equal length in S^3 metric but not so in E^3 metric. Entire S^3 requires two-sheeted surface and is also possible to realize in TGD context. This would allow to realize the 600-cell consisting of 600 tetrahedrons.

These conjecture might be testable. A successful test would also provide support for sub-manifold gravity.

1. The correspondence $(\alpha, \theta, \phi) = (arcsin(r/R), \theta, \phi)$ between Euler angles of S^3 and spherical coordinates of E^3 allows a precise identification of E^3 coordinates of tetrahedra and thus precisely quantifies the deviation of regular S^3 tetrahedra from regular E^3 tetrahedra.

2. In E^3 metric the tetrahedrons do not look regular, and an interesting possibility is that the icosahedral structures encountered in water clusters could be interpreted in terms of regular S^3 tetrahedrons, when the 3-space is not identified as E^3 but consists of icosahedral pieces of S^3 glued together along S^3 faces.

3. For a more general family of functions $k(r)$ similar conditions hold true and would allow quite a large number of quasi-lattice like structures consisting of icosahedra decomposing to 20 regular tetrahedrons. For a given $k(r)$ satisfying the differential equation deduced in Appendix A and the regularity condition $s = \int kdr$, one can predict the precise positions for the vertices of icosahedral tetrahedrons in the spherical coordinates for E^3. The functions $k(r)$ could make possible to code information to the deformations of E^3 tetrahedrons from regularity. Again water provides the test bench.

Single isosahedron consisting of regular tetrahedrons or even 600-cell or half of it is not enough. One would like to have also Penrose tiling having icosahedrons as supercells. Since the proposed deformation of the metric is not translationally invariant but has the center of the icosahedron as a center of symmetry, the only possibility is to glue together their E^3 (realized as $t = constant$ hyper-plane of M^4) translates of the basic icosahedron along common faces. One must allow also intersections of icosahedrons and therefore also of tetrahedrons as already for the triangles of the ordinary Penrose tiling obtained by replacing 20 tetrahedrons with 10 triangles and icosahedron with 10-gon. Clearly, tetrahedral Penrose tiling must be distinguished from tetrahedral dense packing. Physically this means that tetrahedral supercells can have common atoms.

The following arguments try to demonstrate that in TGD framework there are good hopes for tetrahedral Penrose tiling having by definition regular S^3 tetrahedrons as possibly overlapping super-cells and induced from icosahedral Penrose tiling. Whether the icosahedral QC containing icosahedra, dodecahedra, and icosidodecahedra as super-cells allows a decomposition to regular tetrahedra for all 3 super-cells, is probably easy to answer. The argument is of course purely mathematical, and the question whether the construction is also physically realizable remains open.

2.1 Strong gravitation is possible in TGD framework

The basic motivation for the speculations to follow is that many-sheeted space-time makes possible large deviations from gravitation predicted by GRT, which in TGD framework can be seen as a description of gravitation at the long length scale limit. A fundamental distinction between GRT and TGD is indeed that in TGD framework gravitational constant and cosmological constant - actually space-time dependent cosmological "constants" emerge as predictions of the theory rather than as fundamental constants of Nature.

For almost two decades ago I deduced by purely dimensional considerations a formula for gravitational constant G in terms of p-adic length scale and exponent of Kähler action for CP_2 type vacuum extremal defining the line of generalized Feynman diagram representing graviton [K3]. The prediction was that G should have an entire spectrum of values and approach p-adic length scale squared $L_p^2 = pR_{CP_2}^2$ when the action of the deformed CP_2 type vacuum extremal becomes small: this happens at short length scale limit. In particular, hadronic strings would correspond to strong gravitation limit, and TGD predicts fractally scaled up variants of ordinary hadron physics so that a rich spectrum of strong gravities follows as a prediction. This means that in TGD Universe the the gravitational effects on space-time geometry can be rather dramatic even in condensed matter length scales whereas in GRT the effects are extremely small. With this background philosophy I have discussed the possible differences between General Relativity and TGD-based view about gravitation in [K7]. This chapter should help also to understand the discussion of this section.

The starting point for the following considerations was the question whether the flat geometry for a piece of E^3 could be modified by gravitational effects so that it becomes a piece of S^3 allowing the decomposition of icosahedron to 20 regular tetrahedra (in E^3 geometry the tetrahedra cannot be regular). This kind of decomposition is actually possible for much more general deformations of E^3 geometry and one ends up with the vision about quasi-lattice like structures having piece of S^3 or hyperbolic space H^3 as a basic building brick. This notion makes sense in condensed matter length scales only if gravitational constant can be of order $G \sim L_p^2$ since Schwartschild radius $r_S = 2GM$ is the natural scale for the radius of S^3.

The cosmic honeycomb having voids with size of order 10^8 ly as basic building bricks is one possible quasi-lattice like structure suggested by these considerations. In condensed matter length scales strong gravitation could allow similar quasi-lattice like structures and icosahedral water clusters having tetrahedrons as building bricks could be examples of structures of this kind.

Cosmic honeycombs and their possible counterparts for water clusters modeled as consisting of icosahedral pieces of S^3 bring in mind foams. Soap film foam is perhaps the most familiar example about foam. Plateau's laws govern the structure of many foams. Mean curvature is constant for each film and physically derives from area minimization assuming constant pressure difference over the film. 3 films meet at angle of 120 degrees along a line known as Plateau border and 4 Plateau borders meet at each vertex at tetrahedral angle of $arcos(-1/3) \simeq 109.47$ degrees (tetrahedral angle is defined as the angle between radii drawn from the center of tetrahedron to its vertices). This suggests spherical tetrahedron as a basic building brick in a model as a honeycomb built from pieces of S^3. Plateau's laws can be derived mathematically for foams, for which films are minimal surfaces (pressure difference vanishes).

2.2 Construction of deformed metrics in zero energy ontology

In TGD framework the view about 3-space generalizes considerably. One assigns to physical systems space-time sheets and space-time sheets can deviate metrically from flat space much more than in general relativity. The can have even Euclidian signature of metric: this signature is assigned to the 4-D lines of generalized Feynman diagrams. which can have macroscopis scale and can be tentatively identified as the space-time regions characterizing physical objects as we "see" them.

Vacuum extremals [K2] provide some illustrative examples. Consider a geodesic circle of CP_2 with angle coordinate Φ and the 4-surface $\Phi = \omega t$, where t is M^4 time coordinate. The induced metric is $(1 - R^2\omega^2, -1, -1, -1)$ and obviously flat. There is no gravitational field present but there is anomalous time dilation, which can be detected if it is possible to use M^4 coordinates of the M^4 factor of $M^4 \times CP_2$ as preferred coordinates - as the fact that Poincare symmetries are associated with imbedding space rather than space-time surface suggests.

Causal diamonds (CDs) define an essential element of zero energy ontology (ZEO). CD is a double pyramid (with spherical cross section) defined by the intersection of future and past directed light-cones of M^4 and has two light-like boundaries. $CD \times CP_2$ defines what might be called a spot light of consciousness in TGD inspired theory of consciousness. Zero energy states correspond to pairs of positive and negative energy states having opposite total quantum numbers, and assignable to the opposite light-like boundaries $\delta M^4_\pm \times CP_2$. Zero energy states correspond to quantum superpositions of pairs of 3-surfaces at the boundaries of $\delta_\pm CD \times CP_2$: by holography they can be also regarded as superpositions of preferred extremals. Strong form of holography allows to express physical states in terms of information associated with partonic 2-surfaces and their 4-D tangent space data.

2.2.1 Deformation of E^3 metric to guarantee the regularity of icosahedral tetrahedrons

Usually one thinks that in every-day length scales 3-space is flat E^3 apart from very small gravitational effects. In cosmological scales 3-space is known to be flat in good approximation on basic of CMB data. In short scales mass densities can be however much higher than in cosmic scales (one proton per cubic meter roughly) so that a local compactification to overcritical cosmology consisting of a piece of S^3 could take place. The compactification could be interpreted as being due to the presence of topologically condensed matter. These pieces could in turn be glued together along their boundaries to obtain lattices and quasi-lattice like structures. As shown in Appendix A, this kind of local S^3 compactifications can be both static and expanding. In the latter case, the cosmic time evolution is essentially unique and leads to a singularity, for which the induced metric has Euclidian signature and has interpretation as a TGD counterpart of a blackhole.

Remark: The discussion of Appendix A shows that also static deformations of E^3 to hyperbolic space H^3 are possible, and in this case one would obtain an infinite number of tesselations defined by discrete subgroups of $SO(1,3)$ including 8 honeycomb structures. Icosahedral quasi-lattice structures encountered in the physics of water provide again a test bench. Now the correspondence $(\eta, \theta, \phi) = (arsinh(r/R), \theta, \phi)$ between hyperbolic "Euler angles" and spherical coordinates of E^3 would allow to deduce how the H^3 icosahedra (say) and the honeycombs made of them differ from their E^3 counterparts.

Before continuing, some background in TGD is needed. In TGD framework macroscopic objects correspond to 4-surfaces with effective boundaries defined by light-like 3-surfaces at which the signature of the induced metric changes from Minkowskian to Euclidian. The above observations suggest the possibility of large physical effects on metric in the absence of effects on curvature. But even large effects on curvature cannot be excluded.

One can indeed consider more general metrics for which $g_{rr} = k^2(r)$ holds true. By previous argument, these metrics are just what one wants if one is interested in icosahedral Penrose tiling with icosahedrons decomposed to regular tetrahedra. Note that the metrics in question are highly analogous to those for Robertson-Walker cosmologies with over-critical mass density: the only difference is that time coordinate is replaced with radial coordinate r and S^3 with S^2.

