ABEL RESUMMATION , REGULARIZATION, RENORMALIZATION AND INFINITE SERIES

Jose Javier Garcia Moreta
Graduate student of Physics at the UPV/EHU (University of Basque country)
In Solid State Physics
Addres: Practicantes Adan y Grijalba 2 5 G
P.O 644 48920 Portugalete Vizcaya (Spain)
Phone: (00) 34 685 77 16 53
E-mail: josegarc2002@yahoo.es

- ABSTRACT: We Study the use of Abel summation applied to the evaluation of infinite series and infinite (divergent) integrals , we give several examples of how we can obtain a regularization for the case of divergent sums and integrals.
- *Keywords: =* Abel sum formula,Abel-Plana formula, poles , infinities, renormalization, regularization, multiple integrals, Casimir effect.

Abel summation for divergent series:

Given a power series of the form $\sum_{n=0}^{\infty} a_n x^n$ which is convergent on the region $|x|<1$ we define the Abel resummation of the series $\sum_{n=0}^{\infty} a_n$ as the limit

$\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = A(S)$, if such limit exist we will say that the series $\sum_{n=0}^{\infty} a_n$ is 'Abel summable' to the value A(s).

As an example let be the series [6]

$$\sum_{n=0}^{\infty} (-1) n^k = 1 - 2^k + 3^k - .... = \left(x \frac{d}{dx} \right)^k \frac{1}{1+x} = \frac{2^{k+1}-1}{k+1} B_{k+1} \qquad (1)$$

Unfortunately the series $\sum_{n=0}^{\infty} n^k$ is NOT Abel summable, this is due to the pole at x=1 of the function $(1-x)^{-1}$, however Guo [5] studied this series and gave the following identity using an exponential regulator.

$$\sum_{n=0}^{\infty} n^k e^{-\varepsilon n} = \left(-\frac{d}{d\varepsilon} \right)^k \frac{1}{1-e^{-\varepsilon}} = \frac{\Gamma(k+1)}{\varepsilon^{k+1}} + \sum_{j=0}^{\infty} \frac{Z(-k-j)}{j!} (-\varepsilon)^k \qquad k \neq -1 \qquad (2)$$

Where we have used inside (2) the Taylor expansion involving Bernoulli's number $\frac{x}{e^x-1}=\sum_{j=0}^{\infty}B_j\frac{x^j}{j!}$ and the expression for negative values of the Riemann zeta function $\zeta(1-k)=-\frac{B_k}{k}$.

To evaluate the Riemann zeta inside (2) for negative values we will need the Riemann's functional equation defined by $\zeta(1-s)=2\left(2\pi\right)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$,

with $\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin(\pi s)}$

They introduce an small parameter 'epsilon' and after calculations take $\varepsilon\to 0$, unfortunately for k= -1 Guo's method gives only an infinite answer $\sum_{n=1}^{\infty}n^{-1}e^{-\varepsilon n}=\log\left(\frac{1}{\varepsilon}\right)$, this is because the following expressions for the n-th Harmonic number and for the Laplace transform of the logarithm

$$H_n=1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}\approx\gamma+\log n \qquad \int_0^{\infty}dte^{-\varepsilon t}\log t=-\frac{\gamma+\log\varepsilon}{\varepsilon} \qquad (3)$$

Where $\gamma=0.57721..$ is the Euler-Mascheroni constant.

If we take (2) and ignore the pole part we have that $f.p\left(\sum_{n=0}^{\infty}n^k e^{-\varepsilon n}\right)=\zeta(-k)$ for every k ecept k=-1 , this is precisely the value of the series obtained via Zeta regularization, so Abel resummation and Zeta regularization are linked and give the same answer for the divergent series provide we ignore the poles ε^{-k-1}

To study an example of how the regularization and renormalization of the poles is made we will study the Casimir Effect

- *Casimir effect:*

The Casimir effect is a physical force due to the quantizatio of Electromagnetic fields , see [7] , in the simplest version of the Casimir effect the vacuum Energy of the system per unit of Area 'A' is given by

$$\frac{\langle E\rangle}{A}=\frac{\hbar c}{4\pi^2}\sum_n\int_0^{\infty}2\pi r dr\left|r^2+\frac{n^2\pi^2}{a^2}\right|^{1/2}=-\frac{\hbar c\pi^2}{6a^3}\sum_n|n|^3 \qquad (4)$$

Here, $\hbar=\frac{h}{2\pi}=1.054\times 10^{-34}J.s$ is the reduced Planck's constant and $c=3\times 10^8 m/s$ is the speed of light in the vacuum.

