Conservation of Entanglement ?

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Dedicated to Marie-Louise Nykamp

Abstract

Recently, [3], it was shown that in certain composite quantum systems with time independent potentials, the extent of the entanglement in an initial state is conserved during the time evolution under the Schrödinger equation, and thus in the absence of any measurement. Here the *extent* of entanglement is meant in the sense of the *grading* function introduced and studied in [1,2]. Based on the celebrated Stone theorem on one parameter groups of unitary operators on Hilbert spaces, the question is raised whether the mentioned conservation of the extent entanglement may hold for composite quantum systems with *arbitrary* potential.

> "Has any thought been given to the number of things that must remain active in men's soul in order that there may still continue to be 'men of science' in real truth ?
> Is it seriously thought that as long as there are dollars there will be science ?
> This notion in which so many find rest is only a

further proof of primitivism."

Jose Ortega y Gassett, "The Revolt of the Masses" (1930)

"Science is nowadays not done scientifically, since it is mostly done by non-scientists ..."

Anonymous

"Science is nowadays not done scientifically, since it is mostly done by ... scientists ..."

Anonymous

"There have been four sorts of ages in the world's history. There have been ages when everybody thought they knew everything, ages when nobody thought they knew anything, ages when clever people thought they knew much and stupid people thought they knew little, and ages when stupid people thought they knew much and clever people thought they knew little. The first sort of age is one of stability, the second of slow decay, the third of progress, and the fourth of disaster.

Bertrand Russel, "On modern uncertainty" (20 July 1932) in Mortals and Others, p. 103-104.

"History is written with the feet ..."

Ex-Chairman Mao, of the Long March fame ...

“Of all things, good sense is the most fairly distributed : everyone thinks he is so well supplied with it that even those who are the hardest to satisfy in every other respect never desire more of it than they already have.” :-) :-) :-)

R Descartes, Discourse de la Méthode

“Creativity often consists of finding hidden assumptions. And removing those assumptions can open up a new set of possibilities ...”

Henry R Sturman

“Physics is too important to be left only to physicists ...”

Anonymous

“Is the claim about the validity of the so called 'physical intuition' but a present day version of medieval claims about the sacro-sanct validity of theoal revelations ?”

Anonymous

“A physical understanding is a completely unmathematical, imprecise, and inexact thing, but absolutely necessary for a physicist ...”

R. Feynman

“I am looking forward very much to getting back to Cambridge, and being able to say what I think and not to mean what I say: two things which at home are impossible. Cambridge is one of the few places where one can talk unlimited nonsense and generalities without anyone pulling one up or confronting one with them when one says just the opposite the next day.”

Bertrand Russell, Letter to Alys Pearsall Smith; published in The Selected Letters of Bertrand Russell, Volume 1: The Private Years (18841914), edited by Nicholas Griffin.

“Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of anything, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true ... If our hypothesis is about anything, and not about some one or more particular things, then our deductions constitute mathematics.
Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

People who have been puzzled by the beginnings of mathematics will, I hope, find comfort in this definition, and will probably agree that it is accurate."

Bertrand Russell, Recent Work on the Principles of Mathematics, published in International Monthly, vol. 4 (1901).

A "mathematical problem" ?

For quite sometime by now, American mathematicians have decided to hide their date of birth and not to mention it in their own academic CV. Why are they so blatantly against transparency in such an academically related matter ?
Can one, therefore, trust American mathematicians, or for that matter, any other professional who behaves like that ?

Amusingly, Hollywood actors and actresses have their birth date easily available on Wikipedia.
On the other hand, Hollywood movies have also for long by now been hiding the date of their production ...

A bemused non-American mathematician

1. Preliminaries

Recently, [1, 2], a non-negative integer valued *grading* function was considered on tensor products in order to distinguish between non-entangled and entangled elements. The essential property of this grad-

ing function is that it gives the *minimally* entangled expression for all entangled elements in a tensor product. A main interest in such a minimal entanglement is in the study of the *variation* of that minimum when the respective elements are *time dependent*, like for instance, when we have a composite quantum system and its state evolves according to a corresponding Schrödinger equation, and does so in the absence of any measurement.
The general case, obviously, is that of the study of entanglement dynamics in arbitrary dynamical systems which evolve in a *tensor product*. It appears that such a case has not been considered so far, not even in the particular situation of composite quantum systems.