For the case $k = k(r)$ case the expressions of Ricci and Einstein tensor are given by the expressions

$$
\begin{aligned}
&(R_{rr}, R_{\theta\theta}, R_{\phi\phi}) = (X, Y, sin^2(\theta)Y), \ \ , \\
&(G_{rr}, G_{\theta\theta}, G_{\phi\phi}) = (X - \tfrac{k^2}{r^2}Y, -\tfrac{1}{2}\tfrac{r^2}{k^2}X, -\tfrac{1}{2}\tfrac{r^2 sin^2(\theta)}{k^2}X) \ \ . \\
&X = -4\tfrac{dlog(k)}{dlog(r)}X \ \ , \ \ Y = k^{-2} - 1 - \tfrac{1}{2}\tfrac{dk^{-2}}{dlog(r))} \ \ .
\end{aligned}
\tag{2.3}
$$

For $k = constant$ the Ricci tensor and Einstein tensor have components

$$
\begin{aligned}
&(R_{rr}, R_{\theta\theta}, R_{\phi\phi}) = (0, r^2, r^2 sin^2(\theta)) \times \tfrac{(k^{-2}-1)}{r^2} \ \ , \\
&(G_{rr}, G_{\theta\theta}, G_{\phi\phi}) = ((1-k^2)/r^2, 0, 0) \ \ .
\end{aligned}
\tag{2.4}
$$

The vanishing of $G_{\theta\theta}$ and $G_{\phi\phi}$ can be understood in terms of symmetries. For 4-D solution obtained by adding M^4 time coordinate as fourth coordinate this would mean that energy density T^{tt} is non-vanishing and proportional to $(k^{-2}-1))/r^2$.

2.2.2 The modification of Einstein's equations suggested by TGD

The deformation of E^3 can be generalized in trivial manner to 4-D situation giving $G_{tt} = k^{-2}G_{rr}$ is in general not consistent with the assumption that preferred extremals (4-surfaces) at the limit of vacuum satisfy Einstein-Maxwell equations with cosmological term satisfying $T = \kappa G + \Lambda g$, where$T$ is the energy momentum tensor associated with Kähler action satisfying $T^{\alpha}_{\alpha} = 0$ since Maxwell action is invariant under conformal scalings (in TGD this symmetry is actually broken since Maxwell field is not primary field).

This form of the conditions is however un-necessarily strong. The only condition on preferred extremals is that T has vanishing divergence:

$$
D_{\beta}T^{\alpha\beta} = 0 \ \ . \tag{2.5}
$$

One can however go further and ask whether there might be general ansätze allowing to satisfy this condition.

The first thing to observe that the condition reduces to

$$
j^{\alpha}J_{\alpha\beta} \ = \ 0 \ \ . \tag{2.6}
$$

One can of course ask whether this condition satisfied also by the extremals of the ordinary Maxwell action is the most general condition that one can deduce from the local conservation of energy momentum. The condition states that the Lorentz force on and the work done by Kähler current in the induced Kähler field vanish. For Maxwell's equations the condition $j^{\alpha} = 0$ guarantee the condition. In TGD framework light-likeness of Kähler current holding for massless extremals is a more general manner to satisfy the condition. An open question is whether more general solutions to the condition exist.

One however ask whether some condition on metric could be assigned with the condition on Kähler current. The vanishing of the divergence can be indeed satisfied by assuming only

$$
\begin{aligned}
T^{\alpha\beta} &= \kappa G^{\alpha\beta} + \sum_i \Lambda_i P_i^{\alpha\beta} \ \ , \\
\kappa &= \frac{1}{8\pi G} \ \ .
\end{aligned}
\tag{2.7}
$$

Here P_i are projectors to mutually orthogonal sub-spaces of the tangent space of the space-time surface. The distributions of the sub-spaces must be integrable to slicings of the space-time surface by sub-manifolds and define d_i-dimensional sub-manifolds of the space-time surface ($\sum_i d_i = d$). Energy momentum tensor of Kähler action has vanishing divergence if one has

$$D_\beta(\Lambda_i P_i^{\alpha\beta}) = 0 \ . \tag{2.8}$$

Λ_i need not be constant functions anymore so that cosmological constant is replaced by two - presumably slowly varying - cosmological "constants". For $\Lambda_i = \Lambda$ one obtains just the ordinary cosmological constant, which must be a genuine constant.

The condition $T^\alpha_\alpha = 0$ gives

$$\kappa R = \sum_i \Lambda_i d_i \ . \tag{2.9}$$

Here R denotes curvature scalar and d_i denotes the dimension of sub-space to which P_i projects. For $\Lambda_i = \Lambda$ one obtains $\kappa R = \Lambda$ so that curvature scalar would be constant, which looks too strong a condition.

In TGD framework the most natural identification of the term $\sum \Lambda_i P_i$ would be in terms of topologically condensed matter consisting of topological in-homogenities smaller than the scales of length and time resolution. In zero energy ontology the interpretation would be in terms of zero energy states in scales below the measurement resolution and interpreted as quantum fluctuations in QFT context.

In Appendix A the situation is considered in more detail.

1. It is shown that for Robertson-Walker cosmologies 1+3 decomposition is natural.

2. For preferred extremals with what I have called Hamilton-Jacobi structure [K6] in Minkowskian regions one would have 2+2 decomposition. Maybe the 1+3 decomposition corresponds to the quaternionic solution ansatz. The motivation for this belief is that $SO(3)$ has a natural action on quaternions as their automorphism group. For Hamilton-Jacobi structure $SO(3)$ is replaced with the little group of Poincare group assignable to massless particles so that these two kinds of extremals might basically correspond to massive and massless representations of Poincare group.

3. For vacuum extremals the inverse images of points of at most 2-D CP_2 projection define 2+2 slicing of the space-time surface, and one can hope that this slicing could give rise to the decomposition $\sum_{i=1,2} \Lambda_i P_i$. It is of course quite possible that not all vacuum extremals can be regarded as limits of preferred extremals with this decomposition.

In the recent case one is interested on having such $k(r)$ that one has vacuum extremal with 1+1+2 decomposition satisfying

$$\begin{aligned} \kappa G^{\alpha\beta} + \sum_{i=1,2,3} \Lambda_i P_i^{\alpha\beta} &= 0 \ , \\ D_\beta \Lambda_i P_i^{\alpha\beta} &= 0 \ . \end{aligned} \tag{2.10}$$

$1+1+2$ decomposition corresponds to coordinate lines for t and r and to spheres (θ, ϕ). In Appendix B the differential equations for $k(r)$ guaranteeing that $\Lambda_i P^i$ have vanishing divergence are deduced.

These equations can be integrated and give rise to a family of metrics characterized by $k(r)$. Near the center of icosahedron the deformation corresponds to S^3 in a good approximation. S^3 and H^3 are obtained as particular solutions. For S^3 the decomposition of icosahedron to regular tetrahedrons is possible as already shown. The important and definitely new point of view is that these metrics are vacuum extremals for both Kähler action and in gravitational sense. Einstein's equations are replaced by a more general and weaker condition that the energy momentum tensor of Kähler action has a vanishing divergence.

2.3 Various realizations of E^3 and its deformations in TGD framework

TGD allows several realizations of E^3 and its deformations as a surface in $M^4 \times CP_2$.

2.3.1 Deformation of the imbedding of E^3 to $\delta M^4_\pm \times CP_2$

Flat 3-metric is obviously something very fundamental, and one can ask whether one could realize flat 3-surfaces as surfaces in $\delta_\pm CD \times CP_2$. The metric at $\delta_\pm CD$ is metrically 2-D since the radial direction is light-like, and one can write the metric as

$$ds^2 = -r^2 d\Omega^2 \ .$$

For a given value of the radial coordinate r the metric of 2-sphere of radius r is in question.

One can have 3-surfaces in $\delta_\pm CD \times CP_2$ with non-degenerate 3-metric by assuming that CP_2 coordinates depend on the coordinates of $\delta_\pm CD$ - that is light-cone boundary. If CP_2 coordinates depend on r only, the induced metric is still rotationally symmetric, and the induced metric reads as

$$ds^2 = g_{rr}dr^2 - r^2 d\Omega^2 \ \ , \ \ g_{rr} = s_{kl}\frac{ds^k}{dr}\frac{ds^l}{dr} \ .$$

For $g_{rr} = -1$ one obtains flat metric of E^3. This condition has a huge number of solutions since three CP_2 coordinates can be chosen to be almost arbitrary functions of r and the fourth one can be solved from the condition for the metric. The restrictions come only from the condition $g_{rr} = -1$ combined with the Euclidian character of CP_2 metric. The simplest solution is obtained by taking the CP_2 projection to be a geodesic circle so that one obtains $r = \Phi/R$ indeed giving $g_{rr} = -1$.

One can obtain also a deformation allowing decomposition of the icosahedron to regular tetrahedrons by modifying the basic condition to $g_{rr} = -k^2$, $k = 1+\delta$. It is however not obvious whether this deformation can be continued to a 4-surface with vanishing Einstein tensor. A natural continuation of the 3-surface at light-cone boundary to a 4-D space-time surface is obtained by using a slicing of the future light-cone by parallel light-cones along time-like line and having identical 3-metric. Since the coordinate r corresponds to a light-like coordinate v in the pair of $(u = t-r, v = t+r)$ of the standard light-like coordinates of M^4, the line element would be

$$ds^2 = 2dudv - k^2 dv^2 - r^2 d\Omega^2 \ ,$$

and Einstein tensor vanishes for a constant value of the parameter k also now.

2.3.2 The deformations of standard imbeddings of E^3 and M^4

One can also consider the deformation of the standard imbedding of E^3 as $t = constant$ hyper-surface of M^4) obtained by decomposing E^3 to icosahedra consisting of 20 irregular tetrahedra. Also now one can deform the imbedding near he center of each icosahedron so that the radial edges have same length as the surface edges. One starts from the metric of E^3

$$ds^2 = -dr^2 - r^2 d\Omega^2 \ ,$$

and deforms it to

$$ds^2 \ = \ -k^2(r)dr^2 - r^2 d\Omega^2 \ . \tag{2.11}$$

For small deformations one has $k = +\epsilon(r)$. The deformation is of the same form as before and satisfies the conditions:

$$s^k = s^k(r) \ \ , \ \ s_{kl}\frac{ds^k}{dr}\frac{ds^l}{dr} = 1 - k^2(r) = -2\delta(r) - \delta^2(r) \ \ , \int \delta(r)dr = \epsilon \simeq .05 \ . \tag{2.12}$$

In this case the icosahedral Penrose tiling is essentially identical with the standard one except for the modification of the metric near the center of the icosahedron. Besides this $k$must satisfy also the differential equation deduced in Appendix B.

This metric can be generalized to a deformation of M^4 metric by adding the time coordinate so that one has

$$ds^2 \ = \ dt^2 - k^2(r)dr^2 - r^2 d\Omega^2 \ . \tag{2.13}$$

2.3.3 Deformation of E^3 to hyperbolic space H^3

One can also consider a deformation of M^4 replacing E^3 hyperbolic 3-space H^3.