If we use Zeta regularization [3] we find the value $\sum_{n=1}^{\infty} n^3 = \frac{1}{120}$, if we insert this value inside (4) we get the correct experimental value of Casimir effect $\frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{720 a^3}$ so $\frac{F_c}{A} = -\frac{d}{da}\frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{240 a^4}$.

The physicists approach to 'Casimir effect' is a bit more complicate, for example they use renormalization and compute the quantity

$$\langle \delta E \rangle = \langle E_{discrete} \rangle - \langle E \rangle = -\frac{\hbar c \pi^2}{6a^3}\left(\sum_{n=0}^{\infty} n^3 e^{-n\varepsilon} - \int_0^{\infty} dt t^3 e^{-\varepsilon t} \right) \quad (5)$$

This difference can be computed with the aid of the Euler-Maclaurin sum formula

$$\sum_{n=0}^{\infty} f(n) - \int_a^{\infty} f(x)dx = -\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0) \quad f(x) = x^3 e^{-\varepsilon x} \qquad (6)$$

Or using the Abel-Plana sum formula with $\varepsilon \to 0$

$$\sum_{n=0}^{\infty} f(n) - \int_a^{\infty} f(x)dx = \frac{f(0)}{2} + i\int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt \quad f(x) = x^3 e^{-\varepsilon x} \quad (7)$$

If we return to Guo's formula (2) , and we use the identity $\int_0^{\infty} dt e^{-t\varepsilon} t^k = \frac{\Gamma(k+1)}{\varepsilon^{k+1}}$ we find the following.

$$\sum_{n=0}^{\infty} n^k e^{-\varepsilon n} == \sum_{j=0}^{\infty} \frac{Z(-k-j)}{j!} (-\varepsilon)^k + \int_0^{\infty} dt e^{-t\varepsilon} t^k = \frac{\Gamma(k+1)}{\varepsilon^{k+1}} \quad (8)$$

So although the Abel regularization is not valid for the series $\sum_{n=0}^{\infty} n^k$, the difference

$$\Delta = \sum_{n=0}^{\infty} n^k e^{-n\varepsilon} - \int_0^{\infty} dt e^{-t\varepsilon} t^k = \zeta(-k) \quad \varepsilon \to 0 \quad (9)$$

Makes perfect sense and is always FINITE, also for the case k=-1 we find that the Harmonic series is 'summable' and its sum is equal to Euler-Mascheroni constant $\sum_{n=0}^{\infty} n^{-1} = \gamma$ after removing the regulator $e^{-\varepsilon}$.

So, both methods 'renormalization' and zeta regularization gives the same finite answer, however Zeta regularization is an easier and faster method and can be generalized to the case of more general operators, for example

$$E=\frac{1}{2}\hbar c Trace\left(\sqrt{-\Delta}\right) \qquad \Delta=\frac{1}{\sqrt{|g|}}\sum_{i,j}\partial_i\left(\sqrt{|g|}g^{i,j}\partial_j\right) \qquad (10)$$

The operator inside (10) is the Laplace-Beltrami operator and $g=\det\begin{vmatrix} g_{1,1} & g_{1,2} \\ g_{1,2} & g_{2,2}\end{vmatrix}$ is a determinant of a 2x2 matrix , equation (10) is the expression for the vacuum energy of the Laplacian operator in two dimensions.

