Circularly polarized beam carries the double angular momentum

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We consider two different types of angular momentum of electromagnetic radiation. 1) Spin; its origin is a circular polarization. 2) Moment of linear momentum, which is an orbital angular momentum. It is shown that a circularly polarized light beam with plane phase front and the dipole radiation carry angular momentum of both types, contrary to the standard electrodynamics. These two types of angular momentum are spatially separated.

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1. Spin of light

It was suggested as early as 1899 by Sadowsky [1] and as 1909 by Poynting [2] that circularly polarized light has angular momentum density. If j_z [J.s/m^3] and w [J/m^3] are the z-component of the angular momentum and energy volume density, μ_z [J/m^2] and f_z [W/m^2] are the z-components of the angular momentum and energy flux density, then, according to Poynting,

$$\frac{j_z}{w} = \frac{\mu_z}{f_z} = \frac{1}{\omega} \tag{1.1}$$

($\mathbf{f}$ denotes the Poynting vector [3, p.96]). So, a torque τ [J] acts on a body, which absorbs at least a part of the light or/and changes its polarization state, and μ_z is z-component of the torque density, $\tau_z = \int \mu_z da$.

Beth [4] wrote: "The moment of force or torque exerted on a doubly refracting medium by a light wave passing through it. The torque per unit volume produced by the action of the electric field on the polarization of the medium is $\tau / V = \mathbf{P} \times \mathbf{E}$". So, if a half-wave plate is rotated in its own plane, work is in progress. This amount of work must reappear as an alteration in the frequency of the light (in the energy of the photons), which will result in moving interference fringes in any suitable interference experiment [5,6].

Feynman [7, 17–4] explains that circularly polarized light carries an angular momentum and energy in proportion to $1/\omega$ because photons carry spin angular momentum $\hbar$ and energy $\hbar\omega$. So, the angular momentum density j_z is the spin density, $s_z = j_z$, $S_z = \int j_z dV$, and μ_z is z-component of the spin torque density ($\mathbf{S}$ is spin). So we may rewrite (1.1) as

$$\frac{s_z}{w} = \frac{\mu_z}{f_z} = \frac{1}{\omega}. \tag{1.2}$$

Also Carrara [8] wrote: "If a circularly polarized wave is absorbed by a screen or is transformed into a linearly polarized wave, the angular momentum vanishes. Therefore the screen must be subjected to a torque per unit surface equal to the variation of the angular momentum per unit time. The intensity of this torque is $\pm f/\omega$". We noted [9] that the spin torque density μ_z produces a specific mechanical stress in the absorbing screen, and this effect may be tested experimentally [10].

We used the Beth's formula in [11] for a circularly polarized plane wave, which died down in a dielectric (the mark brave indicates complex numbers):

$$\breve{\mathbf{E}} = \exp[i(\breve{k}z - \omega t)](\mathbf{x} + i\mathbf{y})E_0, \quad \breve{\mathbf{B}} = -i\sqrt{\breve{\varepsilon}}\breve{\mathbf{E}},$$
$$\breve{\varepsilon} = \breve{k}^2/\omega^2, \quad \breve{k} = k' + ik'', \quad \breve{k}^2 = k'^2 - k''^2 + 2ik'k'', \quad c = \varepsilon_0 = \mu_0 = 1. \tag{1.3}$$

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$$\tau^z/V = \mathbf{P}\times\mathbf{E} = e^{xyz}\Re\{\breve{P}_{[x}\bar{E}_{y]}\} = \Re\{(\breve{\varepsilon}-1)\breve{E}_{[x}\bar{E}_{y]}\} = \exp(-2k''z)\Re\{(\breve{\varepsilon}-1)(-i-i)\}E_0^2/2$$
$$= \exp(-2k''z)\Im(\breve{\varepsilon}-1)E_0^2 = \exp(-2k''z)2k'k''E_0^2/\omega^2. \qquad (1.4)$$

Then the torque per unit surface $z=0$ is

$$\tau/a = \int_0^\infty \exp(-2k''z)2k'k''E_0^2 dz/\omega^2 = k'E_0^2/\omega^2 \qquad (1.5)$$

