A Geometric Understanding of Fermion Rest Masses Linking the Einstein and Dirac Equations via Weyl's Gauge Theory

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Abstract: We demonstrate how fermion rest masses may be understood on a strictly geometric footing, by showing how the Dirac equation is just another form of the Einstein equation for gravitation in curved spacetime, in view of Weyl's theory of gauge (phase) invariance.

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1. Introduction

It is well-understood that the Dirac equation $(i\not{\partial}-m)\psi=0$ may be thought of as the non-trivial square root of the relativistic energy relationship $p_\sigma p^\sigma - m^2 = 0$. For, if one writes this in flat spacetime as $\eta^{\sigma\tau} p_\sigma p_\tau = m^2$ and then applies $\eta^{\sigma\tau} = \frac{1}{2}\left(\gamma^\sigma\gamma^\tau + \gamma^\tau\gamma^\sigma\right) = \frac{1}{2}\left\{\gamma^\sigma,\gamma^\tau\right\}$ where η^{ab} is the contravariant Minkowski metric tensor, one first obtains $\frac{1}{2}\left(\gamma^\sigma\gamma^\tau + \gamma^\tau\gamma^\sigma\right) p_\sigma p_\tau - m^2 = 0$. Using the dagger notation $\not{p} \equiv \gamma^\sigma p_\sigma$ this becomes $\not{p}\not{p} = m^2$. Then, separating the two parts of this square root and using the resulting expression to operate from the left on a Dirac spinor *u*, yields the equation $(\not{p}-m)u=0$ in which the mass *m* represents the eigenvalues of the daggered momentum matrix $\not{p}$. Upon promoting the spinor to a wavefunction $u \to \psi$ simultaneously with making the substitution $\not{p} \to i\not{\partial}$, the new wavefunction equation becomes $(i\not{\partial}-m)\psi=0$, which is Dirac's equation. In essence, this is the path that Dirac followed to derive his equation in [1], [2].

This, in turn is based on the equation $d\tau^2 = g_{\sigma\tau}dx^\sigma dx^\tau$ for the spacetime metric / proper time. For, if one simply converts this to $1 = g_{\sigma\tau}\left(dx^\sigma / d\tau\right)\left(dx^\tau / d\tau\right) = g_{\sigma\tau}u^\sigma \mu^\tau$ where $u^\sigma \equiv dx^\sigma / d\tau$ defines the velocity vector, and then multiplies through by a square mass *m*, then upon further defining the momentum vector $p^\sigma \equiv mu^\sigma$, one obtains $p_\sigma p^\sigma - m^2 = 0$, which is the starting point for obtaining the Dirac equation. However, as one can readily see from this well-known derivation, the mass *m* is introduced entirely by hand. It would be desirable to find a way to obtain Dirac's equation without the hand-introduction of a mass, but rather, to have mass arise spontaneously, based strictly on a deeper understanding of the spacetime geometry.

It turns out that an exercise similar to the above using the Einstein equation $-\kappa \mathrm{T}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ does enable us to do exactly that, namely, to obtain a strictly geometric interpretation of the fermion rest mass in $(i\not{\partial}-m)\psi=0$. At the same time, we come to view Dirac's equation as just a variant of the Einstein equation. Let us now see how this is done.

2. Connecting the Dirac Equation to the Einstein Equation via Weyl's Gauge Theory

The geometric foundation of Einstein's equation springs from the Bianchi identity $\partial_{;\sigma}R^{\alpha\beta}{}_{\mu\nu}+\partial_{;\mu}R^{\alpha\beta}{}_{\nu\sigma}+\partial_{;\nu}R^{\alpha\beta}{}_{\sigma\mu}=0$ of Riemannian geometry, where $\partial_{;\nu}$ is the gravitationally-covariant derivative which makes well-known use of the Christoffel connections $\Gamma^{\mu}{}_{\alpha\beta}$. For, if one does a first index contraction of this identity while noting that $R^{\alpha\beta}{}_{\mu\nu}$ is antisymmetric in μ,ν, one obtains $\partial_{;\sigma}R^{\beta}{}_{\mu}-\partial_{;\mu}R^{\beta}{}_{\sigma}+\partial_{;\alpha}R^{\alpha\beta}{}_{\sigma\mu}=0$ whereby two of the three terms are contracted to the Ricci tensor via $R^{\alpha\beta}{}_{\mu\alpha}=R^{\beta}{}_{\mu}$. A second contraction yields $\partial_{;\sigma}R-\partial_{;\beta}R^{\beta}{}_{\sigma}-\partial_{;\alpha}R^{\alpha}{}_{\sigma}=0$ whereby the Ricci scalar $R=R^{\beta}{}_{\beta}$. With simple index gymnastics this converts over to the very well-known $\partial_{;\alpha}\left(R^{\alpha}{}_{\sigma}-\frac{1}{2}\delta^{\alpha}{}_{\sigma}R\right)=0$. Because we also know that the local conservation of energy is represented via the mixed energy tensor $\mathrm{T}^{\alpha}{}_{\sigma}$ by the equation $\partial_{;\alpha}\mathrm{T}^{\alpha}{}_{\sigma}=0$, one connects this to the contracted Bianchi identity in the form $-\kappa\partial_{;\alpha}\mathrm{T}^{\alpha}{}_{\sigma}=\partial_{;\alpha}\left(R^{\alpha}{}_{\sigma}-\frac{1}{2}\delta^{\alpha}{}_{\sigma}R\right)=0$ which upon integration sans cosmological constant yields $-\kappa\mathrm{T}_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$. [3]

