Super-Clifford Gravity, Higher Spins, Generalized Supergeometry and much more

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Abstract

An *extended* Orthogonal-Symplectic Clifford Algebraic formalism is developed which allows the novel construction of a graded Clifford gauge field theory of gravity. It has a direct relationship to higher spin gauge fields, bimetric gravity, antisymmetric metrics and biconnections. In one particular case it allows a plausible mechanism to cancel the cosmological constant contribution to the action. The possibility of embedding these Orthogonal-Symplectic Clifford algebras into an infinite dimensional algebra, coined *Super*-Clifford Algebra is described. Finally, some physical applications of the geometry of *Super*-Clifford spaces to Generalized Supergeometries, Double Field Theories, U-duality, $11D$ supergravity, M-theory, and E_7, E_8, E_{11} algebras are outlined.

Keywords : Super-Clifford algebras; orthogonal Clifford algebras; symplectic Clifford algebras; supersymmetry; Higher spins; Bimetric gravity; Biconnections; Antisymmetric metrics; Cosmological constant; Super Clifford spaces; Generalized Super Geometry; Exceptional algebras.

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1 Orthogonal-Symplectic Clifford Algebras

In the past years it has become more clear that extensions and/or generalizations of supersymmetry are needed. Clifford algebras are essential ingredients to attain such goals. A classification of Hermitian versus holomorphic complex and quaternionic generalized supersymmetries of M-theory was attained in [12], including the 12-dim Euclidean generalized supersymmetric F algebra and the 11-dim M theory superalgebra.

Polyvector super Poincare algebras like

$$\{ Q_\alpha, \ Q_\beta \} \ = \ \sum_k \ (C\gamma^{\mu_1\mu_2....\mu_k})_{\alpha\beta} \ W^{(k)}_{\mu_1\mu_2.....\mu_k} \tag{1}$$

were studied by [13]. The summation over k must obey certain crucial restrictions to match the degrees of freedom with the terms in the left hand side and to ensure that there is symmetry under the exchange of spinorial α, β indices. C is the charge conjugation matrix and $W^{(k)}_{\mu_1\mu_2.....\mu_k}$ are the polyvector-valued momentum like generators. Polyvector valued extensions of supersymmetry in Clifford Spaces involving spinor-tensorial supercharge generators $Q_\alpha^{\mu_1\mu_2....\mu_n}$ and momentum polyvectors $P_{\mu_1\mu_2....\mu_n}$ were analyzed in [15], [16]. Clifford-Superspace is an extension of Clifford-space and whose symmetry transformations are generalized polyvector-valued supersymmetries.

The superconformal algebra $su(2,2|1)$ in $4D$ can be realized in terms of 5×5 matrices [1] and whose entries are given explicitly in terms of the gamma matrices γ_a. The momentum, conformal boost, Lorentz and dilation generators are realized as the entries of the 4×4 matrices embedded into 5×5 matrices by setting all the entries of the 5-th column and 5-th rows to zero while identifying the entries of the 4×4 matrices by

$$(P_a)_\alpha^{\ \beta} \ = \ - \ \frac{1}{2} \ \gamma_a \ (1 \ - \ \gamma_5)_\alpha^{\ \beta}, \quad (K_a)_\alpha^{\ \beta} \ = \ - \ \frac{1}{2} \ \gamma_a \ (1 \ + \ \gamma_5)_\alpha^{\ \beta}, \quad a,b = 1,2,3,4. \tag{2a}$$

The Lorentz and dilation generator are

$$(J_{ab})_\alpha^{\ \beta} \ = \ \frac{1}{2} \ \gamma_{ab} \ = \ \frac{1}{4} \ [\gamma_a, \ \gamma_b]_\alpha^{\ \beta}, \quad (D)_\alpha^{\ \beta} \ = \ - \ \frac{1}{2} \ (\gamma_5)_\alpha^{\ \beta}, \quad \alpha,\beta = 1,2,3,4. \tag{2b}$$

The axial charge A generator is represented by $-\frac{i}{4}$ times a diagonal 5×5 matrix whose entries are $(1,1,1,1,4)$.

The 8 fermionic generators Q_α, S_α are represented by 5×5 matrices with zeros along the 4×4 block matrices and whose only nonzero entries are along the 5-th column and 5-th rows as follows

$$(Q_\alpha)^{5\beta} \ = \ - \ \frac{1}{2} \ (1 \ - \ \gamma_5)_\alpha^{\ \beta}, \quad (Q_\alpha)^{\beta 5} \ = \ \frac{1}{2} \ [\ (1 \ + \ \gamma_5) \ C]_\alpha^{\ \beta}, \quad (Q_\alpha)^{55} \ = \ 0 \tag{3a}$$

$$(S_\alpha)^{5\beta} \ = \ \frac{1}{2} \ (1 \ + \ \gamma_5)_\alpha^{\ \beta}, \quad (S_\alpha)^{\beta 5} \ = \ - \ \frac{1}{2} \ [\ (1 \ - \ \gamma_5) \ C]_\alpha^{\ \beta}, \quad (S_\alpha)^{55} \ = \ 0 \tag{3b}$$

where C is the charge conjugation matrix obeying $C = -C^{-1} = -C^T, \ C\gamma_a C^{-1} = -(\gamma_a)^T$ where T denotes the transpose. In the representation chosen above $C = \gamma_0$. The authors [1] have shown explicitly that the above realization of the bosonic and fermionic generators in terms of gamma matrices obey the superconformal $su(2,2|1)$ algebra graded commutator relations. More recently, a 6×6 matrix realization of the $osp(1|4)$ superalgebra was provide by [5]. The noncompact symplectic algebra $sp(2,2)$ is isomorphic to $so(3,2)$. The (Anti) de Sitter group is $so(3,2), so(4,1)$ respectively.

A $(4+N)\times(4+N)$ matrix realization of the N-extended superconformal algebra $su(2,2|N)$ algebra (whose even part is $su(2,2)\oplus u(N)$) is also possible. In particular, a $(4+N)\times(4+N)$ matrix realization of the N spinorial supercharges $Q^i_\alpha, S^i_\alpha; i=1,2,....,N$ can be given by a generalization of eqs-(3a, 3b)

$$(Q^i_\alpha)^{AB} = -\frac{1}{2}(1-\gamma_5)_\alpha^{\ B}\ \delta^{4+i\ A},\ \ (Q^i_\alpha)^{BA} = \frac{1}{2}[(1+\gamma_5)\ C]^B_\alpha\ \delta^{4+i\ A} \qquad (4a)$$

$$(S^i_\alpha)^{AB} = \frac{1}{2}(1+\gamma_5)^B_\alpha\ \delta^{4+i\ A},\ \ (S^i_\alpha)^{BA} = -\frac{1}{2}[(1-\gamma_5)\ C]^B_\alpha\ \delta^{4+i\ A} \qquad (4b)$$

the other matrix components are zero, for instance

$$(Q^i_\alpha)^{4+j\ 4+j} = 0, \quad (S^i_\alpha)^{4+j\ 4+j} = 0, \quad i,j=1,2,...N \qquad (4c)$$

when $i,j=1$ one recovers the 5×5 matrix realization of eqs-(3a, 3b).