1. In this case the imbedding, induced metric, and conditions on it are given by the following equations:

$$\begin{array}{lll} m^0 = \Lambda t + h(u) \ , & \Phi = f(u) + \omega t \ , & u = \frac{r}{R} \ , \\ g_{tt} = \Lambda^2 - R^2\omega^2 = 1 \ , & g_{rr} = -1 - (\frac{df}{du})^2 + (\frac{dh}{du})^2 = -\frac{1}{1+r^2} \ , & g_{tr} = -R\omega_1\frac{df}{du} + \Lambda\frac{dh}{du} = 0 \ . \end{array} \tag{2.14}$$

2. The solution to the conditions reads as

$$\Lambda^2 = 1 - R^2\omega^2 \ , \quad h = \frac{R\omega_1}{\Lambda}f(u) \ , \quad f(u) = f_0 + \frac{1}{2}(1 - (\omega_1 R/\Lambda)^2)^{-1/2}log(\frac{1+r^2}{1+r_0^2}) \ . \tag{2.15}$$

   The deformation is well-defined everywhere. There exists actually infinite number of deformations of this kind since the geodesic circle can be replaced with an almost arbitrary curve of CP_2.

2.3.4 What about corrections to g_{tt} component of the stationary metric?

What about g_{tt} component of the metric? Should it be given in the Newtonian approximation $g_{tt} = 1 - 2\phi_{gr}$. This looks reasonable. The correction has several consequences. The 1+1+2 decomposition would be replaced with $2+2$ decomposition, the g_{tt} metric would be slightly modified, and he imbedding to CP_2 must be replaced with 2-D vacuum extremal. Homologically trivial geodesic sphere $S^2 \subset CP_2$ would provide the simplest imbedding as a vacuum extremal, and the imbedding is of the same form as that for H^3 metric already discussed

$$m^0 = \Lambda t + h(u) \ , \quad \Theta = \Theta(u) \ , \quad \Phi = f(u) + \omega t \ , \quad u = \frac{r}{R} \ , \tag{2.16}$$

giving

$$\begin{array}{ll} g_{tt} = \Lambda^2 - R^2\omega^2 sin^2(\Theta)\omega^2 = 1 + 2\Phi_{gr} \ , & g_{rr} = -1 - sin^2(\Theta)(\frac{df}{du})^2 + (\frac{dh}{du})^2 - (\frac{d\Theta}{du})^2 = -\frac{1}{1-(r/R)^2} \ , \\ g_{tr} = -R\Lambda\omega sin(\Theta)\frac{df}{du} + \Lambda\frac{dh}{du} = 0 \ , & \Phi_{gr} = K(u^2 - u_0^2) < 0 \ , \\ K = nGR^2\rho & . \end{array} \tag{2.17}$$

n is a numerical constant. Note that the sign of Φ_{gr} should be negative as a sum of negative Coulomb contributions and the additive constant $u_0^2 > 1$ not affecting Newton's equations guarantees this for S^3. The deviation of g_{tt} from flat metric is however very small in general.

The condition on g_{tt} gives

$$sin^2(\Theta) = \frac{\Lambda^2 - 1 + 2K(u^2 - u_0^2)}{R^2\omega^2}$$

so that the imbedding fails at certain radii r corresponding to $sin^2(\Theta) = 0$ and $sin^2(\Theta) = 1$. $u = 1$ could correspond to $r = R$ and to $sin^2(\Theta) = 1$ and $u = 0$ to $sin^2(\Theta) = 1$.

In the hyperbolic case the sign of T^{tt} is negative which suggests that the sign of K in Φ_{gr} is also opposite so that gravitational attraction would transform to repulsion. This can make sense only if the topologically condensed matter corresponds to the negative energy part of zero energy state with non-standard arrow of geometric time. Also now the imbedding fails above some critical value of u^2.

What is remarkable that the deviation of the 3- metric from that of E^3 can be large although the effect on geodesic lines with small average radius r is very small since the non-constancy of g_{rr} becomes visible only when the radial velocity dr/dt is non-vanishing. For instance, the deviation of g_{rr}

from its expression in flat or Schwartschild metric does not effect at all circular geodesic lines which are determined by g_{tt}. Hence one can imagine the possibility of having large gravitational effects on 3-metric not visible as large gravitational binding energies.

The cold shower is encountered when one assumes Einstein's equation for the mass density responsible for

$$\rho = T_t^t \simeq \frac{R_3}{16\pi G} = \frac{3}{8\pi G R^2} = \frac{3M}{2\pi R^3} . \tag{2.18}$$

This gives $R = 4GM = 2r_S$, r_S the Schwartshild radius, which is extremely small distance unless the density of the topologically condensed matter is huge.

In TGD framework one way out of the problem could be based on the fact that both G and Λ_i are predictions rather than fundamental constants in TGD framework. If one accepts this, one an consider the possibility that $G \sim R/M$ holds true. In fact, I have proposed for a long time ago [K5, K3] a formula for G in terms of p-adic length scale L_p and exponential of Kähler action for CP_2 type vacuum extremals as

$$G = \frac{L_p^2}{\hbar} exp(-S_K(CP_2)) = \frac{1}{\hbar} p R_{CP_2}^2 exp(-S_K(CP_2)) . \tag{2.19}$$

Here R_{CP_2} denotes the length of CP_2 geodesic circle. For $p = M_{127} = 2^{127} - 1$ - the Mersenne prime characterizing electron and the largest Mersenne prime, which does not correspond to a completely super-astrophysical length scale - one obtains the Newtonian value of G [K5, K3].

The Kähler action assignable to the deformation of CP_2 type vacuum extremal corresponds to the generalized line of Feynman diagram assignable to graviton. S_K can vary due to the finite length of the line so that a full vacuum extremal is not in question as well as due to the deformation of the extremal. The exponential dependence on S_K can give rise to a huge variation range of G, and in the extreme situation one G and be near to the the upper bound $G_{max} = L_p^2/\hbar$. $G = G_{max}/2n$ $M \simeq n\hbar/L_p$ would give $R = L_p = \hbar/M$. This would make sense in hadronic, nuclear and condensed matter length scales. I have indeed proposed that hadronic string tension has interpretation in terms of $G \sim L_p^2$: the interpretation would be in terms of strong gravitation made possible by spin 2 meson exchanges. I have also proposed that fractally scaled up variants of QCD like theory of strong interactions appear in biological length scales [K1] so that also strong gravity would appear in these length scales.

The above approximate construction has certain ad hoc character. The dependence of g_{tt} on r can be however constrained by requiring 1+1+2 of Einstein tensor such form $G_i^{\alpha\beta}$ having the form $\Lambda_i g_i^{\alpha\beta}$, $= 1,2$ and has a vanishing divergence. As a result one obtains a differential equation for the gravitational potential Φ_{gr}.

The differential equation for $G^{tt} = \Lambda_1 g^{tt}$ is trivially satisfied due to time translational invariance. The integration of the differential equation for G^{rr} gives at the first step

$$\begin{aligned} G^{rr} &= \frac{C}{g_{grr} g_{tt}} \equiv -\frac{C}{AB} , \\ g_{tt} &= B , \quad g_{rr} = -A . \end{aligned} \tag{2.20}$$

The expressions for G^{rr} and other components of Einstein tensor [K5] are

$$\begin{aligned} G^{rr} &= \frac{1}{A^2}(-\frac{\partial_r B}{Br} + \frac{(A-1)}{r^2}) , \\ G^{\theta\theta} &= \frac{1}{r^2}[-\frac{\partial_r^2 B}{2BA} + \frac{1}{2Ar}(\frac{\partial_r A}{A} - \frac{\partial_r B}{B}) \\ &+ \frac{\partial_r B}{4AB}(\frac{\partial_r A}{A} + \frac{\partial_r B}{B}] , \\ G^{tt} &= \frac{1}{AB}(-\frac{\partial_r A}{Ar} + \frac{(1-A)}{r^2}) . \end{aligned} \tag{2.21}$$

The above equation for G^{rr} gives the differential equation

$$2\frac{dB}{du} + \frac{CA}{B} = \frac{(A-1)}{u} \ , \ u = r^2 \ . \tag{2.22}$$

If gravitational potential $2\Phi_{gr} = B - 1$ is given one can solve $A - 1$ from this equation purely algebraically and obtains spherically symmetric metric.

For the S^3 ($k = 1$) and H^3 ($k = -1$) metric one has

$$A - 1 = \frac{1}{1-kv} - 1 = \frac{kv}{1-kv} \ , \ v = \frac{u}{R^2} \ .$$

One obtains

$$\frac{dB}{dv} + \frac{CR^2}{2B}(1+kv) = \frac{k}{1-kv} \ . \tag{2.23}$$

Near origin$kvu \simeq 0$ holds true in the first approximation. This gives the approximate expression

$$B = 1 + B_1 log(\frac{B+B_1}{1+B_1}) + \frac{r^2}{R^2} \ , \ B_1 = \frac{CR^2k}{2} \ . \tag{2.24}$$

Apart from the slowly varying logarithmic term $g_{tt} = B$ is of the same form as Newtonian approximation would predict. Note that the result is same for both $k = 1$ and $k = -1$. Near $r = R$ the right-hand side diverges for $k = 1$ and implies logarithmic behavior

$$B \simeq B_0 - \frac{k}{2} log(\frac{1-kv}{1-kv_0}) \ . \tag{2.25}$$

For S^3 the metric therefore develops Euclidian signature. For $k = -1$ the g_{tt} becomes very large for large values of r so that the imbeddability to $M^4 \times CP_2$ eventually fails. The results suggests that for both S^3 and H^3 metric expansion is necessary.

2.3.5 Constant deformation does cannot give a decomposition of icosahedron to 20 regular tetrahedrons

At the first glimpse the most attractive option from TGD point of view is $k = 1 + \delta$, where δ is constant in the entire region but one must demonstrate that for a suitable choice of the value of δ the lengths of radial and surface geodesics are identical. As shown in Appendix C, it is possible to express the condition fixing the angular distance $\Delta\phi$ between neighboring vertices of icosahedron and the equality $s = k$ of the radial and surface edge lengths in terms of elementary functions. It however turns out that the only solution to the conditions is $k = \infty$. The interpretation is following. As k giving the length of the radial edge increases, also the length of the surface edge increases and does it so fast that infinite value of k is required to obtain equality.