Abel summation and divergent integrals:

Abel summation formula can be extended to obtain finite results for divergent integrals too, first we need the formula

$$\begin{aligned}\int_a^\infty x^m dx &= \frac{m}{2}\int_a^\infty x^{m-1}dx+\left(\sum_{i=1}^{\infty} i^m\right)-\sum_{i=1}^{a} i^m + a^m \\ &-\sum_{r=1}^{\infty}\frac{B_{2r}\Gamma(m+1)}{(2r)!\Gamma(m-2r+2)}(m-2r+1)\int_a^\infty x^{m-2r}dx\end{aligned} \qquad (11)$$

Where ‘a’ is a positive integer, and the infinite sum inside (9) must be understood in the sense of Abel regularization , so $\sum_{i=1}^{\infty} i^k \to \sum_{n=0}^{\infty} n^k e^{-\varepsilon n}$

Also this recurrence (11) is finite for ‘k’ a positive integer, due to the poles of the Gamma function at the negative integers, in case ‘k’ is a positive and real number the recurrence (11) is infinite and it must be truncated , in this case we can also use the identity $\int_a^\infty \frac{dx}{x^m}=\frac{1}{a^{m-1}}\frac{1}{m-1}$ valid for $\mathrm{Re}(m)>1$

The case m=-1 is not included and must be taken separately, if we take the finite part $f.p\left(\sum_{n=0}^{\infty} n^{-1}e^{-n\varepsilon}\right)=\gamma$ or if we use the expression $f(x)=\frac{e^{-x\varepsilon}}{x+1}$ inside the Euler-Maclaurin summation formula

$$\sum_{n=a+1}^{\infty} f(n)=\int_a^\infty f(x)dx-\frac{f(a)+f(\infty)}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(\infty)-f^{(2k-1)}(a)\right) \qquad (12)$$

And taking into account the following series expansion for the Digamma function

$$\Psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}=\log x+\frac{1}{2x}-\sum_{r=1}^{\infty}\frac{B_{2r}}{2n}\frac{1}{x^{2n}} \qquad \Psi(1)=-\gamma \qquad (13)$$

We get the renormalized result for the integral with a logarithmic divergence in the form $\int_a^\infty \frac{dx}{x+a}\Big|_{renorm} = -\log a$, this means that in a regularized/renormalized sense the 3 integrals $\int_1^\infty \frac{dx}{x+1}$ $\int_0^1 \frac{dx}{x}$ and $\int_0^\infty \frac{dx}{x}$ are equal to 0

For the case k=0 , which is the first term inside the recurrence formula (11) we find that $\int_0^\infty dx = \frac{f(a)+f(\infty)}{2} + \left(\sum_{n=1}^\infty n^0 e^{-n\varepsilon}\right) = \frac{1}{2}$ this is because the value $\zeta(0) = -\frac{1}{2}$ for the Riemann zeta function , so if we take the finite part of the divergent sum we get the finite value $f.p\left(\sum_{n=0}^\infty n^0 e^{-\varepsilon n}\right) = \zeta(0)$

- *Renormalization/regularization theory from divergent series:*

Using Abel summtion and formula (11) we can give an easy method to regularize divergent integrals of the form $\int_0^\infty f(x)dx$, which any person could understand since it uses very simple mathematics, this method of renormalization regularization is based on the resummation of divergent series of power of the positive integers and also on a relationship in the form of a recurrence equation between the divergent integral $\int_0^a x^k dx$ and its discrete divergent series counterpart $\sum_{n=1}^\infty n^k$, the method is the following.

- Split the integral above into a finite part $\int_0^a f(x)dx$ plus a divergent part $\int_a^\infty f(x)dx$, this can always be made
- Expand the integrand inside $\int_a^\infty f(x)dx$ into a Laurent series of the form $\sum_{n=-\infty}^k a_n x^n$ with coefficients given by an integral over the complex plane using Cauchy's theorem [1] $a_n = \frac{1}{2\pi i}\int_C dz \frac{f(z)}{z^{n+1}}$
- Apply integration on each term of $\sum_{n=-\infty}^{-2} a_n x^n$ and the formula $\int_a^\infty \frac{dx}{x^m} = \frac{a^{-m+1}}{m-1}$ which is valid and well defined for $m \geq 2$