In [2], a brief mention of such a *dynamics of entanglement* was made, based on earlier unpublished work of the present author. Here, some of the related details are now presented.

For convenience, first we recall here briefly the way this grading function classifies entangled elements. Namely, the larger the grade of such an element, the higher the extent to which it is entangled, and of course, the other way round. In essence, this is done as follows. Let X and Y be two vector spaces over a field $\mathbb{K}$, then we define

$$(1.1) \qquad gr : X \bigotimes Y \longrightarrow \mathbb{N}$$

where for $u \in X \bigotimes Y$, we have

$$(1.2) \qquad gr(u) = \min\{\, n \mid u = \sum_{i=1}^{n} x_i \otimes y_i, \quad x_i \in X, \quad y_i \in Y \,\}$$

with the convention that $gr(0 \otimes 0) = 0$.

One of the relevant results is that, given $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \bigotimes Y$, then

$$(1.3) \qquad gr(u) = \min\{\, k, h \,\}$$

where k and h are, respectively, the dimensions of the linear span of $\{x_1, \ldots, x_n\}$ in X, and of $\{y_1, \ldots, y_n\}$ in Y.

In particular, $u \in X \bigotimes Y$ is not entangled, if and only if $gr(u) \leq 1$.

Clearly, $gr(u)$ can be computed by well known methods in linear algebra, for instance, methods which give the rank of a matrix.

Also, if X and Y are finite dimensional, then for $u \in X \bigotimes Y$, we have

(1.4) $\quad gr(u) \leq \min\{dim\, X, dim\, Y\}$

A specific feature of the grading function (1.1) - (1.3) is that it is defined exclusively in terms of the respective tensor product $X \bigotimes Y$, and in view of (1.3), in fact, in terms of X and Y alone.

As for obtaining for a given

$$u = \sum_{i=1}^{n} x_i \otimes y_i \in X \bigotimes Y$$

a corresponding *minimum* representation

$$u = \sum_{j=1}^{m} u_j \otimes v_j \in X \bigotimes Y$$

where $m = gr(u) \leq n$, we have the following result, see [1].

Proposition 1.1.

Let X and Y be two vector spaces over a field $\mathbb{K}$, and let $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \bigotimes Y$. If

(1.5) $\quad gr(u) = m < n$,

(1.6) the dimension of the linear span of $\{x_1, \ldots, x_n\}$ is m, and it is less or equal with the dimension of the linear span of $\{y_1, \ldots, y_n\}$,

(1.7) $\{x_1, \ldots, x_m\}$ are linearly independent

then

(1.8) $\quad u = \sum_{i=1}^{m} x_i \otimes v_i$

where

(1.9) $\quad \{v_1, \ldots, v_m\}$ is linearly independent, and it is contained in the linear span of $\{y_1, \ldots, y_n\}$

Furthermore, as seen next in the Proof, one can obtain an *explicit* expression for the linearly independent vectors $\{v_1, \ldots, v_m\}$, as seen in (1.10) below.

Proof.

In view of (1.6), (1.7), we have

$$x_j = \sum_{i=1}^{m} \mu_{j,i}\, x_i, \qquad m < j \leq n$$

where $\mu_{j,i} \in \mathbb{K}$. Hence

$$u = \sum_{i=1}^{m} x_i \otimes y_i + \sum_{j=m+1}^{n} \sum_{i=1}^{m} \mu_{j,i}\, x_i \otimes y_j =$$

$$= \sum_{i=1}^{m} x_i \otimes y_i + \sum_{i=1}^{m} \sum_{j=m+1}^{n} \mu_{j,i}\, x_i \otimes y_j =$$

$$= \sum_{i=1}^{m} x_i \otimes (y_i + \sum_{j=m+1}^{n} \mu_{j,i}\, y_j)$$

Consequently

(1.10) $\quad v_i = y_i + \sum_{j=m+1}^{n} \mu_{j,i}\, y_j, \qquad 1 \leq i \leq m$

and $\{v_1, \ldots, v_m\}$ must be linearly independent in view of (1.8), (1.5).