One however obtains an infinite faily of deformations for which $k(r)$ is not constant. In TGD framework k satisfies the differential equations deduced in Appendix B. As a special case one obtains deformations transforming piece of E^3 to a piece of S^3.

2.4 Cosmic honeycombs?

The deformed metric still possesses the crucial $SO(3)$ symmetry. Translational symmetry is lost but the loss can be located near the center of inside each icosahedron. Icosahedral Penrose tiling must be constructed by gluing together disjoint 3-surfaces along the faces of the icosahedra. Also intersections of icosahedra and therefore also tetrahedrons must be allowed. Since the correction to the radial distances comes from the region near the center of the icosahedron, the construction of icosahedral Penrose tiling for E^3 option proceeds just as it does usually.

For light-cone boundary option the basic icosahedron is replaced by the 2-D outer boundary of solid icosahedron expanding with light velocity in M^4. In 2-D case a simple analogy would be the replacement of triangle with the boundary of expanding triangle: on just draws the center point of triangle so that one obtains cone with triangular cross section. The expanding icosahedra start expansion simultaneously in $t = 0$ hyperplane. At $t = t_0$ some of them meet along common face and some of them start to intersect already earlier. After this the resulting state would be a fusion of the icosahedra to 3-D Penrose tiling which do not expand anymore.

A possible cosmological application would be an explanation for the honeycomb structure in scales of 10^8 light years. Galaxies seem to be concentrated on approximately spherical surfaces of about this radius known as cosmic voids [E1]. One can imagine three options depending on whether one assigns to the local cosmology sub-critical, critical, or over-critical mass density in these scales (this is of course the GRT based interpretation). If one accepts the TGD based interpretation based on the modification of Einstein equations discussed in Appendix A, the cosmic voids would be genuine vacua in good approximation.

TGD allows also the option for which these voids carry Kähler energy having interpretation as dark energy. I have indeed proposed that the magnetic energy of Kähler magnetic flux tubes could be identified as dark energy. What makes the situation difficult is that both vacuum and non-vacuum options can give rise to accelerated cosmic expansion so that for vacuum option no dark energy would be needed. Note also that for critical and sub-critical vacuum option non-trivial long range gauge fields - in particular electromagnetic fields - are present in the vacuum. This is somewhat frustrating: I had already thought that the issue of dark energy is finally resolved in TGD framework!

All these honeycomb like structures could be realized in TGD by gluing together pieces of H^3, E^3 or S^3.

1. For TGD inspired interpretation these cosmologies are vacua and therefore also the voids identifiable as pieces of these cosmologies. The discontinuities of the normal component of metric g_{nn} at 3-D gluing regions would give rise to surface mass densities providing idealization for the mass carried by galaxies (Einstein tensor involves derivatives of metric).

2. For the standard interpretation these cosmologies carry a mass density perhaps identifiable as dark energy density besides the mass densities related to the discontinuities of g_{nn} at the facets at which gluing takes place.

3. The earlier TGD based interpretation would be that the small deformations of the cosmology manifested as particles would give rise to average energy momentum tensor satisfying Einstein's equations.

In TGD framework these options differ from each other only in that $a = constant$ hyperboloid is deformed in different manner in CP_2 directions.

1. There are infinite number of hyperbolic tesselations. If one requires interpretation in terms of Platonic solids there are only a finite number of tesselations (honeycoms) given in Schöndflies notation by $(6,3,3)$, $(5,3,4)$, $(6,3,4)$, $(4,4,3)$, $(4,3,5)$, $(3,5,3)$, $(5,3,5)$, $(6,3,5)$. Also icosahedral and dodecahedral honeycombs ($(3,5,3)$, $(5,3,3)$ and $(5,3,4)$) are present. Hyperbolic cosmologies are sub-critical and cosmic time evolution is not constrained by the imbedding since CP_2 projection is 1-dimensional.

2. 3-D icosahdedral Penrose tiling or icosahedral QCin flat E^3 (by CMB data cosmology is flat in good approximation in large enough scales but is 10^8 light years large enough scale?). CP_2 projection is 2-D and time evolution which completely fixed apart from a parameter characterizing its finite duration before singularity, which in TGD framework as interpretation as a blackhole like space-time region having Euclidian signature of the induced metric.

3. Over-critical local cosmology and gluing together in similar manner hemi-3-spheres along their 2-D equators giving rise to 600-cells consisting of tetrahedrons. Maybe tetrahedron is too simple an object to serve as as the basic unit of the cosmic honeycomb. Or perhaps a counterpart of icosahedral Penrose tiling could be obtained . Also now time duration fixed completely about from its finite duration before singularity.

It is somewhat frustrating to find, that it is difficult to distinguish experimentally between GRT in which Einstein's equations with cosmological constant characterizing genuine density of matter assignable to the vacuum expectation values of inflaton type fields on one hand, and Einstein equations introducing cosmological constant as purely gravitational parameter. In TGD framework even more general modifications are possible and treat cosmological "constants" as parameters characterizing vacua.

2.5 Does the replacement $E^3 \to S^3$ make sense quantum mechanically?

A possible quantum mechanical sensibility test for the S^3 deformation is based on the quantum motion of a test particle in S^3 metric replacing free motion in E^3 metric when one has two-sheeted representation of S^3. If only a piece of S^3 is considered, the situation is more complex since the solutions must be restricted to a piece of S^3.

The first thing to notice is that S^3 geometry requires mass density which is constant being proportional to the curvature scalar R_3. The energy density is given by $T^{tt} = \kappa G^{tt} = -\kappa g^{tt} R_3/2$. R_3 is negative since the sign of 3-metric is negative so that positive energy density is obtained. For hyperbolic stationary case the energy density would be negative, and this might exclude this option unless one can assign this energy density with negative energy parts of zero energy states. Note that in cosmology the situation changes since R_{tt} is non-vanishing. This kind of distinction between positive and negative energy states at space-time level would be rather dramatic.

One could assign constant mass energy density to large enough atomic nuclei, to condensed matter systems involving many enough atoms such as the water molecule clusters to be discussed later, and also to gases, liquids, and solids. This is natural if the length scale resolution of the description does not allow to distinguish the basic building blocks. These criteria are not satisfied for systems such as single atom.

1. The situation is $SO(4) = SO(3) \times SO(3)$ symmetric and the configuration space is isomorphic with the configuration space $SO(3)$ of a rigid body. One can also consider the possibility of rigid body with half-odd integer angular momentum made possible if one replaces $SO(3)$ with its covering group in both factors of $SO(4)$. For $SO(3)$ only integer values $J = 0, 1, 2, ...$ are allowed whereas $SU(2)$ allows half integer values $J = 0, 1/2, 1, ...$ An interesting question is whether one should allow purely geometric half-odd integer spin which does not reduce to ordinary half-odd integer spin. The so called orientation entanglement relation allows to visualize geometrically the fact that 2π rotation is not homotopically trivial whereas 4π rotation is. This might justify the allowance of also half integer values of J.

2. The transition from free linear motion to free rotational motion means that momentum eigenstates with 3-momentum p_i are replaced with angular momentum eigenstates with angular momentum J and two spin components K, L corresponding to angular momentum projections for the two commuting factors of $SO(4)$, which both vary in the range $-J, -J+1, ..., J$, and therefore have $2J+1$ values. The Laplace operator replacing momentum square is just angular momentum squared and has eigenvalues $J(J+1)$ with degeneracy $(2J+1)^2$.

3. The states correspond to unitary irreducible representations of $SO(4) = SO(3) \times SO(3)$ or its covering group $Spin(4) = SU(2) \times SU(2)$ in the group algebra of $SO(3)$ or of its covering $SU(2)$. By general theorems the group algebra of any compact group decomposes to a direct sum of unitary irreducible representations contains all representations such that n-dimensional representation occurs n times. This can be understood as decomposition of the representation of group element g to a direct sum of $n \times n$ matrices belonging to various irreducible representations. For $SU(2) \times SU(2)$ one obtains all angular momenta $J = 0, 1/2, 1, ...$ with $(2J+1)^2$-fold degeneracy. If both both half-odd integer and integer values of J are allowed, the degenerates are given by $n^2 = (2J+1)^2 = 1, 2^2, 3^2, 4^2$ and same as for hydrogen atom.

4. In non-relativistic approximation for energy as $E - m \equiv p^2/2m$, with $E \to i\partial_t$ and $p_i \to iD_i$ and $p_i p^i \to g^{ij} D_i D_j$ one obtains Schrdinger equation for a test particle as

$$i\frac{\partial Psi}{\partial t} = E\Psi = \frac{1}{2mR^2}\nabla^2_{S^3}\Psi \ , \ \nabla^2_{S^3} = D^i D_i \tag{2.26}$$

The energies are given by $E_J = J(J+1)/I$, $I = 2mR^2$, and identical to those of a spherically symmetric rigid body with moment of inertia $I = mR^2$. The scale of energy is given by the radius of S^3. In atomic length scale the estimate for the radius would be size of the atom. This would give energy, which is of the same order of magnitude as zero point kinetic energy of free particle in same volume so that there are no obvious contradictions with existing physics.

5. Could S^3 geometry for free nucleons serve as an alternative for nuclear shell model based on harmonic oscillator Hamiltonian? In shell model single particle energies are $E_n = nE_0$ and such that even/odd integer valued angular momenta $J \leq n$ correspond to given n ($SU(3)$ dynamical symmetry). The harmonic oscillator model predicts correctly the nuclear magic numbers as $2, 8, 20, 28, 50, 82, 126, 184$. For S^3 option the energies would be concentrated on shells with $E_J = J(J+1)E_0$. Now magic numbers would correspond to full shells with $(2J+1)^2 = n^2$ states on each just as in the case of atoms in the first approximation. The magic numbers would be $2, 10, 28, 60, 92, ..$ and not consistent with the experimental ones. Harmonic oscillator Hamiltonian in S^3 geometry however makes sense and predicts splittings of the harmonic multiplets.

6. What happens for free motion if only a finite piece of S^3 is allowed? The simplest situation corresponds to $r = sin(\alpha) = R_1 < R$. The normal derivatives of wave functions should vanish at $r = R_1$ in order to have conservation of probability. The wave functions are matrix elements $D_{J,K,L}(\alpha, \theta, \phi)$ and can be expressed as products of wave functions assignable to the Euler angles α, θ, ϕ. For α the wave function $R_{J,K,L}(\alpha)$ is the S^3 analog of Legendre polynomial $P_{l,m}(\theta)$. The simplest manner to satisfy the boundary conditions is to assume that only those partial waves in S^3 satisfying $dR_{J,K,L}(\alpha)/d\alpha = 0$ for $\alpha = arcsin(R_1/R)$ are allowed. This leads to a quantization of R_1/R and selection of only some values of (J, K, L).