We will go beyond this ordinary description of Lie superalgebras, like $su(2,2|1)$, in terms of the gamma matrices as displayed above, by incorporating both orthogonal *and* symplectic Clifford algebras into the framework of super Clifford algebras and which $differs$ from the notion of super Clifford algebras studied earlier by [3]. Orthogonal Clifford algebras are well known. What is less known is the notion of symplectic Clifford algebras [2]. A Clifford analysis approach to Superspace based on orthogonal and symplectic Clifford algebras was undertaken by [6]. A theory of quantized fields based on orthogonal and symplectic Clifford Algebras' can be found in [8]. Super Clifford algebras with a Z_4 grading and generalized Clifford algebras, orthogonal and symplectic were constructed by [3], [4]. We shall take a quite different approach and extend further the work of these authors. To our knowledge the results of this work are new.

We begin by introducing the ordinary orthogonal Clifford algebra generators $\gamma_a, a = 1,2,3,...m$, and the symplectic Clifford algebra generators $\xi_i, i=1,2,3,.....,2n$ obeying the graded commutation relations

$$\frac{1}{2}\{\ \xi_i,\ \xi_j\ \} = \xi_{ij} = \xi_{ji},\ \ [\ \xi_i,\ \xi_j\ ] = \omega_{ij} = -\ \omega_{ji};\ \ i,j=1,2,3,...,2n \qquad (5a)$$

$$\omega_{2k_1\ 2k_2} = \omega_{2k_1-1\ 2k_2-1} = 0 \qquad (5b)$$

$$\omega_{2k_1-1\ 2k_2} = -\ \omega_{2k_2\ 2k_1-1} = \delta_{k_1k_2} \qquad (5c)$$

ω_{ij} is a $2n\times 2n$ antisymmetric matrix consisting of diagonal blocks of 2×2 antisymmetric matrices whose nonzero entries are ± 1. The relation $\frac{1}{2}\{\ \xi_i,\ \xi_j\ \} = \xi_{ij} = \xi_{ji}$ and its implications was not considered by [6], [7]. The other commutators are

$$\frac{1}{2}\{\ \gamma_a,\ \gamma_b\ \} = g_{ab}\ \mathbf{1};\ \ \frac{1}{2}[\ \gamma_a,\ \gamma_b\ ] = \gamma_{ab} = -\ \gamma_{ba},\ \ a,b=1,2,3,....,m \qquad (6)$$

$$[\ \gamma_a,\ \gamma_{bc}\ ]\ =\ 2\ g_{ab}\ \gamma_c\ -2\ g_{ac}\ \gamma_b \tag{7}$$

$$[\ \gamma_{ab},\ \gamma_{cd}\ ]\ =\ -2\ g_{ac}\ \gamma_{bd}\ +2\ g_{ad}\ \gamma_{bc}\ -2\ g_{bd}\ \gamma_{ac}\ +2\ g_{bc}\ \gamma_{ad} \tag{8}$$

$$[\ \gamma_a,\ \xi_i\ ]\ =\ 0,\ \ [\ \gamma_{ab},\ \xi_i\ ]\ =\ 0,\ \ [\ \gamma_a,\ \xi_{ij}\ ]\ =\ 0,\ \ [\ \gamma_{ab},\ \xi_{ij}\ ]\ =\ 0 \tag{9}$$

$$[\ \xi_i,\ \xi_{jk}\ ]\ =\ \omega_{ij}\ \xi_k\ +\ \omega_{ik}\ \xi_j \tag{10}$$

$$[\ \xi_{ij},\ \xi_{kl}\ ]\ =\ \omega_{ik}\ \xi_{jl}\ +\ \omega_{il}\ \xi_{jk}\ +\ \omega_{jk}\ \xi_{il}\ +\ \omega_{jl}\ \xi_{ik} \tag{11}$$

We take all the generators of the orthogonal Clifford algebra to be of even grade, while the ξ_i generator has *odd* grade and ξ_{ij} has even grade. Afterwards we shall study the case where all the odd-rank generators of the orthogonal Clifford algebra have an *odd* grade; and all the even-rank generators have *even* grade. For instance, if the grade of γ_a is chosen to be odd, one must replace $[\gamma_a, \xi_i] = 0$ with $\{\gamma_a, \xi_i\ \} = 0$ in eq-(9). The graded commutator is defined as

$$[\ A,\ B\ \}\ =\ A\ B\ -\ (-1)^{(grade\ A\ grade\ B)}\ B\ A \tag{12}$$

where the grade of the even part of the superalgebra is 0, and the grade of the odd part of the superalgebra is 1. Denoting the grades of A, B, C respectively by a, b, c, the graded Jacobi identities are given by

$$(-1)^{ac}\ [\ A,\ [\ B,\ C\ \}\ \}\ +(-1)^{ba}\ [\ B,\ [\ C,\ A\ \}\ \}\ +(-1)^{cb}\ [\ C,\ [\ A,\ B\ \}\ \}\ =\ 0 \tag{13}$$

the above graded Jacobi identity can be also rewritten in terms of the superalgebra of derivations as

$$[\ A,\ [\ B,\ C\ \}\ \}\ =\ [\ [\ A,\ B\ \},\ C\ \}\ +(-1)^{ab}\ [\ B,\ [\ A,\ C\ \}\ \} \tag{14}$$

In the Appendix we show explicitly that the graded Jacobi identities corresponding to the superalgebra are satisfied.

For simplicity let us take at the moment $m = 2$ and $2n = 2$ and define

$$\mathbf{A}_\mu\ =\ A_\mu\ +\ A^a_\mu\ \gamma_a\ +\ A^{ab}_\mu\ \gamma_{ab}\ +\ A^i_\mu\ \xi_i\ +\ A^{ij}_\mu\ \xi_{ij};\quad a, b = 1, 2;\quad i, j = 1, 2 \tag{15}$$

since ξ_i belongs to the *odd* part of the algebra, the gauge field component A^i_μ is taken to be anticommuting; i.e. it is Grassmannian odd, an a-number. The other field components are commuting; i.e. they are Grassmannian even, a c-number. Because $\frac{1}{2}\{\xi_i, \xi_j\} = \xi_{ij} = \xi_{ji}$, the latter belongs to the even part of the algebra. Also we have $A^{ab}_\mu = -A^{ba}_\mu$ and $A^{ij}_\mu = A^{ji}_\mu$.