The densities s_z and μ_z (or rather $s_{\wedge}^{xy} = e_{\wedge}^{xyz}s_z, \mu_{\wedge}^{xy} = e_{\wedge}^{xyz}\mu_z$) can be expressed in terms of the electromagnetic fields of the light wave as components of the spin tensor (density) [12,13,9]:

$$\Upsilon_{\wedge}^{\lambda\mu\nu} = (A^{[\lambda}\partial^{|\nu|}A_{\wedge}^{\mu]} + \Pi^{[\lambda}\partial^{|\nu|}\Pi_{\wedge}^{\mu]}),\ \ dS^{\lambda\mu} = \Upsilon_{\wedge}^{\lambda\mu\nu}dV_{\nu}^{\wedge}, \qquad (1.6)$$

where A^λ and Π^λ are magnetic and electric vector potentials, which satisfy $2\partial_{[\mu}A_{\nu]} = F_{\mu\nu}$, $2\partial_{[\mu}\Pi_{\nu]} = -e_{\mu\nu\alpha\beta}F^{\alpha\beta}$ (sometimes we mark densities with the symbol 'wedge' $\wedge$ [9]). So we have

$$s^{xy} = \Upsilon^{xyt} = (\varepsilon_0\mathbf{E}\times\mathbf{A} + \Pi\times\mathbf{H}\mu_0)/2,\ \ dS^{xy} = \Upsilon^{xyt}dV, \qquad (1.7)$$

$$\mu^{xy} = \Upsilon^{xyz} = -A^{[x}\partial_z A^{y]} - \Pi^{[x}\partial_z\Pi^{y]},\ \ d\tau^{xy} = \Upsilon^{xyz}da_z, \qquad (1.8)$$

where $\partial_t\mathbf{A} = -\mathbf{E}$, $\partial_t\Pi = \mathbf{H}$. However, the two addends in (1.7), (1.8) are equal to each other for most cases ($\varepsilon_0\mathbf{E}\times\mathbf{A}$ and $\Pi\times\mathbf{H}\mu_0$ supplement each other in standing waves [12,13]).

The result (1.5) may be repeated as the angular momentum flux density (1.8) if we will consider vacuum at $z\le 0$ [11]. The wave (1.3) is provoked by incident and reflected waves:

$$\breve{\mathbf{E}}_1 = (1+\breve{k}/\omega)\exp[i(\omega z-\omega t)](\mathbf{x}+i\mathbf{y})E_0/2,\quad \breve{\mathbf{B}}_1 = -i\breve{\mathbf{E}}_1, \qquad (1.9)$$

$$\breve{\mathbf{E}}_2 = (1-\breve{k}/\omega)\exp[i(-\omega z-\omega t)](\mathbf{x}+i\mathbf{y})E_0/2,\quad \breve{\mathbf{B}}_2 = i\breve{\mathbf{E}}_2. \qquad (1.10)$$

Now substitute $A^x = -\int E^x dt = -\int(E_1^x + E_2^x)dt,\quad A^y = -\int E^y dt = -\int(E_1^y + E_2^y)dt$ into $\Upsilon^{xyz} = -\Re(\bar{A}^x\partial_z\breve{A}^y - \bar{A}^y\partial_z\breve{A}^x)/2$. Because of $\breve{E}^y = i\breve{E}^x$ and consequently $\breve{A}^y = i\breve{A}^x$,

$$\bar{A}^x\partial_z\breve{A}^y = -\bar{A}^y\partial_z\breve{A}^x = i\bar{A}^x\partial_z\breve{A}^x,\ \text{and}\ \ \Upsilon^{xyz} = -\Re(i\bar{A}^x\partial_z\breve{A}^x), \qquad (1.11)$$

As a result we have sequentially:

$$\breve{A}^x = -i\breve{E}^x/\omega = -i(\breve{E}_1^x + \breve{E}_2^x)/\omega,\quad \partial_z\breve{A}^x = (\breve{E}_1^x - \breve{E}_2^x). \qquad (1.12)$$

$$\Upsilon^{xyz} = \Re\{(\bar{E}_1^x + \bar{E}_2^x)(\breve{E}_1^x - \breve{E}_2^x)\}/\omega = [|E_1^x|^2 - |E_2^x|^2]/\omega + \Re(-\bar{E}_1^x\breve{E}_2^x + \bar{E}_2^x\breve{E}_1^x)/\omega$$
$$= [|(1+\breve{k}/\omega)|^2 - |(1-\breve{k}/\omega)|^2]E_0^2/4\omega \qquad . \qquad (1.13)$$
$$= [(1+k'/\omega)^2 + (k''/\omega)^2 - (1-k'/\omega)^2 - (k''/\omega)^2]E_0^2/4\omega = k'E_0^2/\omega.$$

You see (1.13) coincides with (1.5).