If both both half-odd integer and integer values of J are allowed the degenerates are given by $n = (2J+1) = 1, 2^2, 3^2, ...$ and are same as for hydrogen atom. For spherically symmetric harmonic oscillator appearing in the model of atomic nucleus only even or odd angular momenta are allowed: in this case $SU(3)$ is the dynamical symmetry which happens to be isometry group of CP_2.

1. The exceptionally large degeneracy of energy eigenstates in Coulomb and harmonic oscillator potentials is due to a dynamical symmetry. Besides angular momentum also so called Runge-Lenz vector is conserved in Coulomb potential (so called generalized conserved Runge-Lenz vector can be defined for all central forces). In fact, the motion in Coulomb potential is group theoretically equivalent to the free motion of a particle in S^3! The conserved Runge-Lenz vector is given by $A = p \times L + mke$, where e is radial unit vector at the orbit orthogonal to angular momentum L and p is 3-momentum. Note that $A \cdot L = 0$ holds true. By dividing this with conserved quantity $1/\sqrt{2mE}$ one obtains an operator D with dimensions of angular momentum. The ordinary angular momentum L and D generate $SO(4)$ Lie algebra and the first Casimir operator $C_1 = L^2 + D^2$ for this algebra equals to $mk^2/|E|$. $E \propto 1/n^2$ implies that eigenvalues of S^3 Laplacian are proportional to $C_1 - Id$ and thus proportional to $n^2 - 1 = (2J+1)^2 - 1$. The eigenvalues of C_2 are given by n^2 with $n = 2J+1$, $J = 0, 1/2, 1, ...$ As already noticed, second Casimir operator $C_1 = L \cdot D$ vanishes for the orbits in Coulomb potential.

2. Could the reduction of the motion Coulomb potential to free particle motion in S^3 be more than a mere mathematical curiocity? The relationship $C_1 = -mk^2/H_{Coul}$ between C_1 and Hamiltonian $H_{Coul} = \hbar^2\nabla^2/2m + V(r)$ transforms to

$$H_{S^3} = \frac{\hbar^2 J(J+1)}{2mR^2} = \frac{\hbar^2}{2mR^2}(n^2-1) = \frac{\hbar^2}{2mR^2}(-\frac{mk^2}{H_{Coul}} - 1) \ . \quad (2.27)$$

The relationship looks rather artificial and one can argue that the connection is purely group theoretical and cannot have a genuine geometric meaning.

3. In the case of atom S^3 geometry is not well-motivated. One can still look what happens if one replaces E^3 geometry with S^3 geometry. Coulomb potential has a well-defined S^3 counterpart. The straightforward generalization of Coulomb potential energy to $V(r) = -k/r$, $r = sin(\alpha)$) is

well-defined. It however replaces the singularity at origin with singularities at North and South poles of S^3 unless one considers only a piece of S^3. Physically one would have two-sheeted structure with nucleus at both North and South poles. $SO(4)$ symmetry is reduced to $SO(3)$ symmetry, and is expected to induce the splitting of the energies of states with different values of J but same value of n. Near $r = 0$ the metric is in a good approximation flat so that for states with small n one does not expect large deviations from E^3 and perturbation theoretical treatment using $g^{rr} = 1/(1-(r/R)^2) \simeq 1 + \frac{r}{R}^2$ should make sense. For states of atom with $r_n = (a_0/Z^2)n^2 \leq\leq R$, $a_0 = \hbar/\alpha m_e \simeq .53 \times 10^{-10}$ m, this approximation should be good. For the values of $n > Z^2(R/a_0)$ the deviations from E^3 metric are certainly since the first node of the E^3 radial wave function would be at radius $r > R$.

In principle the effects on the energy eigenvalues of hydrogen atom could be used to derive a lower bound on the value of R if hydrogen atom space-time sheet correspond locally to S^3 (which it very probably does not!). The splitting of states with same n but different values of j can be compared to the splitting predicted by Dirac equation and given exactly by the Sommerfeld formula. The approximate expression for the relativistic splitting reads as

$$\frac{\Delta E_{n,j}}{E_n} \simeq \frac{\alpha^2}{n^2} \times \frac{n}{j+\frac{1}{2}-\frac{3}{4}} \quad .$$

This splitting can be compared with the order of magnitude estimate for the correction coming from S^3 geometry in the lowest order approximation obtained as $g^{rr} = 1-(r/R)^2$ in the Laplacian:

$$\begin{aligned}
\frac{\Delta E_{n,l}}{E_n} &= -\langle \frac{r}{R})^2 \frac{T_r}{E_n}\rangle_{n,l} \ , \\
\langle (\frac{r}{R})^2 T_r\rangle_{n,l} &= \langle \frac{\hbar^2 \partial_r^2}{2m_e}\rangle_{n,l} = -\langle (\frac{r}{R})^2 \left[E_n - \frac{\hbar^2 l(l+1)}{2m_e r^2} - V(r)\right]\rangle_{n,l} \\
-\langle \frac{(\frac{r}{R})^2 T_r}{E_n}\rangle_{n,l} &= -(\frac{a_n}{R})^2 \left[\langle \rho_n^2\rangle_{n,l} + \frac{l(l+1)}{n^2} + \frac{2\pi}{\alpha^2}\langle \rho_n\rangle_{n,l}\right] \\
\rho_n &= \frac{r}{a_n} \ , \ a_n = n^2 a_0 \ , \ a_0 = \frac{\alpha\hbar}{m_e} \ , \ E_n = \frac{\alpha}{2a_n} \ .
\end{aligned} \tag{2.28}$$

The expectation values of ρ_n and ρ_n are of order unity since ρ_n appears as a natural variable in radial wave functions. The condition $R > n^3 a_0/\alpha$ guarantees that the splittings are smaller than given by the Sommerfeld formula. Already for $n = 1$ this would give $R > 137a_0$, which is of order .5 nm.

3 Appendix A: Does TGD force a modification of Einstein equations?

The discovery of preferred extremals [K6, K2] meant a decisive breakthrough in TGD. One of the implications was that Kähler energy momentum tensor must havea vanishing divergence for the preferred extremals. Einstein-Maxwell equations with cosmological term given by

$$T^{\alpha\beta} = \kappa G^{\alpha\beta} + \Lambda g^{\alpha\beta} \quad . \tag{3.1}$$

guarantee this automatically. In TGD framework $T^\alpha_\alpha = 0$ implies that curvature scalar is constant. This implication seems to be too strong. One ends up with problems also with Robertson-Walker cosmologies realized as vacuum extremals if one regards them as limiting cases of preferred extremals. These observations suggest that the condition of Eq. 3.1 is un-necessarily strong This is indeed the case, and a more general condition leads to appearance of several parameters analogous to cosmological constant but not being genuine constants. Even this generalization might be un-necessarily strong and minimalist could argue that just the vanishing of the divergence of Kähler energy momentum tensor might serve as the TGD counterpart of Einstein's equations and Equivalence Principle.

3.1 Does TGD allow several cosmological "constants"?

The introduction of cosmological constant need not be the only solution of the vanishing of the covariant divergence of T_K. If the preferred extremals have special symmetries, one can satisfy the condition by replacing cosmological term $\Lambda g^{\alpha\beta}$ with a more general term:

$$T^{\alpha\beta} = \kappa G^{\alpha\beta} + \sum_i \Lambda_i P_i^{\alpha\beta} \quad . \tag{3.2}$$

Here P_i are identified as projectors to orthogonal sub-spaces of the tangent space of the space-time surface. The distributions of these sub-spaces are assumed to be integrable and define d_i-dimensional sub-manifolds of space-time surface with $\sum d_i = 4$. The maximally symmetric situation corresponds to the canonical imbedding of M^4 with $d_i = 1$.

The condition that energy momentum tensor is divergenceless is satisfied if one has

$$D_\beta(\Lambda_i P_i^{\alpha\beta}) = 0 \quad .$$

Λ_i need not be constant functions anymore so that cosmological constant is replaced by two or more - presumably slowly varying - cosmological "constants". For $\Lambda_i = \Lambda$ one obtains just ordinary cosmological constant which must be genuine constant.

Two remarks are in order.

1. This generalization of the notion of cosmological does not make sense in GRT framework, where Einstein equations are deduced from a variational principle but do so in TGD framework were they characterize preferred extremals and are deduced from the vanishing of covariant divergences of Kähler-Maxwell energy momentum tensor. In TGD framework the acceptance of two cosmological "constants" is just acceptance of TGD and forcing only single one would be too strong ad hoc assumption. In GRT situation would be completely opposite.

2. It should be also noticed that the proposed modification is not physically equivalent to the assumption that cosmological constant characterizes the energy momentum tensor assignable to vacuum expectation values of Higgs like fields (inflaton field). In this case one would have ordinary Einstein equations without cosmological term but energy momentum tensor containing the additional terms:

$$\kappa G^{\alpha\beta} = T^{\alpha\beta} - \sum_i \Lambda_i P_i^{\alpha\beta} \quad . \tag{3.3}$$

   The physical interpretation of these two options is totally different.

3.2 Preferred extremals suggest a generalization of Einstein's equations

There is actually support for the generalization of Einstein's equations in TGD framework.

1. The preferred extremals possessing Hamilton Jacobi structure in Minkowskian regions indeed have the needed 2+2-decomposition of the tangent space to 2 longitudinal and 2 transversal degrees of freedom. The distributions of these tangent spaces are integrable. Physically longitudinal and transversal degrees of freedom correspond to light-like momentum and and orthogonal polarization for the preferred extremals. Also number theoretical vision leads to similar 2+2 decomposition of the quaternionic tangent space. Similar 2+2 decomposition occurs for string like objects. In Euclidian regions Hamilton-Jacobi structure is replaced with complex structure in 4-D and also in this case this kind of decomposition is possible but need not be so unique. The standard Eguchi-Hanson complex coordinate for CP_2 define this kind of decomposition.

   Therefore it would be very natural to assume that the energy momentum tensor has the decomposition given by Eq. 3.5. This would quite generally suggest 2+2 decomposition.