The field strength is defined as $\mathbf{F} = d\mathbf{A} + \frac{1}{2}\mathbf{A} \wedge \mathbf{A}$. One should note that due to the anticommuting nature of the gauge field component A^i_μ one has $A^i_\mu\ A^j_\nu = -A^j_\nu\ A^i_\mu$ so that wedge product

$$\frac{1}{2}\ \Big(\ A^i_\mu\ A^j_\nu\ \xi_i\ \xi_j\ -\ A^j_\nu\ A^i_\mu\ \xi_j\ \xi_i\ \Big)\ dx^\mu \wedge dx^\nu =\ \frac{1}{2}\ A^i_\mu\ A^j_\nu\ \{\ \xi_i,\ \xi_j\ \}\ dx^\mu \wedge dx^\nu =$$

$$A^i_\mu\ A^j_\nu\ \xi_{ij}\ dx^\mu \wedge dx^\nu \tag{16}$$

involves the *anticommutator* $\{\xi_i, \xi_j\}$ rather than the commutator $[\xi_i, \xi_j]$. Hence, the field strength component associated to the ξ_{ij} generator is given by

$$F^{ij}_{\mu\nu}\ =\ \partial_\mu\ A^{ij}_\nu\ -\ \partial_\nu\ A^{ij}_\mu\ +\ A^i{}_{k[\mu}\ A^{kj}_{\nu]}\ + \frac{1}{2}\ (A^i_\mu\ A^j_\nu\ +\ A^j_\mu\ A^i_\nu),\quad A^{ij}_\mu = A^{ji}_\mu \tag{17}$$

such that

$$F^{ij}_{\mu\nu}\ =\ -\ F^{ij}_{\nu\mu},\quad F^{ij}_{\mu\nu}\ =\ F^{ji}_{\nu\mu} \tag{18}$$

due to the Grassmannian odd character $A^i_\mu A^j_\nu = -A^j_\nu A^i_\mu$. The other components are

$$F^{ab}_{\mu\nu}\ =\ \partial_\mu\ A^{ab}_\nu\ -\ \partial_\nu\ A^{ab}_\mu\ + 2\ A^a{}_{c[\mu}\ A^{cb}_{\nu]}\ + \frac{1}{2}\ A^a_{[\mu}\ A^b_{\nu]},\quad A^{ab}_\mu = -A^{ba}_\mu \tag{19}$$

$$F^i_{\mu\nu}\ =\ \partial_\mu\ A^i_\nu\ -\ \partial_\nu\ A^i_\mu\ +\ A^i{}_{j[\mu}\ A^j_{\nu]} \tag{20}$$

$$F^a_{\mu\nu}\ =\ \partial_\mu\ A^a_\nu\ -\ \partial_\nu\ A^a_\mu\ + 2\ A^a{}_{b[\mu}\ A^b_{\nu]} \tag{21}$$

$$F_{\mu\nu}\ =\ \partial_\mu\ A_\nu\ -\ \partial_\nu\ A_\mu \tag{22}$$

where the $[\mu\nu]$ denotes antisymmetrization of the indices with unit weight.

A Yang-Mills like invariant Lagrangian is quadratic in the field strengths $< \mathbf{F}_{\mu\nu}\ \mathbf{F}^{\mu\nu} >$ where the bracket $< \quad >$ denotes extracting the unit element of the superalgebra in the product of two generators as follows

$$< \gamma_a\ \gamma_b\ >\ =\ g_{ab},\quad <\ \gamma_{ab}\ \gamma_{cd}\ >\ =\ g_{ac}\ g_{bd}\ -\ g_{bc}\ g_{ad},\quad <\ \gamma_a\ \gamma_{cd}\ >\ =\ 0 \tag{23a}$$

$$<\ \xi_i\ \xi_j\ >\ =\ \frac{1}{2}\ \omega_{ij},\quad <\ \xi_i\ \xi_{jk}\ >\ =\ 0 \tag{23b}$$

$$<\ \xi_{ij}\ \xi_{kl}\ >\ =\ \frac{1}{4}\ (\ \omega_{ik}\ \omega_{jl}\ +\ \omega_{il}\ \omega_{jk}\ ) \tag{23c}$$

To obtain eqs-(23) one needs the initial (anti) commutators and the relations

$$\{\ \gamma_{ab},\ \gamma_{cd}\ \}\ =\ 2\ \gamma_{abcd}\ - 2\ g_{ac}\ g_{bd}\ - 2\ g_{bc}\ g_{ad},\quad \{\gamma_{ab},\ \gamma_c\}\ =\ 2\ \gamma_{abc} \tag{24}$$

$$\{\ \xi_{ij}\ ,\ \xi_k\ \}\ =\ \xi_{ij,k},\quad \{\ \xi_{ij}\ ,\ \xi_{kl}\ \}\ =\ \xi_{ij,kl}\ + \frac{1}{2}\ (\ \omega_{ik}\ \omega_{jl}\ +\ \omega_{il}\ \omega_{jk}\ ) \tag{25}$$

$$[\xi_{kl,j} \ , \ \xi_i] \ = \ \xi_{ij,kl}, \ \ \{ \ \xi_{kl,j} \ , \ \xi_i \ \} \ = \ \omega_{ki} \ \xi_{lj} \ + \ \omega_{li} \ \xi_{kj} \ + \ \omega_{ji} \ \xi_{kl} \tag{26}$$

obtained from the identities

$$[A, \ BC] \ = \ [A, \ B] \ C \ + \ B \ [A, \ C], \ \ \{A, \ BC\} \ = \ \{A, \ B\} \ C \ - \ B \ [A, \ C] \tag{27}$$

One must introduce new generators in eqs-(25,26) obeying

$$\xi_{ij,k} \ = \ \xi_{ji,k} \ = \ \xi_{k,ij} \ = \ \xi_{k,ji} \tag{28a}$$

$$\xi_{ij,kl} \ = \ \xi_{kl,ij} \ = \ \xi_{ji,kl} \ = \ \xi_{ij,lk} \ = \ \xi_{ji,lk} \ = \ \xi_{lk,ji} \tag{28b}$$

and as a result one may *enlarge* the number of terms in the decomposition of the field $\mathbf{A}_\mu$ in eq-(15), using these new generators $\xi_{ij,k}$ and $\xi_{ij,kl}$. The only *caveat* is that the super algebra will *not* close because the graded commutators

$$[\ \xi_{ij,kl} \ , \ \xi_{mn,p} \ \} \ = \ [\ \xi_{ij,kl} \ , \ \xi_{mn,p} \ ] \ \sim \ \omega_{im} \ \xi_{jn,klp} \ + \ .... \tag{29}$$

$$[\ \xi_{ij,kl} \ , \ \xi_{mn,pq} \ \} \ = \ [\ \xi_{ij,kl} \ , \ \xi_{mn,pq} \ ] \ \sim \ \omega_{im} \ \xi_{jn,klpq} \ + \ .... \tag{30}$$

furnish the *new* generators $\xi_{jn,klp}, \ \xi_{jn,klpq}$ which were *not* included . The other *graded* commutators are fine because no new generators are introduced in

$$[\xi_{kl,j}, \ \xi_i\} \ = \ \{\xi_{kl,j}, \ \xi_i\} \sim \omega_{ki} \ \xi_{lj} \ + \ ...., \ \ [\xi_{ij,k} \ , \ \xi_{lm}] \ \sim \ \omega_{il} \ \xi_{jm,k} \ + \ ....$$

$$[\xi_{ij,kl} \ , \ \xi_{mn}] \ \sim \ \omega_{im} \ \xi_{jn,kl} \ + \ ....., \ \ \{\xi_{ij,k} \ , \ \xi_{lm,n}\} \ \sim \ \omega_{il} \ \xi_{jm,kn} \ + \ ..... \tag{31}$$

However, by just including $A^{ij,k}_\mu \ \xi_{ij,k}$ and excluding $A^{ij,kl}_\mu \ \xi_{ij,kl}$, one will have another problem due to the last anticommutator relation $\{\xi_{ij,k} \ , \ \xi_{lm,n}\} \ \sim \ \omega_{il} \ \xi_{jm,kn}$ which will preclude the exclusion of $\xi_{jm,kn}$. Therefore one will be forced to introduce an *infinite* number of generators beyond ξ_i and ξ_{ij}. The new field $A^{ij,k}_\mu$, for example, is Grassmannian *odd*, it is an a-number. Its addition will also modify the expression for the terms in $F^{ij}_{\mu\nu} \ \xi_{ij}$ due to the anticommutator contribution of the first terms of eq(31).