Hermann Weyl teaches [4], [5], [6] that whenever we have a field equation or a Lagrangian for a scalar ϕ or fermion ψ field which includes a term $\partial_{;\mu}\phi$ or $\partial_{;\mu}\psi$, we should subject the field to the *local* gauge (phase) transformation $\phi\rightarrow e^{i\,\theta(x)}\phi$ or $\psi\rightarrow e^{i\,\theta(x)}\psi$ and insist that the field equation or Lagrangian remain invariant under this transformation. What does one do to ensure such invariance? Replace $\partial_{;\mu}\rightarrow D_{;\mu}=\partial_{;\mu}-iG_{;\mu}$. So now, one changes $\partial_{;\mu}\phi\rightarrow D_{;\mu}\phi$ and $\partial_{;\mu}\psi\rightarrow D_{;\mu}\psi$ with the consequence that ϕ or ψ acquires an interaction with the gauge field G_{μ}. If we apply Weyl's gauge recipe to $-\kappa\partial_{;\alpha}\mathrm{T}^{\alpha}{}_{\sigma}=\partial_{;\alpha}\left(R^{\alpha}{}_{\sigma}-\frac{1}{2}\delta^{\alpha}{}_{\sigma}R\right)=0$, then we should promote $\partial_{;\alpha}\rightarrow D_{;\alpha}$ and write this with a bit of index gymnastics as:

$$-\kappa g^{\sigma\tau}D_{;\sigma}\mathrm{T}_{\tau\nu}=g^{\sigma\tau}D_{;\sigma}\left(R_{\tau\nu}-\tfrac{1}{2}g_{\tau\nu}R\right)=0\,. \tag{1}$$

Now, using the $g^{\sigma\tau}$ expressly shown in the above, let us follow the exact same recipe that Dirac used [1], [2] to convert $g^{\sigma\tau}p_{\sigma}p_{\tau}=m^2$ into $(i\not\partial-m)\psi=0$ and see what happens.

Working in curved spacetime, we make use of a *vierbein* field $e^{\sigma}{}_{a}$, [6] where the Greek indexes label general spacetime coordinates and the Latin indexes label the local Lorentz / Minkowski coordinates. The metric tensor is then related to this in the customary manner as:

$$g^{\sigma\tau}=e^{\sigma}{}_{a}e^{\tau}{}_{b}\eta^{ab}=\tfrac{1}{2}e^{\sigma}{}_{a}e^{\tau}{}_{b}\left(\gamma^{a}\gamma^{b}+\gamma^{b}\gamma^{a}\right)=\tfrac{1}{2}\left(\Gamma^{\sigma}\Gamma^{\tau}+\Gamma^{\tau}\Gamma^{\sigma}\right)=\tfrac{1}{2}\left\{\Gamma^{\sigma},\Gamma^{\tau}\right\}, \tag{2}$$

where we define $\Gamma^{\sigma}\equiv\gamma^{a}e^{\sigma}{}_{a}$. This is simply a generalization of $\eta^{\sigma\tau}=\frac{1}{2}\left\{\gamma^{\sigma},\gamma^{\tau}\right\}$ into curved spacetime. We shall continue to employ the usual "Dirac-Dagger" notation, but shall now define this in curved spacetime such that for any arbitrary vector B_{σ}, we have $\not B\equiv\Gamma^{\sigma}B_{\sigma}$.

Let us now turn back to (1) and make use of $g^{\sigma\tau}D_{;\sigma}\mathrm{T}_{\tau\nu}=0$. Making use of (2) in the $g^{\sigma\tau}D_{;\sigma}\mathrm{T}_{\tau\nu}=0$ portion of (1) now enables us to write the anticommutator equation:

$$g^{\sigma\tau}D_{;\sigma}\mathrm{T}_{\tau\nu}=\tfrac{1}{2}\left(\Gamma^{\sigma}\Gamma^{\tau}+\Gamma^{\tau}\Gamma^{\sigma}\right)D_{;\sigma}\mathrm{T}_{\tau\nu}=\tfrac{1}{2}\left(\mathbb{D}\mathbf{T}_{\nu}+\mathbf{T}_{\nu}\mathbb{D}\right)=\tfrac{1}{2}\left\{\mathbb{D},\mathbf{T}_{\nu}\right\}=0\ , \tag{3}$$

where $\mathbf{T}_{\nu}=\Gamma^{\tau}\mathrm{T}_{\tau\nu}$ is the "half-daggered" energy tensor daggered in one index while retaining one free index. We then use this to operate from the left on a Dirac wavefunction as such:

$$\tfrac{1}{2}\left(\mathbf{T}_{\nu}\mathbb{D}+\mathbb{D}\mathbf{T}_{\nu}\right)\psi=\tfrac{1}{2}\left\{\mathbf{T}_{\nu},\mathbb{D}\right\}\psi=0\ . \tag{4}$$