3.3 Also vacuum extremals suggest a generalization of Einstein's equations

The vacuum extremals of Kähler action are expected to be very important piece of TGD since their small deformations are expected to give rise to physical non-vacuum extremals. These extremals have vanishing induced Kähler field but they are not vacua in gravitational sense. One can hope that at least some class of these vacuum extremals could allow a realization as limits of non-vacuum preferred extremals and would therefore satisfy a generalization of Einstein's vacuum equations.

One cannot be make these solution gravitational vacua by introducing cosmological constant so that one would have

$$\kappa G^{\alpha\beta} + \Lambda g^{\alpha\beta} = 0 \ . \tag{3.4}$$

One can however consider more general equations

$$\begin{aligned} \kappa G^{\alpha\beta} = -\sum_i \Lambda_i P_i^{\alpha\beta} \ , \\ D_\beta(\Lambda_i P_i^{\alpha\beta}) = 0 \ . \end{aligned} \tag{3.5}$$

Especially interesting vacuum extremals are defined by the imbeddings of Robertson-Walker cosmology [K4].

1. Robertson-Walker cosmologies for sub-critical, critical, and over-critical mass density correspond to 3-space, which is constant curvature space. For negative curvature one has hyperbolic space H^3, for vanishing curvature Euclidian 3-space E^3, and for positive curvature 3-sphere S^3. The metric in these three cases is given by

$$ds^2 = g_{aa}da^2 - a^2(\frac{dr^2}{1+kr^2} + r^2 d\Omega^2) \ , \tag{3.6}$$

   where $k = 1, -, 0, 1$ corresponds to H^3, E^3, S^3.

2. All these cosmologies are imbeddable to $M^4 \times CP_2$.

   (a) For H^3 signature the imbedding is obtained by assuming almost arbitrary 1-D CP_2 projection with CP_2 coordinates arbitrary functions of cosmic time a defined by M^4 light-cone proper time: $s^k = f^k(a)$. $a = constant$ surfaces correspond to hyperboloids of future light-cone $M^4_+ \subset M^4$.

   (b) For E^3 and S^3 one must assume that CP_2 projection is 2-dimensional. The simplest option corresponds to a CP_2 projection which is homologically trivial geodesic sphere $S^2 \subset CP_2$ with vanishing induced Kähler form. Denoting by (Θ, Φ) the coordinates of $S^2 \subset CP_2$, the imbedding must be of form

$$sin(\Theta) = \frac{a}{\tau} \ , \ \ \Phi = f(r) \ . \tag{3.7}$$

   The dependence of Θ on cosmic time is dicrated by the condition that the contribution of $sin^2(\Theta)(\partial_r\Phi)^2)$ is proportional to a^2. $f(r)$ is fixed from the condition that one obtains $k = 0$ or $k = -1$ meaning that hyperbolic metric of 3-space transforms to Eucldian or spherical one by deformation in CP_2 directions. The resulting cosmologies have only their duration τ as a free parameter [K4].

   Euclidian and spherical cosmologies end up with singularity as the induced metric transforms to Euclidian signature. In TGD framework,where blackholes like objects corresponds to space-time regions with Euclidian signature, the interpretation would be in terms of blackhole collapse. What is interesting is that the "pressure" in Einstein tensor in negative so that one obtains accelerated cosmic expansion for critical mass density.

3. For these cosmologies 1+3 decomposition of the tangent space looks more natural than 2+2 decomposition. 1+3 decomposition corresponds to the decomposition of Einstein's tensor to $G = \rho P_1 - pP_3$, where $P_1 \leftrightarrow g^{aa}$ projects to the direction of cosmic time a and $P_3 \leftrightarrow g_3^{ij}$ to 3-space $a = constant$. It is easy to see that pg_3^{ij} has vanishing divergence and Einstein's equations implies that also ρP_1 does so. This does not conform with the idea that Hamilton-Jacobi structure dictates the decomposition, which would therefore be 2+2. If this is really true, then on expects a second family of solutions for which one has $1+3$ decomposition of the tangent space. Maybe the conjectured quaternionic space-time surfaces could correspond to this family of preferred extremals. The earlier conjecture has been that preferred extremals with Hamilton-Jacobi structure are equivalent with quaternionic ones.

Note that this kind of decomposition takes place for the deformation of E^3 or equivalently M^4 discussed above. One must deduce the conditions under which the divergence free decomposition holds true. This must pose a differential equation on $k(r)$.

One can also ask what the situation is for Schwartschild metric and Reissner-Nodström metric. For Schwartschild metric Einstein tensor vanishes so that $\Lambda_i = 0$ holds true trivially. A little calculation shows that for Reissner-Nordström metric Einstein tensor has the decomposition $E^2 g_1 - E^2 g_2$, where g_1 and g_2 are the projectors to (t, r) plane and (θ, ϕ) sphere. $E^2 = Q^2/r^4$ implies that the conditions $D_\beta(\Lambda_i P_i^{\alpha\beta}) = 0$ cannot be satisfied.

3.4 What could the modification of Einstein's equations mean from the point of view of dark energy?

The modification of Einstein's equations has highly non-trivial implications concerning the notion of dark energy. In GRT based interpretation cosmological constant does not correspond to energy density whereas in the models assigning it to vacuum expectations of Higgs like inflaton fields a genuine energy density is in question: this allows the variation of cosmological constant with time whereas in Einstein's theory Λ would be a constant of Nature subject only to coupling constant evolution. In TGD framework G, Λ would depend on space-time sheet and the set of cosmological "constants" Λ_i would depend on position for a given space-time sheet.

One must of course remember that in TGD vacuum extremals can be only limiting situations. The real space-time sheets are definitely not vacuum extremals. For instance, elementary particles correspond to space-time regions with 4-D CP_2 projection and Euclidian signature of the induced metric so that their nearby Minkowskian environment has 3-D CP_2 projection and is also non-vacuum extremal. Same applies to cosmic strings, which in the ideal situation have 2-D CP_2 projection which corresponds to homologically non-trivial 2-surfaces in CP_2.The evolution of cosmic strings would mean gradual thickening of their originally infinitely thin E^3 projection and the remnants of cosmic strings would explain also the magnetic fields filling the Universe.

It is not easy to find any killer argument against the proposed modification of Einstein's equations.

1. What happens to Equivalence Principle if the modification of Einstein's equations is accepted? The basic manifestation of Equivalence Principle is as the geodesic motion of test particles. In Newtonian framework the analog of this is the cancellation of the dependence on the mass of the particles due to the identical values of inertial and gravitational masses. In this respect nothing changes since the motion is determined completely by the geometry. The independence of the effects of geometry on test particle on what one assumes about the energetics is analogous to the disappearance of inertial and rest masses from the equation of motion for Newtonian test particle.

   Also the effects caused by the background geometry on particles - say redshift or spectrum of microwave temperature fluctuations of microwave background - are same irrespective of the energetic interpretation.

2. Both vacuum and non-vacuum options can give rise to accelerated cosmic expansion so that for vacuum option no dark energy would be needed. Does this mean that dark energy thought to

be responsible for the accelerated cosmic expansion or it is pure vacuum fiction? And could one explain the velocity spectrum of distant stars rotating around galactic nuclei by assuming that galaxies are like pearls in necklace, which is vacuum flux tube instead of Kähler magnetically charged flux tube carrying huge energy density characterized by string tension defined by CP_2 scale? Note also that for critical and sub-critical vacuum option non-trivial long range gauge fields - in particular electromagnetic fields - are present in the vacuum. The situation is admittedly somewhat frustrating: I had already thought that the issue of dark energy is finally resolved in TGD framework!

Could one see vacuum extremals and non-vacuum extremals - not as options between which to choose - but descriptions applying in different length scales. In zero energy ontology this might be possible.

1. The original hypothesis was that the small deformations of vacuum extremals, which microscopically correspond to the generation of particles, have average energy momentum tensor given by the Einstein tensor. If this is the case, vacuum extremals would carry information about matter topologically condensed at them coded to their own geometry. The energy momentum current of the topologically condensed matter represents simplest information of this kind. Could the energy momentum tensor defined by Einstein tensor of vacuum extremal be identified with the energy momentum tensor of the topologically condensed matter? More generally, could the sum of Kähler energy momentum tensor and of the terms corresponding to projection operators representing topologically condensed matter correspond to Einstein tensor in the case of non-vacuum extremals? If so, the deviation from Einstein's equations woul have quite generally interpretation in terms of topologically condensed matter. Topologically condensed matter would replace the contribution of vacuum expectations of inflaton fields.

2. Could the vanishing of the actual energy momentum current for vacuum extremals be interpreted as saying that the causal diamonds assignable to the zero energy states are small in the scale of vacuum extremal so that the topologically condensed matter has interpretation as quantum fluctuations? Vacuum extremal would however carry information about the topologically condensed matter (vacuum fluctuations) and make it manifest as effects like redshift, accelerated expansion, and temperature fluctuations of CMB.

One must be however always ready to invent a counter argument.

1. The vacuum degeneracy of Kähler action is enormous. Any space-time surface with CP_2 projection belonging to a Lagrangian sub-manifold of CP_2 has vanishing induced Kähler form and is therefore vacuum extremal. Symplectic transformations of CP_2 give rise to new Lagrangian manifolds and diffeomorphisms of M^4 give rise to new vacuum extremals. Is it really possible do identify a canonical decomposition of Einstein tensor for these vacuum extremals to at least two non-vanishing pieces characterized by cosmological "constants" Λ_i?

2. One could circumvent the objection by noticing that there is no need for every vacuume extremal to define a limit of a preferred extremal.

3. The only possible hope about the decomposition is given by the two-dimensional character of CP_2 projection. The inverse image of a given point of CP_2 belonging to the space-time surface is in the generic case a 2-D sub-manifold of the space-time surface and as the point of CP_2 varies one obtains a slicing of the space-time surface by these 2-surfaces. Could the Einstein tensor have a representation $G = \Lambda_1 P_1 + \Lambda_2 P_2$ as a sum of contributions associated with the tangent space and space-time complement?

4 Appendix B: Conditions on function $k(r)$ from the generalization of Einstein's equations for vacuum extremals

In this Appendix the conditions on the function $k(r)$ guaranteeing that the metric $ds^2 = dt^2 - k^2 dr^2 - r^2 d\Omega^2$ is imbeddable to $M^4 \times CP_2$ as vacuum extremal obtained by deforming $t = constant$ hypersurface E^3 of M^4 in CP_2 direction are discussed. The vacuum extremal is taken to be the simplest possible one having CP_2 projection to a geodesic circle S^1 of CP_2 having angular coordinate Φ as coordinate so that one has $\Phi = f(r)$ and $k^2 = 1 + R^2(df/dr)^2$.