In the orthogonal Clifford algebra $Cl(m)$ case the introduction of new generators stops when the number of different factors in the antisymmetric product of the gammas equals to m, the dimension of spacetime; i.e the rank of an antisymmetric tensor cannot exceed the spacetime dimension.

Concluding, by just keeping ξ_i, ξ_{ij} one can explicitly verify that the super algebra closes with respect to the graded commutation relations and the graded Jacobi identities are obeyed as shown in the Appendix. In this case we have the orthogonal Clifford algebra generators $\mathbf{1}, \gamma_a, \gamma_{ab}$ and the ξ_i, ξ_{ij} generators associated with the symplectic Clifford algebra. In the most general case when $m > 2$, $2n > 2$ we have a super-Clifford valued gauge field associated with the ortho-symplectic Clifford algebras of the form

$$\mathbf{A}_\mu \ = \ A_\mu \ + \ A^a_\mu \ \gamma_a \ + \ A^{a_1a_2}_\mu \ \gamma_{a_1a_2} \ + \ A^{a_1a_2a_3} \ \gamma_{a_1a_2a_3} \ + \ ..... \ + \ A^{a_1a_2a_3...a_m} \ \gamma_{a_1a_2a_3....a_m} \ +$$

$$A^i_\mu \ \xi_i \ + \ A^{i_1i_2}_\mu \ \xi_{i_1i_2} \ + \ A^{i_1i_2,j_1}_\mu \ \xi_{i_1i_2,j_1} \ + \ A^{i_1i_2,j_1j_2}_\mu \ \xi_{i_1i_2,j_1j_2} \ + \ ...... \tag{32}$$

The range of indices is $a_1, a_2, a_3, ... = 1, 2, 3, ..., m$ and $i_1, i_2, ...j_1, j_2... = 1, 2, 3, ..., 2n$. The (anti) commutators of the $\gamma_a, \gamma_{a_1 a_2}, ..., \gamma_{a_1 a_2 .....a_m}$ generators can be found in the monograph [23]. The (anti) commutators of the infinite number of ξ's generators would require a computer algebra package. One expects to have analogous formulae to those in eqs-(5,10,11,25,26,29,30,31).

The infinite number of terms in (32) resembles the expansion based on the frame-like formalism, developed by [17], [18], describing the higher spin fields dynamics in terms of higher spin connection gauge fields that generalize objects such as vielbeins and spin connections in gravity. Constructing consistent gauge theories of interacting higher spin fields is a difficult and unsolved problem.

Higher spin fields [18] have been conjectured to be part of the spectrum of *tensionless* strings. Higher conformal spin field currents generate the W_∞ (super) algebras which are the higher spin extensions of the (super) Virasoro algebra in $2D$. The analog of "photons" in the Extended Relativity theory in C-spaces (Clifford spaces) correspond to *tensionless* strings and branes [11]. The quantum Virasoro algebra generators can be constructed in terms of operators of the generalized Clifford algebras as shown by [24]. For these reasons the gauge field theory proposed in this work deserves further scrutiny.

For simplicity purposes and without loss of generality we shall retain the original decomposition of the field $\mathbf{A}_\mu$ displayed in eq-(15). To sum up, from eqs-(17-22) one has for the quadratic Yang-Mills like invariant Lagrangian the following

$$\mathcal{L} \;=\; \frac{1}{2}\,\omega_{ij}\;F^i_{\mu\nu}\;F^j_{\rho\sigma}\;g^{\mu\rho}\;g^{\nu\sigma}\;+\frac{1}{2}\,\omega_{ik}\,\omega_{jl}\;F^{ij}_{\mu\nu}\;F^{kl}_{\rho\sigma}\;g^{\mu\rho}\;g^{\nu\sigma}\;+$$
$$F^a_{\mu\nu}\;F^{\mu\nu}_a\;-2\;F^{ab}_{\mu\nu}\;F^{\mu\nu}_{ab}\;+\;F_{\mu\nu}\;F^{\mu\nu} \tag{33}$$

One can rewrite

$$\omega_{ij}\;F^i_{\mu\nu}\;F^j_{\rho\sigma}\;g^{\mu\rho}\;g^{\nu\sigma}\;=\;F^i_{\mu\nu}\;F^{\mu\nu}_i \tag{34}$$

and

$$\omega_{ik}\;\omega_{jl}\;F^{ij}_{\mu\nu}\;F^{kl}_{\rho\sigma}\;g^{\mu\rho}\;g^{\nu\sigma}\;=\;-\;F^{ij}_{\mu\nu}\;F^{\mu\nu}_{ij} \tag{35}$$

the *change* in sign in (35) is due to the subtle ordering of indices in the contraction

$$\omega_{ik}\;F^{kl}\;\omega_{jl}\;=\;-\;\omega_{ik}\;F^{kl}\;\omega_{lj}\;=\;-\;F_{ij} \tag{36}$$

the index contraction is performed among adjacent indices. One should note that $\delta_{ij}F^i_{\mu\nu}F^{j\mu\nu} = 0$ due to the Grassmannian odd character of $F^i_{\mu\nu}$. For this reason one must contract the i, j indices with $\omega_{ij} = -\omega_{ji}$. The contraction of the vector indices in F^aF^b requires the metric g_{ab} and the contraction of *bivector* indices in $F^{ab}F^{cd}$ requires $g_{ac}g_{bd} - g_{bc}g_{ad}$ as displayed in eq-(23a).

The Lagrangian (33) is invariant under the super gauge transformations involving the super Clifford algebra valued gauge parameter

$$\Lambda\;=\;\lambda\;+\;\lambda^a\;\gamma_a\;+\;\lambda^{ab}\;\gamma_{ab}\;+\;\lambda^i\;\xi_i\;+\;\lambda^{ij}\;\xi_{ij};\quad \lambda^{ab} = -\lambda^{ba},\quad \lambda^{ij} = \lambda^{ji} \tag{37}$$

$$\delta\mathbf{A}_\mu\;=\;\partial_\mu\Lambda\;+\;[\mathbf{A}_\mu,\;\Lambda\},\quad \delta\mathbf{F}_{\mu\nu}\;=\;[\mathbf{F}_{\mu\nu},\;\Lambda\} \tag{38}$$