This is the Einstein equation used as an operator equation for fermions, no more, and no less. The bears exactly the same relationship to the energy conservation relation $-\kappa g^{\sigma\tau}D_{;\sigma}\mathrm{T}_{\tau\nu}=0$ linked to geometry via $g^{\sigma\tau}D_{;\sigma}\left(R_{\tau\nu}-\tfrac{1}{2}g_{\tau\nu}R\right)=0$ in (1), which Dirac's equation $\left(i\partial\!\!\!/-m\right)\psi=0$ bears to the metric equation $d\tau^{2}=g_{\sigma\tau}dx^{\sigma}dx^{\tau}$.

This, however, raises the prospect of introducing a fermion rest mass *without* having to do so by hand, and in the process, of obtaining a geometric understanding of this mass which links Einstein's equation to Dirac's. This is in contrast to when we multiply $1=g_{\sigma\tau}u^{\sigma}\mu^{\tau}$ by a hand-added m^{2} without ever explaining anything about the mass *per se*. Specifically, we take Dirac's equation $\left(i\partial\!\!\!/-m\right)\psi=0$, regard this in curved spacetime such that $\partial\!\!\!/=\gamma^{\sigma}\partial_{;\sigma}$, and use Weyl's gauge prescription to introduce a gauge interaction by promoting $\partial\!\!\!/\rightarrow\mathbb{D}$ to write:

$$\left(\mathbb{D}+im\right)\psi=0\ . \tag{5}$$

We have also multiplied through by –*i*, which allows us to contrast (5) directly with (4), as one should now do.

Contrasting, we see that both (4) and (5) contain two added terms. The left hand terms have the respective forms $\mathbf{T}_{\nu}\mathbb{D}\psi$ and $\mathbb{D}\psi$. The right hand terms are $\mathbb{D}\mathbf{T}_{\nu}\psi$ and $im\psi$. This suggests that perhaps the fermion mass can be interpreted via a commutator $\left[\mathbf{T}_{\nu},\mathbb{D}\right]$. Specifically: If we take both (4) and (5) to be true equations, with (4) being the Einstein equation represented in Dirac form with a Weyl supplement $\partial_{;\alpha}\rightarrow D_{;\alpha}$, and with (5) being Dirac's equation with a gauge field in curved spacetime, then we see that (4) and (5) are in fact *one and the same equation*, if and only if:

$$\left[\mathbb{D},\mathbf{T}_{\nu}\right]\psi=\left(\mathbb{D}\mathbf{T}_{\nu}-\mathbf{T}_{\nu}\mathbb{D}\right)\psi=2i\mathbf{T}_{\nu}m\psi\ . \tag{6}$$

Specifically, if we now substitute (6) into (4), we see that

$$\tfrac{1}{2}\left(T_\nu \not{D} + \not{D} T_\nu\right)\psi = \tfrac{1}{2}\left(T_\nu \not{D} + T_\nu \not{D} + 2iT_\nu m\right)\psi = T_\nu\left(\not{D} + im\right)\psi = 0 \ . \tag{7}$$

We then factor out T_ν from the above, and the result is identically equivalent to Dirac's equation represented in the form of (5). So, consolidating (6), we see that:

$$T_\nu m\psi = \tfrac{1}{2}\left[T_\nu, i\not{D}\right]\psi \ . \tag{8}$$

If (8) is true, then Dirac's equation with a gauge field coupling $\partial \to \not{D}$ as in (5) is just a special case of the Einstein equation (1) with Weyl's gauge supplement $\partial \to \not{D}$ as represented in the Dirac form (4).

3. Conclusion

Using the foregoing approach, the fermion rest mass is given a strictly geometric interpretation in terms of the commutation $\left[T_\nu, i\not{D}\right]$ of a half-daggered energy tensor T_ν with the daggered gauge-covariant derivative $\not{D}$ of Weyl. The fermion mass no longer needs to be regarded as something that is added "by hand," and this should help to understand how to "reveal" a fermion rest mass via spontaneous symmetry breaking. The correspondence $\not{p} \leftrightarrow i\not{\partial}$ which is such a familiar part of the quantum mechanical landscape is now shown to have an analogous correspondence $m \leftrightarrow i\not{D}$ with rest mass when used in the form of (8). Finally, with the fermion rest mass now understood as in (8), Dirac's equation written in the form of (4) as $\tfrac{1}{2}\left\{T_\nu, \not{D}\right\}\psi = 0$, is simply a variant of Einstein's equation $-\kappa T_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R$. This is completely analogous to how the usual Dirac equation $\left(i\not{\partial} - m\right)\psi = 0$ is just a variant of the metric relationship $d\tau^2 = g_{\sigma\tau} dx^\sigma dx^\tau$.

References

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