4.1 Differential equation for $k(r)$ from generalized Einstein equations

As shown, the Einstein tensor G of the solution of the metric $ds^2 = dt^1 - k^2dr^2 - r^2d\Omega^2$ is in general vanishing and this is not consistent with the vacuum extremal property of the imbedding meaning that the induced Kähler field vanishes. These vacuum extremals are also in conflict with the assumption that preferred extremals (4-surfaces) at the limit of vacuum satisfy Einstein-Maxwell equations with cosmological term satisfying $T_K = \kappa G + \Lambda g$. This condition is however un-necessarily strong and not actually prediction of TGD. The only condition on preferred extremals in TGD is that T_K has vanishing divergence and this condition can be satisfied by assuming only

$$\begin{aligned} T^{\alpha\beta} &= \kappa G^{\alpha\beta} + \sum_i \Lambda_i P_i^{\alpha\beta} \ , \\ D_\beta(\Lambda_i P_i^{\alpha\beta}) &= 0 \ \ , \ \ \kappa = \frac{1}{8\pi G} \ . \end{aligned} \quad (4.1)$$

Λ_i need not to be constant functions anymore so that cosmological constant is replaced by several - presumably slowly varying - cosmological "constants".

In the recent case one is interested on having such $k(r)$ that one has vacuum extremal with 1+1+2 decomposition satisfying

$$\kappa G^{\alpha\beta} = - \sum_{i=1,2,3} \Lambda_i P_i^{\alpha\beta} \ . \quad (4.2)$$

Now one would have three terms P_i in the decomposition which is 1+1+2. corresponding to coordinate lines for t and r and spheres (θ, ϕ). In the following the differential equations for $k(r)$ guaranteeing that $\Lambda_i P^i$ has a vanishing divergence, are deduced. These equations can be integrated and give rise to a family of metrics.

The first thing to notice is that if $\Lambda_1 P_1$ and $\Lambda_2 P_2$ have vanishing divergences then also $\Lambda_3 P_3$ does so since G has vanishing divergences and $T_K = 0$ holds true. There it is enough to show that $(G^{tt}, 0, 0, 0) = (-g^{tt}R_3/2, 0, 0, 0)$ expressible as $\Lambda_1 g^{tt}$ and $(0, G^{rr}, 0, 0) = (0, R^{rr} - g^{rr}R_3/2, 0, 0$ have vanishing divergence.

For $G^{tt} = -g^{tt}R/2$ the divergence vanishes trivially since it involves only ordinary time derivative. For $G^{rr} = R^{rr} - g^{rr}R/2$ the condition is non-trivial and gives rise to a differential equation giving as solutions a family of functions $k(r)$. The vanishing of covariant divergence for $\Lambda_2 P_2 \leftrightarrow G^{rr}$ gives rise to the condition

$$\partial_r G^{rr} + 2\left\{ {r \atop r\ r} \right\} G^{rr} = 0 \ . \quad (4.3)$$

This equation can be integrated

$$G^{rr} = \frac{K}{k^2(r)} = -\frac{\Lambda_2}{\kappa} g^{rr} \ , \quad (4.4)$$

where Λ_2 is 1-D cosmological "constant "with dimensions of length to fourth is k is taken to be dimensionless so that r has dimension of length.

Writing the expression of G^{rr} explicitly one can cast this equation to the differential equations

$$r^2 G^r_r = \frac{1}{k^3}\frac{dk}{dlog(r)} + \frac{k^2 - 1}{k^2} = \frac{\Lambda}{\kappa} r^2 \ . \quad (4.5)$$

This differential equation is non-linear and non-homogenous and can be written also in the form

$$\frac{dk}{dlog(r)} + k(k^2-1) = \frac{\Lambda}{\kappa}k^3r^2 \quad . \tag{4.6}$$

The first thing to notice is that the equation allows $k = constant$ as a solution only if one has $k = 1$. Hence $k = 1 + \epsilon$ solution is not allowed as a gravitational vacuum solution in TGD sense.

4.2 Spherical and hyperbolic metrics satisfy modified Einstein's equations

For $\Lambda_2 = 0$ it can be however integrated. In this case on obtains

$$\int_{k_0}^{k} \frac{1}{k(k^2-1)}dk = \int_0^u du = log(\frac{r}{R}) \quad . \tag{4.7}$$

This gives

$$k^2 = \frac{1}{1-(\frac{r}{R})^2} \quad . \tag{4.8}$$

This is nothing but the metric of S^3 with radius R. As a matter fact, one obtains only one half of the sphere and by gluing the two halves along equator one would obtain two sheeted 4-surface defining the entire 3-sphere.

Also hyperbolic metric with $k = 1/(1+(r/R)^2)$ satisfies the generalized Einstein's equations as is easy to see by a direct calculation.

Could $k(r)$ ansatz allow other than H^3, E^3, andS^3 for Robertson-Walker cosmology? The deviation of $k(r)$ from the form

$$k^2(r) = \frac{1}{(1+\epsilon(r/R)^2} \quad , \quad \epsilon = \pm 1, 0$$

means breaking of $SO(3,1), SO(4)$, or $SO^3 \times T^3$ symmetry. Hence the 1+3 decomposition is replaced with 1+1+2 decomposition so that one obtains two conditions for $k(r)$ corresponding to G^{aa} and G^{rr}. Note that the condition $sin(\Theta) = a/\tau$ must be satisfied unless the situation is hyperbolic. This suggests that only these 3 solutions are possible.

4.3 Approximate solutions for $\Lambda_1 \neq 0$

For non-vanishing values of $\Lambda_{\equiv}\Lambda$ one obtains candidates for solutions which one is searching provided that $k(r)$ approaches rapidly to $k(r) = 1$. Near the origin Eq. 4.6 reduces to that giving S^3 so that the solution looks near the center of the icosahedron. For large values of r it should give flat metric. If $k \to 1$ or $k \to 0$ the equation reduces in good approximation to

$$\frac{1}{k^3}\frac{dk}{dr} = \frac{\Lambda}{\kappa}r \quad . \tag{4.9}$$

This can be integrated to give

$$k^2 = \frac{k_0^2}{\left[1+\frac{\Lambda}{\kappa}k_0^2(r^2-r_0^2)\right]} \quad . \tag{4.10}$$

k approaches zero asymptotically so that the result is consistent with the assumption about the asymptotic behavior. The solution becomes singular at

$$r^2 = r_0^2 - \frac{\kappa}{k_0^2\Lambda} \quad . \tag{4.11}$$

One can get rid of the singularity if one assumes

$$r_0^2 < \frac{\kappa}{k_0^2\Lambda} \quad . \tag{4.12}$$

With this assumption one can write the solution as

$$\begin{aligned} k &= \left[(\frac{r}{r_a})^2+1\right]^{-1/2} \quad , \\ r_a^2 &= \frac{\kappa}{\Lambda} \quad , \quad r_1^2 = k_0^{-2}r_a^2 - r_0^2 \quad . \end{aligned} \tag{4.13}$$

The limiting case corresponds to $r_1 = 0$ taking the singularity to origin. This is nothing but H^3 metric so that the solutions would look like S^3 near origin and like H^3 at larger distances. A local S^3 like bump would be in question.

Consider now that ratio of the distance s to the Euclidian distance R and assume that the metric is non-singular also at origin so that it makes sense to use the approximation holding true at $r \rightarrow \infty$ limit. Assume that R is defined as the radius for which $k(R) = 1$ holds true. This condition gives

$$\frac{R}{r_a} = \sqrt{1-(\frac{r_1}{r_a})^2} \quad . \tag{4.14}$$

The basic solution dependent parameters are k_0 and r_0 whereas r_a would be analogous to a constant of Nature in GRT context: in TGD framework also this parameter can be seen as a parameter characterizing space-time sheet. In any case it is convenient to express everything in terms of k_0 and $r0$. A tedious exercise gives the following formulas for various length ratios:

$$\begin{aligned} \frac{R}{r_1} &= \frac{\sqrt{1-(\frac{r_1}{r_a})^2}}{\frac{r_1}{r_a}} = \sqrt{k_0^2\left[1+(\frac{r_0}{r_1})^2\right]-1} \equiv X \quad , \\ \frac{r_1}{r_a} &= \frac{\sqrt{1-(\frac{k_0r_0}{r_a})^2}}{k_0} \quad , \quad \frac{r_a}{R} = \frac{k_0}{X} \quad . \end{aligned} \tag{4.15}$$

The ratio increases exponentially with R/r_1.

The goal of the calculation is to deduce the ratio s/R of the deformed distance to Euclidian distance. One can estimate $s(R) = \int_0^R k(r)dr$ by approximating the metric with its asymptotic form for large values of r. This gives

$$\frac{s}{R} = \frac{r_a}{R}arsinh(\frac{R}{r_1}) \simeq \frac{k_0}{X}arsinh(X) \quad , \quad X = \sqrt{k_0^2\left[1+(\frac{r_0}{r_1})^2\right]-1} \quad . \tag{4.16}$$

In the recent TGD-based cosmology one has the order of magnitude estimate $\Lambda \sim \kappa/a^2$, a cosmic time defined by the light-cone proper time. In condensed matter length scale the value of a would be much smaller than in cosmology. r_a would be of order a and $r_1/r_a \sim 1/k_0$ would hold true. One would have $r_0/r_1 \simeq 0$, and $R/r_1 \sim \sqrt{k_0^2-1}$ giving $r_a/R = (r_a/r_1)(r_1/R) = k_0/\sqrt{k_0^2-1}$ giving

$$\frac{s}{R} = \frac{r_a}{R}arsinh(\frac{R}{r_1}) \simeq \frac{k_0}{\sqrt{k_0^2-1}}arsinh(\sqrt{k_0^2-1}) \equiv coth(U_0)\times U_0 \quad , \quad k_0 = cosh(U_0) \quad . \tag{4.17}$$

Large enough value for k_0 gives large ration s/R. Hence it seems possible to have rather large values of s/R. This gives also excellent hopes about equal lengths for the edges of icosahedral tetrahedrons. It should be relatively easy to estimate numerically the length of geodesic edges of the "surface" edges in the proposed metric. In the approximation $k(r) = 1$ in the region containing the edges, the lengths of surfaces edges are just the Euclidian lengths.