In the same time, the first addend of spin volume density (1.7), $\varepsilon_0\mathbf{E}\times\mathbf{A}$, is widely used. Jackson [14]: "The term $\varepsilon_0\int d^3x(\mathbf{E}\times\mathbf{A})$ is identified with the 'spin' of the photon". Ohanian [15]: "The term

$$\mathbf{S} = \varepsilon_0\int \mathbf{E}\times\mathbf{A}\, d^3x \qquad (1.14)$$

represents the spin". Friese et al. [16]: "The angular momentum of **a plane electromagnetic wave** can be found from the electric field $\mathbf{E}$ and its complex conjugate $\mathbf{E}^*$ by integrating over all spatial elements d^3r giving $\mathbf{J} = (\varepsilon_0/(2i\omega))\int d^3r\,\mathbf{E}^*\times\mathbf{E}$". Crichton & Marston [17]: "The spin angular momentum density, $s_i = E_j^*(-i\varepsilon_{ijk})E_k/(8\pi\omega)$, is appropriately named in that there is **no moment arm**".

2. Orbital angular momentum of a light beam with plane phase front

However, an angular momentum of another nature exists at the lateral surface of a circularly polarized wave, i.e. at the surface of a circularly polarized beam. The point is that there are longitudinal components of electromagnetic fields near the lateral surface of a wave because the field lines are closed loops [15]. It entails a rotary mass-energy flow and, correspondingly, an orbital angular momentum volume density $\mathbf{l} = (\mathbf{r} \times \mathbf{f})/c^2$, which is determined by the moment arm $\mathbf{r}$.

$$\mathbf{L} = \frac{1}{c^2} \int (\mathbf{r} \times \mathbf{f}) dV \tag{2.1}$$

is the *orbital* angular momentum of the beam.

Heitler [18]: "It can be shown that the wall of a wave packet gives a finite contribution to $\mathbf{L}$". Simmonds and Guttmann [19]: "The electric and magnetic fields can have a nonzero z-component only within the skin region of this wave. Having z-components within this region implies the possibility of a nonzero z-component of angular momentum within this region".

The cylindrical beam has the form [14]

$$\mathbf{E} = \exp(ikz - i\omega t)[\mathbf{x} + i\mathbf{y} + \frac{\mathbf{z}}{k}(i\partial_x - \partial_y)]E_0(r), \quad r^2 = x^2 + y^2, \quad \mathbf{B} = -i\mathbf{E}/c, \tag{2.2}$$

z-component the orbital angular momentum volume density was found to be [20,21]

$$l_z = -\varepsilon_0 r \partial_r E_0^2(r)/2\omega. \tag{2.3}$$

Energy volume density in the beam (2.2) is

$$w = \varepsilon_0 E_0^2. \tag{2.4}$$

Therefore the ratio between the densities,

$$\frac{l_z}{w} = -\frac{r\partial_r E_0^2(r)}{2\omega E_0^2(r)}, \tag{2.5}$$

has a sharp maximum near the beam boundary, in contrast to (1.2).

Despite of the difference in the distributions, spin (1.14) and orbital angular momentum (2.1) of a piece of the beam are equal to each other:

$$\mathbf{S} = \mathbf{L}, \quad \varepsilon_0 \int (\mathbf{E} \times \mathbf{A}) dV = \frac{1}{c^2} \int (\mathbf{r} \times \mathbf{f}) dV. \tag{2.6}$$

Integrating of energy density (2.4) over the same piece as well gives

$$S = L = W/\omega. \tag{2.7}$$

Thus the total angular momentum

$$J = S + L = 2W/\omega. \tag{2.8}$$

3. Moment of momentum is not spin

Famous equality (2.6) is usually referred to as a Humblet equality [22]. On the ground of the equality, an inference was made that spin (1.14) and orbital angular momentum (2.1) are the same *matter* in spite of the fact that they are spatially separated. Ohanian: "This angular momentum (2.1) is the spin of the wave" [15].

Jackson [14] and Becker [23] tried to generalize the equation (2.6) to a free electromagnetic radiation produced by a source localized in a finite region of space. They applied the Humblet transformation with the integration by parts for fields produced a finite time in the past and obtain the equality (2.6).

But they were mistaken! The integration by parts cannot be used when radiating into space. A straight calculation presented in [24] for the radiation of a rotating dipole gives

$$2\mathbf{S} = \mathbf{L}, \quad 2\varepsilon_0 \int \mathbf{E} \times \mathbf{A}\, dV = \frac{1}{c^2} \int (\mathbf{r} \times \mathbf{f}) dV. \tag{3.1}$$

Somewhat such result must be expected because when radiating into space photons are variously directed, and their spins are not parallel to each other as in a beam. As a result, equality (3.1) proves the moment of momentum is not the spin!