In component form one has

$$\delta A^i_\mu \;=\; \partial_\mu \lambda^i \;+2\; A^{ij}_\mu\; \lambda_j \;+2\; A^j_\mu\; \lambda_j^{\ i}, \;\; \delta F^i_{\mu\nu} \;=\; 2\; F^{ij}_{\mu\nu}\; \lambda_j \;+2\; F^j_{\mu\nu}\; \lambda_j^{\ i} \tag{39}$$

$$\delta A^a_\mu \;=\; \partial_\mu \lambda^a \;+4\; A^{ab}_\mu\; \lambda_b \;+4\; A^b_\mu\; \lambda_b^{\ a}, \;\; \delta F^a_{\mu\nu} \;=\; 4\; F^{ab}_{\mu\nu}\; \lambda_b \;+4\; F^b_{\mu\nu}\; \lambda_b^{\ a} \tag{40}$$

$$\delta A^{ab}_\mu \;=\; \partial_\mu \lambda^{ab} \;+4\; A^{ac}_\mu\; \lambda_c^{\ b} \;-4\; A^{bc}_\mu\; \lambda_c^{\ a} \;+\; A^a_\mu\; \lambda^b \;-\; A^b_\mu\; \lambda^a \tag{41}$$

$$\delta F^{ab}_\mu \;=\; 4\; F^{ac}_{\mu\nu}\; \lambda_c^{\ b} \;-4\; F^{bc}_{\mu\nu}\; \lambda_c^{\ a} \;+\; F^a_{\mu\nu}\; \lambda^b \;-\; F^b_{\mu\nu}\; \lambda^a \tag{42}$$

$$\delta A^{ij}_\mu \;=\; \partial_\mu \lambda^{ij} \;+2\; A^{ik}_\mu\; \lambda_k^{\ j} \;+2\; A^{jk}_\mu\; \lambda_k^{\ i} \;+\; A^i_\mu\; \lambda^j \;+\; A^j_\mu\; \lambda^i \tag{43}$$

$$\delta F^{ij}_{\mu\nu} \;=\; 2\; F^{ik}_{\mu\nu}\; \lambda_k^{\ j} \;+2\; F^{jk}_{\mu\nu}\; \lambda_k^{\ i} \;+\; F^i_{\mu\nu}\; \lambda^j \;+\; F^j_{\mu\nu}\; \lambda^i \tag{44}$$

$$\delta A_\mu \;=\; \partial_\mu \lambda, \;\; \delta F_{\mu\nu} \;=\; 0 \tag{45}$$

In Appendix B we show explicitly that the Lagrangian (33) is invariant under the graded gauge transformations (39-45).

2 Bimetric Gravity, Biconnection, Antisymmetric Metrics and Cosmological Constant

In this section we discuss the physical implications of the field theory constructed above. If $A^a_\mu \leftrightarrow e^a_\mu$, is identified with the vielbein then the contraction $e^a_\mu e^b_\nu g_{ab} = g_{\mu\nu}$ yields the spacetime metric. The partner $A^i_\mu \leftrightarrow e^i_\mu$ is Grassmannian *odd* so that the contraction

$$h_{\mu\nu} \;=\; e^i_\mu\; e^j_\nu\; \omega_{ij} \;=\; -\; e^j_\nu\; e^i_\mu\; \omega_{ij} \;=\; e^j_\nu\; e^i_\mu\; \omega_{ji} \;=\; h_{\nu\mu} \tag{47}$$

furnishes another symmetric rank two tensor $h_{\mu\nu} = h_{\nu\mu}$, that can be interpreted as another metric tensor typical of bi-metric gravity [10]. The actual contraction should be performed as $e^i_\mu\; \omega_{ij}\; e^j_\nu = h_{\mu\nu}$ so the indices are adjacent.

The spin connection $A^{ab}_\mu \leftrightarrow \omega^{ab}_\mu$ has for partner $A^{ij}_\mu \leftrightarrow \omega^{ij}_\mu$ which is also Grassmannian even. The pair of connections $\omega^{ab}_\mu, \omega^{ij}_\mu$ can be used in formulations of bi-connection extensions of ordinary gravity [9] in the same fashion that $g_{\mu\nu}, h_{\mu\nu}$ are the elements in bi-metric gravity [10].

When $A^a_\mu \leftrightarrow e^a_\mu$, $A^{ab}_\mu \leftrightarrow \omega^{ab}_\mu$, one has that $F^{ab}_{\mu\nu}\; dx^\mu \wedge dx^\nu$ coincides with the 2-form $(R^{ab}_{\mu\nu} + e^a_{[\mu}\; e^b_{\nu]})\; dx^\mu \wedge dx^\nu$ where $R^{ab}_{\mu\nu}\; dx^\mu \wedge dx^\nu$ is the curvature 2-form. $F^a_{\mu\nu}\; dx^\mu \wedge dx^\nu$ becomes the Torsion 2-form. The Macdowell-Mansouri-Chamsedinne-West (MMCW) Lagrangian in $4D$ is given by

$$L_{MMCW} \;=\; (R^{ab} \;+\; e^a \wedge e^b) \;\wedge\; (R^{cd} \;+\; e^c \wedge e^d)$$

which in component form becomes

$$\epsilon^{\mu\nu\rho\sigma}\; (R^{ab}_{\mu\nu} \;+\; e^a_{[\mu}\; e^b_{\nu]})\; (R^{cd}_{\rho\sigma} \;+\; e^c_{[\rho}\; e^d_{\sigma]})\; \epsilon_{abcd} \tag{48}$$

The $R \wedge R$ terms are the topological Gauss-Bonnet curvature squared terms. $R \wedge e \wedge e$ is the Einstein-Hilbert term and $e \wedge e \wedge e \wedge e$ is the cosmological constant term (after one introduces a proper length scale to match units). Since the connection A^a_μ has units of $(length)^{-1}$ the proper correspondence is $A^a_\mu \leftrightarrow e^a_\mu / L$ so there is an overall factor L^{-4} in front of the spacetime volume term $L^{-4} \int \sqrt{g}\ d^4x$ in the action. The length scale L can be set to the Planck scale.

In $4D$ the following contraction involving $e^{(i}_\mu\ e^{j)}_\nu = e^i_\mu\ e^j_\nu + e^j_\mu\ e^i_\nu$; ω_{ij} and $\epsilon^{\mu\nu\rho\sigma}$ is trivially zero

$$L^{-4}\ \omega_{ik}\ \omega_{jl}\ e^{(i}_\mu\ e^{j)}_\nu\ e^{(k}_\rho\ e^{l)}_\sigma\ \epsilon^{\mu\nu\rho\sigma}\ =\ 2L^{-4}\ \epsilon^{\mu\nu\rho\sigma}\ (\ h_{\mu\rho}\ h_{\nu\sigma}\ +\ h_{\nu\rho}\ h_{\mu\sigma}\ )\ =\ 0 \qquad (49)$$

due to the antisymmetry of the epsilon tensor density. While the actual contraction involving the metric tensor in the Yang-Mills terms of eq-(35) is nonvanishing

$$L^{-4}\ \omega_{ik}\ \omega_{jl}\ e^{(i}_\mu\ e^{j)}_\nu\ e^{(k}_\rho\ e^{l)}_\sigma\ g^{\mu\rho}\ g^{\nu\sigma}\ =\ 2L^{-4}\ g^{\mu\rho}\ g^{\nu\sigma}\ (\ h_{\mu\rho}\ h_{\nu\sigma}\ +\ h_{\nu\rho}\ h_{\mu\sigma}\ )\ \neq\ 0 \qquad (50)$$

the latter nonvanishing contraction has the form of mass-like terms for the symmetric rank 2 tensor $h_{\mu\nu}$

$$\mathcal{L}_{mass}\ =\ 2L^{-4}\ (\ h^{\mu\sigma}\ h_{\mu\sigma}\ +\ (h^\mu_\mu)^2\ )\ >\ 0,\ \ h^\mu_\mu\ =\ trace\ (h_{\mu\rho})\ =\ g^{\mu\rho}\ h_{\mu\rho} \qquad (51)$$