□

In this paper the above grading function will be applied to the study of the dynamics of composite quantum systems. Namely, let X, Y be complex Hilbert spaces and let S be a quantum system with the state space $X \bigotimes Y$. Then its evolution is given by a one parameter family of *unitary* operators $U(t)$, with $t \in [0, \infty)$, where

$$(1.11) \qquad X \bigotimes Y \ni |\,\psi > \longmapsto U(t)(\,|\,\psi >\,) \in X \bigotimes Y$$

Namely, given any preparation $|\,\psi_0 >$ of the system S at time $t = 0$, then the state of the system at a time moment $t \geq 0$ will be

$$(1.12) \qquad |\,\psi_t > \;=\; U(t)(\,|\,\psi_0 >\,)$$

The *problem* under study in this paper is as follows. We obviously have

$$(1.13) \qquad |\,\psi_0 > \;=\; \sum_{i=1}^{n(0)} x_i(0) \otimes y_i(0) \in X \bigotimes Y$$

while, for $t \geq 0$, we shall have

$$(1.14) \qquad |\,\psi_t > \;=\; U(t)(\,|\,\psi_0 >\,) = \sum_{i=1}^{n(t)} u_i(t) \otimes v_i(t) \in X \bigotimes Y$$

where both $n(0)$ and $n(t)$ are supposed to be *minimal*, namely, we assume that

$$(1.15) \qquad gr(|\,\psi_0 >) \;=\; n(0)$$

$$(1.16) \qquad gr(|\,\psi_t >) \;=\; n(t)$$

and note that $n(t)$ may in general be a *variable* non-negative interger, depending on the time t.

Thus in general

- the state $|\,\psi_t >$ of the composite system S at any moment of time $t \geq 0$ may be *entangled*, namely, whenever $gr(|\,\psi_t >) \;=\; n(t) \geq 2$,

- the extent $gr(|\,\psi_t >) \;=\; n(t)$ of that entanglement may *vary* from one moment of time to another.

We therefore intend to study this variation of the *extent of entanglement*, which in terms of the above notation, is given by the mapping

$$(1.17) \qquad [0, \infty\} \ni t \;\longrightarrow\; gr(|\,\psi_t >) \in \mathbb{N}$$

that is, do so with the help of the grading function gr.

Here one can note from the beginning that, since the grading function gr only takes non-negative integer values, the mapping (1.15) will in general have *discontinuities*. And the closer study of these discontinuities can have mathematical, as well as quantum physical interest.

Let us therefore give a seemingly general definition, as follows :

Definition 1.1.

We call *entanglement dynamics* the situation when given a regular enough, for instance, continuous mapping

$$(1.18) \qquad \mathbb{R} \ni t \longmapsto F(t) \in X \otimes Y$$

with

$$(1.19) \qquad F(t) = x_1(t) \otimes y_1(t) + \ldots + x_{n(t)}(t) \otimes y_{n(t)}(t)$$

where

$$(1.20) \qquad gr(F(t)) = n(t), \qquad t \in \mathbb{R}$$

there may occur a variation in $n(t)$, as t ranges over $\mathbb{R}$.

Remark 1.1.

It is important to clarify the necessary minimal complexity of the notation in (2.4) in the sequel, used for the general form of the solution $F(t)$ of an evolution equation (2.1) - (2.3) in a tensor product. Namely, given two moments of time $0 \leq t_1 < t_2$, we obviously have in general

$$(1.21) \quad F(t_1) = a_1 \otimes b_1 + \cdots + a_n \otimes b_n, \quad F(t_2) = c_1 \otimes d_1 + \cdots + c_m \otimes d_m$$

where $a_i, c_j \in X,\; b_i, d_j \in Y$. Now obviously, a_i, b_i and n may depend

on t_1, while c_j, d_j and m may depend on t_2.
It follows therefore that the notation in (2.4) for the general form of the solution $F(t)$ is minimal in its complexity, although is may be replaced, in case it would be convenient, with the equally minimally complex notation

$$(1.22) \qquad F(t) = x_{t,\ 1} \otimes y_{t,\ 1} + \ldots + x_{t,\ n(t)} \otimes y_{t,\ n(t)}$$

It should be noted that it is the novelty of dynamical systems in tensor products which leads to the usefulness of such a clarification. Dynamical systems in Cartesian products, thus corresponding to classical - and not quantum - composite systems, have a well established and considerably simpler notation for the evolution of their states.