The spatial separation of quantities $\varepsilon_0 \mathbf{E} \times \mathbf{A}$ and $(\mathbf{r} \times \mathbf{f})/c^2$ is obvious for a light beam. The separation for the radiation of a rotating dipole is depicted in Figure (partly from [25]). In this case moment of momentum, $(\mathbf{r} \times \mathbf{f})/c^2$, is radiated mainly near the plane of rotating of the dipole (Fig.

a), while spin, $\varepsilon_0 \mathbf{E}\times\mathbf{A}$, exists near the axis of rotation (Fig. c), where the radiation is circularly or elliptically polarized [26].

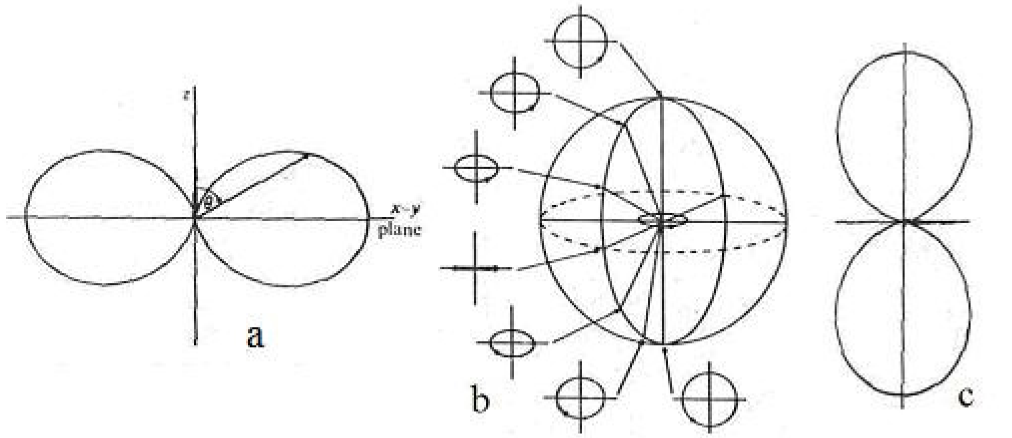

Figure. **(a)** Angular distribution of z-component of the moment of momentum flux, $dL_z / dtd\Omega \propto \sin^2\theta$. **(b)** Polarization of the electric field seen by looking from different directions at the rotating dipole. **(c)** Angular distribution of z-component of the spin flux, $dS_z / dtd\Omega \propto \cos^2\theta$.

Note that our result, $dS_z / dtd\Omega \propto \cos^2\theta$, for the angular distribution of z-component of the spin flux was obtained by Feynman [7] beyond the standard electrodynamics. Really, the amplitudes that a RHC photon and a LHC photon are emitted in the direction θ into a certain small solid angle $d\Omega$ are [7, (18.1), (18.2)]

$$a(1+\cos\theta)/2 \quad \text{and} \quad -a(1-\cos\theta)/2. \tag{3.2}$$

So, in the direction θ, the spin flux density is proportional to

$$[a(1+\cos\theta)/2]^2 - [a(1-\cos\theta)/2]^2 = a^2\cos\theta. \tag{3.3}$$

The projection of the spin flux density on z-axis is

$$dS_z / dtd\Omega \propto a^2\cos^2\theta. \tag{3.4}$$

Thus, according to Feynman, spin (3.3), (3.4) is not a moment of momentum.

There is another important circumstance, which prevents the interpretation of $(\mathbf{r}\times\mathbf{f})/c^2$ as spin density of a *radiation*. The Poynting vector of a radiation is parallel to the wave vector and to the position vector, $(\mathbf{E}\times\mathbf{H})\times\mathbf{k} = \mathbf{f}\times\mathbf{k} = \mathbf{f}\times\mathbf{r} = 0$. Therefore $\mathbf{E}$ & $\mathbf{H}$ - fields used in $\mathbf{L} = \int(\mathbf{r}\times\mathbf{f})dV/c^2$ must be non-radiative fields; they are proportional to $1/r^2$ in the case of a radiation into space. This indicate non-radiative nature of the moment of momentum while spin is an attribute of a radiation and must be calculated by the use of fields, which are proportional to $1/r$ only. Heitler, when defending the spin nature of the moment of momentum, refers to a subtle interference effect on this subject [18]. But this explanation seems to be not convincing.