Since $\mathcal{L}_{mass} > 0$ is positive definite these terms cannot be used to cancel out the cosmological constant contribution

$$L^{-4}\ \epsilon_{abcd}\ e^a_\mu\ e^b_\nu\ e^c_\rho\ e^d_\sigma\ \epsilon^{\mu\nu\rho\sigma}\ =\ L^{-4}\ |det\ e|\ =\ L^{-4}\ \sqrt{|g|} \qquad (52)$$

There is another possibility worth exploring if one takes the grade of γ_a to be *odd* while the grade of γ_{ab} still remains even. In this case we will have that $A^a_\mu \leftrightarrow e^a_\mu$ is now Grassmannian *odd* (instead of even), and a-number so that $e^a_\mu\ e^b_\nu\ = -e^b_\nu\ e^a_\mu$ and the contraction $e^a_\mu\ e^b_\nu\ g_{ab}\ =\ g_{[\mu\nu]}$ yields an *antisymmetric* metric tensor $g_{[\mu\nu]}$. In this case, a symmetric tensor is only obtained from the contraction in eq-(47) $h_{\mu\nu} = e^i_\mu\ e^j_\nu\ \omega_{ij} = h_{\nu\mu}$. If we were to identify $h_{\mu\nu}$ with a spacetime metric $g_{(\mu\nu)}$, in this case the "emergent" metric $h_{\mu\nu}$ can be seen as the "condensate" of two Grassmannian *odd* valued fields $h_{\mu\nu} = e^i_\mu\ e^j_\nu\ \omega_{ij} = h_{\nu\mu}$.

A symmetric and antisymmetric metric can both be accommodated within a Hermitian complex metric $\mathbf{g}_{\mu\nu} = g_{(\mu\nu)} + ig_{[\mu\nu]}$ obeying

$$\mathbf{g}^*_{\mu\nu}\ =\ \mathbf{g}_{\nu\mu}\ \Rightarrow\ \mathbf{g}^\dagger_{\mu\nu}\ =\ \mathbf{g}_{\mu\nu} \qquad (53)$$

When the grade of γ_a is taken to be *odd* we have now that the graded commutator must be $[\gamma_a, \xi_i\} = \{\gamma_a, \xi_i\} = 0$, instead of having $[\gamma_a, \xi_i] = 0$ as displayed before in eq-(9). Due to the Grassmannian *odd* nature of A^a_μ we have also *modifications* to the expressions for the following field strength components

$$F^{ab}_{\mu\nu}\ =\ \partial_\mu\ A^{ab}_\nu\ -\ \partial_\nu\ A^{ab}_\mu\ + 2\ A^{ac}_{[\mu}\ A^{cb}_{\nu]} \qquad (54)$$

$$F_{\mu\nu} \;=\; \partial_\mu \; A_\nu \;-\; \partial_\nu \; A_\mu \;+\; A^a_\mu \; A^b_\nu \; g_{ab} \tag{55}$$

since $[\gamma_a, \gamma_b\} = \{\gamma_a, \gamma_b\} = 2g_{ab}\mathbf{1}$, instead of having $[\gamma_a, \gamma_b] = 2\gamma_{ab}$. One should compared eqs-(54,55) with the prior expressions in eqs-(19,22).

Using the definition $e^a_\mu \; e^b_\nu \; g_{ab} \;=\; g_{[\mu\nu]}$ the quadratic terms $F_{\mu\nu}F^{\mu\nu}$ will yield mass terms for Kalb-Ramond like fields of the form $g_{[\mu\nu]} \; g^{[\mu\nu]} \;=\; B_{\mu\nu} \; B^{\mu\nu}$, after identifying $B_{\mu\nu} = g_{[\mu\nu]}$ and raising the indices with the metric $h^{\mu\nu} = g^{(\mu\nu)}$. Another possibility is to raise the indices with the inverse Hermitian complex metric

$$(\mathbf{g}_{\mu\nu})^{-1} \;=\; \mathbf{g}^{\mu\nu} \;=\; \tilde g^{(\mu\nu)} \;+ i \; \tilde g^{[\mu\nu]} \;\neq\; g^{(\mu\nu)} \;+ i \; g^{[\mu\nu]} \tag{56a}$$

we specifically used the tilde symbol and the inequality to emphasize that both $\tilde g^{(\mu\nu)}, \tilde g^{[\mu\nu]}$ are *nontrivial* functions of both $g_{\mu\nu}$ and $g_{[\mu\nu]}$. However, when a complex metric is involved one has to add the complex (Hermitian) conjugate of any given term in order to have a real-valued action. The measure associated with such Hermitian complex metric $\mathbf{g}_{\mu\nu}$ is

$$(|| \; det \; \mathbf{g}_{\mu\nu} \; ||)^{\frac{1}{2}} \;=\; (\; (det \; \mathbf{g}_{\mu\nu}) \; (det \; \mathbf{g}_{\mu\nu})^*)^{\frac{1}{4}} \tag{56b}$$

Furthermore one can also use the epsilon tensor in $4D$ and contract indices in the following expression which can be included in the action

$$L^{-4} \; B_{\mu\nu} \; B_{\rho\sigma} \; \epsilon^{\mu\nu\rho\sigma} \;=\; L^{-4}\sqrt{|det \; g_{(\mu\nu)}|} \; \left(\; B_{\mu\nu} \; B_{\rho\sigma} \; \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{|det \; g_{(\mu\nu)}|}} \right) \tag{57a}$$

One of the most salient features of choosing the grade of γ_a to be odd, and consequently when e^a_μ is an a-number, is that it is possible for the terms in eq-(57a) to *cancel* out a cosmological constant term constructed now in terms of $h_{\mu\nu}$ as

$$L^{-4}\sqrt{|det \; h_{\mu\nu}|} \;=\; L^{-4}\sqrt{|det \; g_{(\mu\nu)}|} \tag{57b}$$

the contributions from $h_{\mu\nu} = g_{(\mu\nu)}$ and $B_{\mu\nu} = g_{[\mu\nu]}$ in eqs-(57a, 57b) may cancel each other when the on-shell dynamics of the fields (associated with the action) satisfies

$$B_{\mu\nu} \; B_{\rho\sigma} \; \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{|det \; g_{(\mu\nu)}|}} \;+\; 1 \;=\; 0 \tag{57c}$$

after solving the field equations for $h_{\mu\nu} = g_{(\mu\nu)}$ and $B_{\mu\nu} = g_{[\mu\nu]}$ and inserting their solutions back into eq-(57c). This problem deserves further investigation.

3 Super-Clifford Algebras and Generalized Super-Geometry

Clifford gauge field theories based on orthogonal Clifford $Cl(m)$ algebras have been explored in [14] and references therein. One may assign an *odd* grade to the odd-valued rank polyvector basis generators $\gamma_a, \gamma_{a_1a_2a_3}, ...$, and an *even* grade to the even-valued rank, like the unit element $\mathbf{1}$ and $\gamma_{a_1a_2}, \gamma_{a_1a_2a_3,a_4}, ...$ There is an addition the odd grade $\xi_i, \xi_{i_1i_2,j}, ...$ and even grade $\xi_{ij}, \xi_{i_1i_2,j_1j_2}, ...$ generators associated with the symplectic Clifford algebra. In this way one can generalize our construction to the case when $m > 2; 2n > 2$ and build a super gauge field theory.