2. A Simple Instance of Possible Entanglement Dynamics

We recall that the evolution of quantum systems which are not subject to measurement is supposed to take place according to the Schrödinger equation. In other words, the state $|\,\psi>$ of a quantum system - a state which is a vector in a suitable Hilbert space H, and which is a square integrable function on a corresponding configuration space given by a finite dimensional Euclidian space E - satisfies a linear partial differential equation, namely the Schrödinger equation, in which the independent variables are the time $t \in \mathbb{R}$, as well as the coordinates $x \in E$ of the respective configuration space.

Our interest here being in *entanglement dynamics*, see its definition at the end of this section, we focus on composite quantum systems which, therefore, have their state space given by suitable tensor products.

At the same time, however, the core of the develepment to follow can easily be extended to general dynamical systems in tensor product spaces.

In view of the above, however, it will help first to have a look at the following more general mathematical formulations of the entanglement dynamics. Indeed, the Schrödinger equation is, in the language of partial differential equations, an *evolution equation*, and then, it can be written as a first order differential equation in the time t, which describes a dynamics taking place in a suitable space of functions in the coordinates $x \in E$ of the corresponding configuration space E of the quantum system considered. And this space of functions is in fact the Hilbert space $\mathcal{L}^2(E)$.

Here however, it will be convenient to star by considering the evolution equations in the more general Banach spaces, and at the convenient stages, to return to the particular case of Hilbert spaces.
Let therefore $(X, || \, ||), (Y, || \, ||)$ be two Banach spaces over a field $\mathbb{K}$. In particular, they can be finite dimensional Euclidean spaces. We first consider *autonomous* first order ODEs in the tensor product space $X \bigotimes Y$, namely of the form

(2.1) $\quad dF(t)/dt = A(F(t)), \quad t \in [0, \infty)$

where

(2.2) $\quad [0, \infty) \ni t \longmapsto F(t) \in X \bigotimes Y$

while

(2.3) $\quad A : X \bigotimes Y \longrightarrow X \bigotimes Y$

The problem is that, in terms of X and Y, the solution of (2.1) - (2.3) will in general be of the form

(2.4) $\quad F(t) = x_1(t) \otimes y_1(t) + \ldots + x_{n(t)}(t) \otimes y_{n(t)}(t)$

And it is quite likely that $x_i(t) \in X, \; y_i(t) \in Y$, as well as $n(t) \in \mathbb{N}$, do indeed all of them depend on t. Thus the situation is of considerable difficulty, since (2.4) means that the ODE in (2.1) - (2.3), when considered in terms of X and Y, will have a *variable* number of unknowns and equations. Furthermore, the representation of the solution $F(t)$ in (2.4) is not unique.

Of course, when instead of (2.1) - (2.4), we have the *classical*, and not quantum, case of the composition of two systems with the respective state spaces X and Y, namely

$$(2.5) \qquad [0, \infty) \ni t \longmapsto F(t) \in X \times Y$$

then instead of (2.4) we have the much simpler form of solution, given by

$$(2.6) \qquad F(t) = (x(t), y(t)) \in X \times Y$$

and thus we simply have a usual system of two ODEs in $X \times Y$, which avoids the possibility of a *variable* number of unknown functions - and thus, equations - as it may in general happen in (2.4).

3. Conservation of the Extent of Entanglement in the Case of a Simple Composite Quantum System

Let us consider two one dimensional quantum systems S and T, with the respective state spaces $X = Y = \mathcal{L}^2(\mathbb{R})$. Then their composite quantum system Q will have the state space $Z = X \bigotimes Y = \mathcal{L}^2(\mathbb{R}) \bigotimes \mathcal{L}^2(\mathbb{R})$. Correspondingly, the evolution of the composite quantum system Q is given by the Schrödinger equation

$$(3.1) \qquad i\hbar \frac{\partial}{\partial t} \psi(x, y, t) = -\big[\frac{\hbar^2}{2m} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + V(x, y, t) \big] \psi(x, y, t)$$

with $x, y \in \mathbb{R}$, $t \in [0, \infty)$, where at any moment of time t, the state of the composite system is given by $| \, \psi_t > \in Z = X \bigotimes Y = \mathcal{L}^2(\mathbb{R}) \bigotimes \mathcal{L}^2(\mathbb{R})$.