Conclusion

Simmonds and Guttmann [19] claimed: "A classical quantity associated with the electromagnetic field does not necessarily indicate the value of that quantity which will be measured. The angular momentum density of the wave was zero at the center, yet when we attempted to measure it there the classical field adjusted themselves and produced a nonzero measurement". We explain this magic trick.

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References

1. Sadowsky A. *Acta et Comm. Imp. Universitatis Jurievensis* **7**, No. 1-3 (1899)
2. Poynting J. H., The wave motion of a revolving shaft, and a suggestion as to the angular momentum in a beam of circularly polarised light. *Proc. R. Soc. Lond.* A **82**, 560-567.
3. Becker R., Electromagnetic Fields and Interactions, V. 2, (NY, Dover, 1964)
4. Beth R. A., "Mechanical detection and measurement of the angular momentum of light" *Phys. Rev*. **50**, 115 (1936).
5. Wood R. W. Physical Optics, 2nd ed, p. 341
6. Atkinson R. d'E., "Energy and Angular Momentum in Certain Optical Problems", *Phys. Rev*. **47**, 623-627 (1935).
7. Feynman R. P., R. B. Leighton, M. Sands, The Feynman Lectures on Physics (Addison–Wesley, London, 1965) Vol. 3.
8. Carrara, N. Torque and Angular Momentum of Centimetre Electromagnetic Waves. *Nature* **164** (1949) 882.
9. Khrapko R. I., "Mechanical stresses produced by a light beam," *J. Modern Optics* **55**, 1487-1500 (2008)
10. Khrapko R I, "On the possibility of an experiment on 'nonlocality' of electrodynamics", *Quantum Electron*, 2012, **42** (12), 1133–1136. See also http://khrapkori.wmsite.ru/ftpgetfile.php?id=46&module=files (replies of journals are presented), http://www.mai.ru/science/trudy/published.php?ID=28833 (2012).
11. Khrapko R.I. "Circularly polarized beam carries the double angular momentum" http://www.mai.ru/science/trudy/published.php?ID=34422 (2003) (in Russian).
12. Khrapko R. I.. "True energy-momentum tensors are unique. Electrodynamics spin tensor is not zero," physics/0102084
13. Khrapko R. I.. "Violation of the gauge equivalence," physics/0105031
14. Jackson J. D., Classical Electrodynamics, (John Wiley, 1999), p. 350.
15. Ohanian H. C., "What is spin?" *Amer. J. Phys*. **54**, 500-505 (1986).
16. Friese M. E. J., Nieminen T. A., Heckenberg N. R. & Rubinsztein-Dunlop H., *Nature* **394**, 348–350 (1998)
17. Crichton J. H. and Marston P. L., "The Measurable Distinction Between the Spin and Orbital Angular Momenta of Electromagnetic Radiation" *Electronic Journal of Differential Equations Conf.* 04, 37 (2000).
18. Heitler W., The Quantum Theory of Radiation (Oxford: Clarendon, 1954) p. 401
19. Simmonds J. W., Guttmann M. J., States, Waves and Photons (Addison-Wesley, Reading, MA, 1970)
20. Allen, L.; Padgett, M.J.; Babiker, M. The orbital angular momentum of light. *Progress in Optics XXXIX*; Elsevier: Amsterdam, 1999, p 299.
21. Zambrini, R.; Barnett, S.M. Local transfer of optical angular momentum to matter. *J. Mod. Opt*. **52**: (2005) 1045–1052.
22. Humblet J., "Sur le moment d'impulsion d'une onde electromagnetique". *Physica (Utrecht)* **10** Issue 7, p.585-603 (1943)
23. Becker R., Electromagnetic Fields and Interactions, V. 2, (NY, Dover, 1964), p.320.
24. Khrapko R. I., Spin is not a moment of momentum http://www.mai.ru/science/trudy/published.php?ID=28834 (2012). See also viXra:1302.0077
25. Corney A., Atomic and Laser Spectroscopy (Oxford University Press, 1977)
26. Khrapko R. I.. "Radiation of spin by a rotator," http://www.mai.ru/science/trudy/published.php?ID=34635 (2001), see also http://www.ma.utexas.edu/cgi-bin/mps?key=03-315 (2003), viXra:0703.0044
27. Khrapko R. I., "Does plane wave not carry a spin?" *Amer. J. Phys*. **69**, 405 (2001).

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