Now we turn to the question of constructing a *super*-Clifford algebra and which is very *different* from the so-called "super Clifford" algebras with a Z_4 grading proposed by [3], [4]. To proceed we turn to the the Lie superalgebra $osp(m|2n)$ (ortho-symplectic superalgebra) whose *even* part is $so(m) \oplus sp(2n)$. In particular $sp(4) \sim so(5)$. Since we've started with an orthogonal and a symplectic Clifford algebra, we may assign now an *even* grade to *all* the generators of the latter orthogonal-symplectic Clifford algebras. Therefore, the *super*-Clifford algebra in question must be a *super* algebra such that its *even* grade part must be given by the orthogonal-symplectic Clifford algebras, and its *odd* grade part must be an algebra whose generators $\mathcal{Q}_u, u = 1, 2, 3, ...$ (*not* to be confused with the spinorial charge generators Q_α, S_α of ordinary super conformal algebras !) should obey the graded commutator relations

$$[\ \gamma_a,\ \mathcal{Q}_u\ ] \sim (M_a)_u^v\ \mathcal{Q}_v,\ \ [\ \gamma_{ab},\ \mathcal{Q}_u\ ] \sim (M_{ab})_u^v\ \mathcal{Q}_v,\ \ [\ \gamma_{abc},\ \mathcal{Q}_u\ ] \sim (M_{abc})_u^v\ \mathcal{Q}_v,\ ... \quad (58)$$

$$[\ \xi_i,\ \mathcal{Q}_u\ ] \sim (N_i)_u^v\ \mathcal{Q}_v,\ \ [\ \xi_{ij},\ \mathcal{Q}_u\ ] \sim (N_{ij})_u^v\ \mathcal{Q}_v,\ [\ \xi_{ij,k}\ ,\ \mathcal{Q}_u\ ] \sim (N_{ij,k})_u^v\ \mathcal{Q}_v,\ ..... \quad (59)$$

$$\{\ \mathcal{Q}_u,\ \mathcal{Q}_v\ \} \sim \delta_{uv}\ \mathbf{1} + (L^a)_{uv}\ \gamma_a + (L^{ab})_{uv}\ \gamma_{ab} + (L^{abc})_{uv}\ \gamma_{abc} + ... +$$
$$(L^i)_{uv}\ \xi_i + (L^{ij})_{uv}\ \xi_{ij} + (L^{i_1i_2,j})_{uv}\ \xi_{i_1i_2,j} + (L^{i_1i_2,j_1j_2})_{uv}\ \xi_{i_1i_2,j_1j_2} + ... \quad (60)$$

where the M, N, L are suitable matrices. An analog of the above equations are the (anti) commutators of the (Anti) de Sitter superalgebra

$$[\ J_a,\ Q_\alpha\ ] \sim (\Gamma_a)_\alpha^\beta\ Q_\beta,\ \ [\ J_{ab},\ Q_\alpha\ ] \sim (\Gamma_{ab})_\alpha^\beta\ Q_\beta,\ \ [\ K,\ Q_\alpha\ ] \sim Q_\alpha, \quad (61)$$

$$\{\ Q_\alpha,\ Q_\beta\ \} \sim \delta_{\alpha\beta}\ K + \Gamma^a_{\alpha\beta}\ J_a + \Gamma^{ab}_{\alpha\beta}\ J_{ab} \quad (62)$$

If one were to set the generators $\mathcal{Q}_u, u = 1, 2, 3, ...$ to coincide with the spinorial charge generators Q_α of ordinary supersymmetry by identifying the u index with the spinorial α index, the $M_a, M_{ab}, M_{abc}, ..., L^a, L^{ab}, L^{abc}$ matrices in (58) could be identified with the gamma matrices $\Gamma_a, \Gamma_{ab}, \Gamma_{abc}, ...$ The matrices $\Gamma^a, \Gamma^{ab}, \Gamma^{abc}$, as usual, are simply obtained by raising the indices of $\Gamma_a, \Gamma_{ab}, \Gamma_{abc}, ...$ via a metric g^{ab} and its antisymmetrized products as depicted in eq-(23a). The difficulty arises then in constructing the matrices $N_i, N_{ij}, N_{ij,k}, ..., L^i, L^{ij}, L^{ij,k}, ...$ in eq-(59,60). We do not know at this stage if these matrices can be expressed in terms of the gamma matrices. This is an interesting problem that warrants further investigation. To sum up, a truly *super* Clifford algebra must be one whose (anti) commutators obey eqs-(58-60), in addition to the commutation relations among the γ's and the ξ's generators (since now we have assigned an *even* grade to *all* of them). To considerably simplify the problem one could set to zero all the ξ's generators except ξ_i, ξ_{ij}.

How does one relate the Grassmannian odd A^i_μ gauge fields to fermions, in particular to the gravitino Ψ^α_μ ?. It is well known to the experts that spinors are right/left ideals elements of the Clifford algebra and consequently spinors are already encoded within the Clifford algebraic structure itself. Hence, a natural *correspondence* (*not* an equality) among A^i_μ and the spin $\frac{3}{2}$-gravitino field Ψ^α_μ could be given as (omitting spinorial indices in the right hand side for simplicity)

$$A^i_\mu \ \omega_{ij} \ A^j_\nu \ \leftrightarrow \ \ c \ \bar\Psi_\mu \ \Psi_\nu \ + c_a \ \bar\Psi_\mu \ \Gamma^a \ \Psi_\nu \ + c_{ab} \ \bar\Psi_\mu \ \Gamma^{ab} \ \Psi_\nu \ + c_{abc} \ \bar\Psi_\mu \ \Gamma^{abc} \ \Psi_\nu \ + \ ...... \quad (63)$$

From the discussion in section **2**, when $A^i_\mu \leftrightarrow e^i_\mu$, the left hand side of (63) becomes $h_{\mu\nu}$ so the mass-like squared terms in eq-(51) will have a correspondence with the quartic fermionic terms $(\bar\Psi_\mu \Psi_\nu)^2 \ + \ .....$

Another correspondence is

$$A^i_{\mu j} \ A^j_\nu \ A^k_{\rho l} \ A^l_\sigma \ \omega_{ik} \ \leftrightarrow \ c_1 \ \bar\Psi_\nu \ A_\mu \ A_\rho \ \Psi_\sigma + c_2 \ \bar\Psi_\nu \ A^a_\mu \ \Gamma_a \ A^b_\rho \ \Gamma_b \ \Psi_\sigma + c_3 \ \bar\Psi_\nu \ A^{ab}_\mu \ \Gamma_{ab} \ A^{cd}_\rho \ \Gamma_{cd} \ \Psi_\sigma + ... \quad (64)$$

$c, c_a, c_{ab}, ..., c_1, c_2, c_3...$ are numerical coefficients. A further analysis of the correspondence in eqs-(64, 64) will be left for future investigations.