- Use the regularization for the Harmonic series $\sum_{n=0}^{\infty} n^{-1} = \gamma$ and of the logarithmic integral $\int_a^{\infty} \frac{dx}{x} = -\log a$ to regularize and give a finite meaning the the divergent logarithmic integral
- Use formula (11) to regularize the divergent integrals $\int_a^{\infty} dx x^m$ for every m=0,1,2,...,k with Abel resummation $\sum_{n=1}^{\infty} n^m e^{-\varepsilon n}$, the 'renormalized' value for every 'm' of this series is just $\left.\sum_{n=1}^{\infty} n^m e^{-\varepsilon n}\right|_{renorm} = \zeta(-m)$ so Abel and Zeta regularization give both the same results, except for the harmonic series
- Another definition of the renormalized infinite series is made with the Abel-plana sum formula, use Abel-Plana formula to compute the renormalized value of the series $\sum_{n=0}^{\infty} e^{-n\varepsilon} - \int_a^{\infty} x^n e^{-x\varepsilon} dx$ when the regulator 'epsilon' is taken to 0 , this results is analogue to zeta regularization.

As an example , let be the divergent integral $\int_0^{\infty} \frac{x^2}{x+c} dx$, with c >0 , the renormalized value of this integral using formula (11) would be

$$\int_0^{\infty} \frac{x^2}{x+c} dx = \int_0^{\infty} x dx - c\int_0^{\infty} dx + c^2 \int_0^{\infty} \frac{dx}{x+c} \to_{reg} \int_0^{\infty} \frac{x^2}{x+c} dx = -c^2 \log c - \frac{c}{2} + \frac{1}{6} \quad (14)$$

A more complicate 2-loop integral $\int_0^{\infty} dx \int_0^{\infty} dy \frac{xy}{x+y+1}$ can be computed with our renormalizatio method based on the regularization and study of divergent series, in this case , the integral has a sub divergence in the variable 'x' which should be renormalized first, the renormalized value of this integral is

$$\int_0^{\infty} dx \int_0^{\infty} dy \frac{x}{x+y+1} = -\int_0^{\infty} dy y^2 (y+1) \int_0^{\infty} \frac{dx}{(x+y+1)(x+1)} + \frac{1}{2}\int_0^{\infty} y dy \qquad (15)$$

The integral inside (15) $\int_0^{\infty} \frac{dx}{(x+y+1)(x+1)} = f(x)$ is finite for every positive 'x' , to simplify the calculations we can replace (approximate) this integral by a quadrature formula with n-points so the sum (quadrature) is easier to work with, for example if we use the Laguerre quadrature formula, valid for $[0,\infty)$ see [1]

$$-y^2(y+1)\int_0^{\infty} \frac{dx}{(x+y+1)(x+1)} \approx -y^2(y+1)\sum_{j=0}^{n} \omega_j \frac{e^{x_j}}{\left(x_j+1+y\right)\left(x_j+1\right)} \quad (16)$$

Now, each term inside (16) depend on 'y' so we have to renormalize the n divergent integrals (n is the number of points of the quadrature formula used) $-\sum_{j=0}^{n}\omega_j\int_0^\infty dy\frac{e^{x_j}(y+1)y^2}{(x_j+1+y)(x_j+1)}$, this has a quartic divergence $\Lambda^4\to\infty$, this can be seen if we introduce a cut-off term in the integral, we have converted a 2-loop integral into an ordinary integral by using a Numerical method and applying the Abel resummation and formula (11) to our original integral $\int_0^\infty dx\int_0^\infty dy\frac{xy}{x+y+1}$

- *Understanding the Casimir effect renormalization and why the divergent series* $\sum_{n=1}^{\infty}n^k=\zeta(-k)$ *has a finite physical value:*

Let be the boundary value problem

$$D_0 f=-\frac{d^2 f}{dx^2}\qquad f(0)=f(\pi)=0\qquad D_0 f=E_n f\qquad E_n=n^2\qquad (17)$$

Then, if we define the operator $T=\sqrt{D_0}$, the sums $\sum_{n=0}^{\infty}n^k$ are the traces of the powers of the operator 'T' in terms of the spectral zeta function of the Energies of the eiganvalue problem inside (17)

$$\sum_{n=0}^{\infty}n^k=Trace\left(T^k\right)=\zeta_T\left(-\frac{k}{2},L=\pi\right)\qquad \zeta_T\left(s,L=\pi\right)=\sum_{n=1}^{\infty}E_n^{-s}=\zeta(2s)\qquad (18)$$