The conclusion is that if one accepts the TGD based vision implying a modification of Einstein's equations, one indeed can have a situation in which the the icosahedral tetrahedra are regular. Whether this has any interesting physical meaning, remains of course open. Perhaps the only real defense for this exercise in Riemannian geometry is that it forced to question the naive assumption that all preferred extremals satisfy Einstein-Maxwell equations with cosmological term.

5 Appendix C: Explicit form for the regular tetrahedron property for $k = constant$ option

The generalization of Platonic solids to the deformation of E^3 is obtained by replacing their edges by geodesic lines in the deformed metric. For spherical tetrahedron at unit sphere the distances between vertices are equal to $\sqrt{8/3} \simeq 1.633$. For the irregular tetrahedrons assignable to an icosahedron the surface edges are much shorter but still by a factor $\sqrt{2^3/(5+\sqrt{5})} \simeq 1.0515$ longer than the radial edges.

In the following only the case $k = 1+\delta$ is considered so that radial edges have length k if unit sphere in Minkowski metric is in question. In order to see whether the condition $s = k$ for the length between the vertices of icosahedron can be satisfied for a suitable choice of δ, one must calculate the geodesic distance between the vertices. It turns out that $k = \infty$ is the only solution. The interpretation is following. As k giving the length of the radial edge increases, also the length of the surface edge increases so fast that the radial edge remains shorter than the surface edge for all finite values of k. As already found, the condition that solution is vacuum extremal also in gravitational sense as it is understood in TGD framework allows only $k = 1$ for constant value of k.

5.1 Equations of geodesic lines for deformed E^3

The equations of geodesic line are in general form

$$\frac{d^2x^k}{dt^2} + \left\{ \begin{array}{c} k \\ l\ m \end{array} \right\} \frac{dx^l}{dt}\frac{dx^m}{dt} \quad .$$

They can be solved by using angular momentum conservation and energy conservation.

1. Rotational symmetry allows to choose the coordinates so that the geodesic line is in $z = 0$ plane so that one has $\theta = \pi/2$. The equations reduce to

$$\begin{aligned} \frac{d^2r}{dt^2} + \left\{ \begin{array}{c} r \\ \phi\ \phi \end{array} \right\} \left(\frac{d\phi}{dt}\right)^2 &= 0 \quad , \\ \frac{d^2\phi}{dt^2} + 2\left\{ \begin{array}{c} \phi \\ r\ \phi \end{array} \right\} \frac{d\phi}{dt}\frac{d\phi}{dt} &= 0 \quad . \end{aligned}$$

   This gives

$$\frac{d^2r}{dt^2} = \frac{r}{k^2}\left(\frac{d\phi}{dt}\right)^2 \quad ,$$

$$\frac{\frac{d^2\phi}{dt^2}}{\frac{d\phi}{dt}} = -2\frac{\frac{dr}{dt}}{r} \quad .$$

2. The latter equation can be integrated just as in E^3 and gives

$$\frac{d\phi}{dt} = \omega_0\left(\frac{r_0}{r}\right)^2 \quad .$$

   $r_0 = 1$ holds true for unit sphere. $r_0 = 1$ is assumed in the following formulas. The interpretation is in terms of angular momentum conservation.

3. Substituting $d\phi/dt$ to the equation for r one obtains

$$\frac{d^2r}{dt^2} = \frac{\omega_0^2}{k^2r^3} \ .$$

Energy conservation becomes explicit by multiplying with dr/dt and integrating to get

$$(\frac{dr}{dt})^2 + \frac{K^2}{r^2} = v_0^2 + K^2 \ ,$$

$$v_0 = \frac{dr}{dt}(0) \ , \ K = \frac{\omega_0}{k} \ .$$

This gives

$$\frac{dr}{dt} = \pm v_0 X \ ,$$

$$X = \sqrt{1 + \frac{K^2}{v_0^2}(1-u^2)} \ ,$$

$$u = \frac{1}{r} \ .$$

The initial values v_0 and ω_0 - or rather their ratio- must be fixed from the condition that the geodesic line connects the neighboring vertices of the icosahedron. This condition boils down to the condition that the angular distance between the points is same as for ordinary icosahedron.

4. Also this equation can be integrated. It is convenient to take ϕ instead of t as the variable by using $dr/d\phi = dr/dt \times dt/d\phi$ and the expression of $d\phi/dt$ in Eq. 2. This gives

$$\Delta\phi = arccos(\frac{\Phi}{2+\Phi}) = \frac{\omega_0}{v_0}\oint_1^1 \frac{1}{\sqrt{X}}du \ ,$$
$$\Phi = \frac{1+\sqrt{5}}{2} \ .$$

$\oint_1^1$ tells that the integral is between points, which are at the surface of the sphere. The integral is two times the integral between $u = u_0 = 1$ and u_{max} at which $dr/dt = 0$ holds true. By taking $u = 1/r$ as an integration variable one obtains

$$\Delta\phi = 2k\sqrt{A} \times I(A, u_{max}) \ ,$$

$$I(A, u_{max}) = \int_1^{u_{max}} \frac{1}{\sqrt{1+A(1-u^2)}}du \ ,$$

$$A = \frac{K^2}{v_0^2} \ .$$

$\Delta\phi$ is fixed as the angular distance between neighboring vertices of icosahedron. u_{max} corresponds to the vanishing of X, and is given by

$$u_{max} = \sqrt{\frac{A+1}{A}} = \sqrt{1 + \frac{k^2v_0^2}{\omega_0^2}} \ .$$

u_{max} increases with k meaning that the geodesic line visits nearer to origin for larger values of k unless the radial kinetic energy at the initial moment is reduced as compared to the rotational one. From these condition one can solve that initial values v_0 and ω_0.

5. The variable change $v = \sqrt{A/(1+A)}u$ allows to express the integral $I(A, u_{max})$ in terms of elementary functions:

$$I(A, u_{max}) = \sqrt{\frac{1}{1+A}}\int_{\sqrt{A/(1+A)}}^1 \frac{1}{\sqrt{1-v^2}}dv = \frac{1}{\sqrt{A}}(\pi/2 - arcsin(\sqrt{A/(1+A)})) \ .$$

This gives the condition

$$
\begin{aligned}
\Delta\phi &= arccos(\frac{\Phi}{2+\Phi}) = 2k(\pi/2 - arcsin(\sqrt{A/(1+A)})) && (5.1)\\
A &= \frac{\omega_0^2}{v_0^2 k^2} \ . && (5.2)
\end{aligned}
$$

This condition relates the parameters ω_0/v_0 and k:

$$
k = \frac{arccos(\frac{\Phi}{2+\Phi})}{2(\pi/2 - arcsin(\sqrt{A/(1+A)})} \ . \quad (5.3)
$$

$$(5.4)$$

The above conditions only fix the geodesic line representing the edge of the icosahedron. Besides the condition fixing $\Delta\phi$, one must apply the condition $s = k$, where s is the geodesic length of the edge. Since the scaling up of g_{rr} increases also the distance between the vertices of the icosahedron, it is not completely clear whether any solution exists unless one gives up that assumption that $k(r)$ is constant.

One can however argue that since the variation of r for the surface edge is much shorter than for radial edge, the length of radial edge increases faster with k as that of the surface edge so that there are hopes that the lengths of the edges can be equal.

5.2 Explicit form for the condition $s = k$

Also the integral involved with the condition $s = k$ can be solved explicitly so that it is rather trivial exercise in numerics to find whether the solution to the condition $k = s$ exists.

1. The expression for the length of the surface edge is given by

$$
s = \int \sqrt{k^2 + (\frac{d\phi}{dr})^2} dr \ .
$$

By substituting $d\phi/dr$ one obtains

$$
\begin{aligned}
&s = 2k\sqrt{A}J(A, u_{max}) \ , \\
&J(A, u_{max}) = \int_1^{u_{max}} \frac{1}{\sqrt{1+A(1-u^2)}} \frac{du}{u^2} \ , \\
&A = \frac{K^2}{v_0^2} \ , \quad u_{max} = \sqrt{\frac{A+1}{A}} = \sqrt{1 + \frac{k^2 v_0^2}{\omega_0^2}} \ .
\end{aligned}
$$

Edge length s is proportional to the radial edge length k and depends also on parameter A, which in turn turn depends on k both explicitly and implicitly.

The first thing to notice is that if A or equivalently u_{max} does not depend on k, both s and radial edge length are proportional to k so that their ratio does not change. Hence A must depend on k both explicitly and implicitly. The constraint of Eq. 4 for $\Delta\phi$ and the non-trivial explicit dependence of $A = x^2/k^2$ on k together imply that also the parameter $x = \omega_0/v_0 = k\sqrt{A}$ must depend on k.

2. The integral $J(A, u_{max})$ can be expressed in terms of elementary function by the same variable change as performed for $I(A, u_{max}$. One obtains

$$
\begin{aligned}
&J(A, u_{max}) = \frac{\sqrt{A}}{(1+A)} \int_{\sqrt{A/(1+A)}}^1 \frac{1}{\sqrt{1-v^2}} \frac{dv}{v^2} \\
&= -\frac{\sqrt{A}}{(1+A)} cot(arcsin(\sqrt{A/(1+A)})) = \frac{1}{1+A} \ .
\end{aligned}
$$

This transforms the condition $s/k = 1$ to the form

$$\begin{aligned} \frac{s}{k} = \frac{2\sqrt{A}}{(1+A)} = 1 \ , \\ A = \frac{K^2}{v_0^2} = \frac{\omega_0^2}{k^2 v_0^2} \ . \end{aligned} \quad (5.5)$$

The condition $s/k = 1$ gives

$$A = 1 \ . \quad (5.6)$$

Combibing this with Eq. 5 for k, one obtains

$$\begin{aligned} A &= 1 \ , \\ k &= \frac{arccos(\frac{\Phi}{2+\Phi})}{2(\pi/2 - arcsin(\sqrt{A/(1+A)}))} = \infty \ , \\ r = \frac{\omega_0}{v_0} &= k\sqrt{A} = \infty \ . \end{aligned} \quad (5.7)$$

The conclusion is that for a finite value of k it is not possible to satisfy the condition $s/k = 1$. The interpretation is that the length of radial edge does not increase fast enough to reach the length of the surface edge since it becomes also longer.

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