Clearly, (3.1) is of the form (2.1) - (2.3), where $A(| \, \psi_t >)$ is the right-hand term in (3.1), divided by the constant $i\hbar$.

Now a general *initial condition* for (2.1) - (2.3) is of the form

$$(3.2) \qquad \begin{aligned} & \psi(x, y, 0) = a(x, y) = \\ & = \textstyle\sum_{1 \leq i \leq n} b_i(x) \otimes c_i(y) \in Z = X \bigotimes Y = \mathcal{L}^2(\mathbb{R}) \bigotimes \mathcal{L}^2(\mathbb{R}) \end{aligned}$$

where $b_i(x) \in X$, $c_i(y) \in Y$. And in view of (1.4), n in (3.2) can be arbitrary large, since X and Y are infinitely dimensional vector spaces.

Clearly, the evolution of the composite quantum system Q will exhibit *entanglement dynamics*, if and only if we shall have

$$(3.3) \qquad gr(a(x,y)) \neq gr(|\psi_t>)$$

for some $t \in (0, \infty)$.

For convenience, let us consider in (3.1) the usual case of the *time independent* potential V, namely

$$(3.4) \qquad i\hbar \frac{\partial}{\partial t}\psi(x,y,t) = -\big[\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + V(x,y)\big]\psi(x,y,t)$$

with $x, y \in \mathbb{R}$, $t \in [0, \infty)$. In this case, as well known, we obtain

$$(3.5) \qquad \psi(x,y,t) = \exp(-\frac{i}{\hbar}Et)\, a(x,y)$$

where $E \in \mathbb{R}$ is a constant such that

$$(3.6) \quad i\hbar \frac{\partial}{\partial t}\psi(x,y,t) = E\psi(x,y,t) = -\big[\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+V(x,y)\big]\psi(x,y,t)$$

Now (3.2), (3.5) give

$$(3.7) \qquad \psi(x,y,t) = \exp(-\frac{i}{\hbar}Et)\, \textstyle\sum_{1 \leq i \leq n} b_i(x) \otimes c_i(y)$$

Since obviously $\exp(-\frac{i}{\hbar}Et) \neq 0$, for $t \in [0, \infty)$, it follows that

$$(3.8) \qquad gr(\psi(x,y,t)) = gr(\textstyle\sum_{1 \leq i \leq n} b_i(x) \otimes c_i(y)) = gr(a(x,y))$$

with $t \in [0, \infty)$, thus according to (3.3), in the case of a *time independent* potential V, the composite quantum system Q does *not* exhibit an entanglement dynamics.

4. The Stone Theorem

Let E be a Hilbert space and U_t, with $t \in \mathbb{R}$, a strongly continuous group of unitary operators on H. Then there exists a self-adjoint operator H on E, such that

$$(4.1) \qquad U_t = \exp(itH), \quad t \in \mathbb{R}$$

Conversely, given the Schödinger equation

$$(4.2) \qquad i\hbar \frac{\partial}{\partial t} \psi_t = H \psi_t, \quad t \in \mathbb{R}, \ \psi_t \in E$$

where H is a self-adjoint Hamiltonian, then the solution is given by the strongly continuous group of unitary operators U_t on H , with $t \in \mathbb{R}$, in (4.1) according to

$$(4.3) \qquad \psi_t = U_t \psi_0, \quad t \in \mathbb{R}$$

5. Is the Extent of Entaglement Conserved in Composite Quantum Systems ?

The above, and in particular, the affirmative result in section 3, leads to the

Question :

Given a composite Hilbert space $E = F \otimes G$ and a strongly continuous group of unitary operators U_t on E, with $t \in \mathbb{R}$. Further, given $\psi \in H$. Is then the case that :

$$(5.1) \qquad gr(U_t \psi) = gr(\psi), \quad t \in \mathbb{R} \ ?$$

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