The spectrum of problem (17) is discrete, since we have imposed the boundary conditions for the eigenfunctions $f(0)=f(L=\pi)=0$, if we take the limit $L\to\infty$ the spectrum is no longer discrete and the traces are given by an integral instead of a discrete sum , $Trace\left(T^k\right)_{L\to\infty}=\int_0^\infty t^k dt=\zeta_T\left(-\frac{k}{2},L\right)$,

This integral is still divergent but if we take the difference between the 2 (an exponential regulator is assumed) , and we can define a 'renormalized' value of the divergent series

$$\zeta_T\left(-\frac{k}{2},L=\pi\right)-\zeta_T\left(-\frac{k}{2},L=\infty\right)=\sum_{n=1}^{\infty}n^k e^{-n\varepsilon}-\int_0^\infty dt t^k e^{-t\varepsilon}=\zeta(-k)\qquad (19)$$

And for the case of the Harmonic series, the difference is $\zeta_T\left(\frac{1}{2},L=\pi\right)-\zeta_T\left(\frac{1}{2},L=\infty\right)=\gamma$ which is again a renormalizatio of the divergent Harmonic series, so in the end we have only a finite value.

This method is use in the evaluation of the functional determinant of an operator with a discrete set of eigenvalues $\det(A)=\prod_{n}\lambda_n$, in general the expression $\sum_{n}\log\lambda_n$ is divergent but we can define the logarithm of the functional determinant as the finite difference (substraction of the divergence).

$$\log A - LogC = -\partial_s Z(0,x)+\partial_s Z(0,0) \qquad Z(s,x)=\sum_{n=0}^{\infty}\left(x+\lambda_n\right)^{-s} \quad (20)$$

And ‘C’ is a finite constant, this method is used for example to expand the Gamma function and the sine function into an infinite product over their zeros.

$$\frac{\sqrt{2\pi}}{\Gamma(x+1)}=\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right) \qquad \frac{\sin(\pi x)}{\pi x}=\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) \qquad (21)$$

APPENDIX A: ON THE SERIES $\sum_{n=0}^{\infty}\frac{1}{n+a}$

In this paper we have proved the regularizad value for the series $\sum_{n=1}^{\infty}\frac{1}{n}=\gamma$ in terms of the Euler-Mascheroni constant , then to evaluate the most general Harmonic series one could have used

$$\sum_{n=0}^{\infty}\frac{1}{n+a}=\sum_{n=0}^{\infty}\left(\frac{1}{n+a}-\frac{1}{n+1}\right)+\gamma=-\Psi(a)=-\frac{\Gamma'(a)}{\Gamma(a)} \qquad (A.1)$$

However, Kowalenko [7] proved that (A.1) would be wrong if we compare it to renormalization, in [7] Kowalenko estudied the divergent series $\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{b}}$ for some positive b >0 he obtained from renormalization theory [7] the value

$$\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{b}}=-\Psi\left(\frac{1}{b}\right)-\log b \quad (A.2)$$

If b=1 both (A.2) and (A.1) agree , since $-\Psi(1)=\gamma$ and the logarithm vanishes, the expression (A.2) can be obtained substracting the divergent integral to the divergent logarithmic series in the form

$$\sum_{n=0}^{\infty}\frac{1}{n+a}_{reg} = \lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{1}{n+a}-\int_0^N\frac{dx}{x+a}\right)=-\Psi(a)+\log(a) \quad (A.3)$$

This method is very similar to what we have done to regularize the divergent series $\sum_{n=0}^{\infty} n^k e^{-n\varepsilon}$, to obtaine a finite value $\zeta(-k)$ we take the renormalized value for the series as follows

$$\sum_{n=0}^{\infty} n^k e^{-n\varepsilon} = \lim_{N\to\infty}\left(\sum_{n=0}^{N} n^k e^{-n\varepsilon} - \int_0^N dt t^k e^{-t\varepsilon}\right) = \zeta_R(-k) \quad (A.4)$$

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