To finalize, following the approach of [6], [7], we introduce anti-commuting coordinates $y^u : y^u y^v = -y^v y^u$ in order to initiate the construction of super-Clifford spaces. When one has commuting spacetime coordinates $x^\mu, \mu = 1, 2, ..., m$ and anticommuting coordinates $y^u, u = 1, 2, ..., 2n$ the exterior wedge product of super differentials is defined as

$$dZ^A \ \wedge \ dZ^B \ = \ - \ (-1)^{(grade \ A \ grade \ B)} \ dZ^B \ \wedge \ dZ^A \ \Rightarrow$$

$$dx^\mu \wedge dx^\nu \ = - \ dx^\nu \wedge dx^\mu, \ \ dy^u \wedge dy^v \ = \ dy^v \wedge dy^u, \ \ dx^\mu \wedge dy^v \ = - \ dy^v \wedge dx^\mu, \quad (65)$$

and the field strength associated with the ortho-symplectic Clifford-algebra-valued super differential one-form $\mathbf{A}_\mu dx^\mu + \mathbf{A}_u dy^u$ is

$$\mathbf{F}_{\mu\nu} \ dx^\mu \wedge dx^\nu \ + \ \mathbf{F}_{uv} \ dy^u \wedge dy^v, \ \ \mathbf{F}_{\mu\nu} = -\mathbf{F}_{\nu\mu}, \ \ \mathbf{F}_{uv} = \mathbf{F}_{vu} \quad (66)$$

$\mathbf{A}_u$ admits an expansion similar to $\mathbf{A}_\mu$ in eqs-(15,32) by replacing the μ index with u.

Now we are in a position to construct the coordinates of *super* Clifford spaces in terms of the polyvector-valued coordinates $x^{[\mu_1\mu_2....\mu_n]}$ which are fully antisymmetric under the exchange of indices, and the other set of coordinates with a *mixed* symmetry $y^{u_1u_2.....u_r, v_1v_2.....v_s}$ under the index exchange. A recent study of the Extended Relativity Theory in Clifford Phase Spaces and modifications of gravity at the Planck/Hubble scales, with many references therein can be found in [19]. We extended the construction of Born's Reciprocal Phase Space Relativity to the case of Clifford Spaces which involve the use of *polyvector*-valued coordinates and momenta $x^{[\mu_1\mu_2....\mu_n]}, p^{[\sigma_1\sigma_2....\sigma_n]}$; generalized vielbeins $E^A_M = E^{[a_1a_2....a_r]}_{[\mu_1\mu_2.....\mu_s]}, E^{[b_1b_2....b_r]}_{[\sigma_1\sigma_2.....\sigma_s]}$ corresponding to the coordinate and momentum space directions, respectively; generalized metrics like $g_{[\mu_1\mu_2....\mu_r] \ [\nu_1\nu_2....\nu_s]}, ...$ and the inclusion of an *lower/upper* (Planck/Hubble) length scale.

A *super* Clifford valued coordinate in *super* Clifford spaces has a decomposition

$$\mathbf{Z} \ = \ x \ \mathbf{1} \ + \ x^\mu \ \gamma_\mu \ + \ x^{\mu_1\mu_2} \ \gamma_{\mu_1\mu_2} \ + \ ...... \ + \ x^{\mu_1\mu_2...\mu_m} \ \gamma_{\mu_1\mu_2...\mu_m} \ +$$

$$y^u \ \xi_u \ + \ y^{u_1 u_2} \ \xi_{u_1 u_2} \ + \ y^{u_1 u_2, v} \ \xi_{u_1 u_2, v} \ + \ y^{u_1 u_2, v_1 v_2} \ \xi_{u_1 u_2, v_1 v_2} \ + \ .... \tag{67}$$

We conclude with some of the salient features of the generalized super-geometry associated with *super* Clifford spaces :

(i) A super-Clifford theory of gravity will amount to a generalized supergravity theory based on an orthogonal-symplectic Clifford algebraic structure and the geometry of super-Clifford spaces.

(ii) The generalized vielbeins like $E^A_M = E^{[a_1 a_2 .... a_r]}_{[\mu_1 \mu_2 ..... \mu_s]}, E^{[b_1 b_2 .... b_r]}_{[\sigma_1 \sigma_2 ..... \sigma_s]}$, will cast more light to the preliminary analysis of the generalized vielbein postulate [20] that reveals tantalizing hints of new structures beyond $D = 11$ supergravity and ordinary space-time covariance. An E_7-valued vielbein in eleven dimensions was analyzed and they discussed the extension of these results to E_8. An E_8 Gauge theory of gravity in $8D$ based on generalized vielbein was proposed in [14].

(iii) When the construction of *super*-Clifford spaces is extended to Clifford Phase Spaces [19], the coordinates X, P are now doubled, and one may connect with the U-duality and double field theory formalism of generalized geometry which has been gaining a lot attraction recently [22].

(iv) The fact that in general one has an $infinite$ number of generators described by the *super* Clifford algebras proposed in this work, one should try to find its relation (if any) to the infinite dimensional hyperbolic Kac-Moody algebras E_{11} which have been conjectured [21] to encode the hidden symmetries of M-theory in $11D$. It was mentioned earlier that the quantum Virasoro algebra generators can be constructed in terms of operators of the generalized Clifford algebras as shown by [24]. By extending this construction using the super-Clifford algebraic approach in this work we should be able to connect with the hyperbolic Kac-Moody algebras E_{11} description of [21].

APPENDIX A : Graded Jacobi Identities

In this Appendix we shall prove that the graded Jacobi identities are satisfied.

We shall firstly take all the generators of the orthogonal Clifford algebra to be of even grade, while the ξ_i generator has odd grade and ξ_{ij} has even grade. Afterwards we shall take the odd rank generators of the orthogonal Clifford algebra to have odd grade; and the even rank generators to have even grade. The grades take the values of $0, 1$ for even, odd grade respectively. Let us begin with

$$(-1)^0 \{ \ \xi_i, \ [\xi_j, \ \xi_{kl}] \ \} \ + (-1)^1 \ \{ \ \xi_j, \ [\xi_{kl}, \ \xi_i] \ \} \ + (-1)^0 \ [\ \xi_{kl}, \ \{\xi_i, \ \xi_j\} \ ] \ =$$

$$\{ \ \xi_i, \ \omega_{jk} \ \xi_l + \omega_{jl} \ \xi_k \ \} \ - \ \{ \ \xi_j, \ \omega_{ki} \ \xi_l + \omega_{li} \ \xi_k \ \} \ + 2 \ [\ \xi_{kl}, \ \xi_{ij} \ ] \ =$$

$$2 \ \omega_{jk} \ \xi_{il} \ + 2 \ \omega_{jl} \ \xi_{ik} \ - 2 \ \omega_{ki} \ \xi_{jl} \ - 2 \ \omega_{li} \ \xi_{jk} \ +$$

$$2 \ (\ \omega_{ki} \ \xi_{lj} \ + \ \omega_{kj} \ \xi_{li} \ + \ \omega_{li} \ \xi_{kj} \ + \ \omega_{lj} \ \xi_{ki} \ ) \ = \ 0 \tag{A.1}$$

due to $\omega_{jk} + \omega_{kj} = 0$, $\omega_{jl} + \omega_{lj} = 0$ and $\xi_{jk} = \xi_{kj}, ...$

The graded Jacobi relation among $\xi_{ij}, \xi_{kl}, \xi_{pq}$ is

$$(-1)^0[\ \xi_{ij},\ [\xi_{kl},\ \xi_{pq}]\ ]\ +(-1)^0\ [\ \xi_{kl},\ [\xi_{pq},\ \xi_{ij}]\ ]\ +(-1)^0\ [\ \xi_{pq},\ [\xi_{ij},\ \xi_{kl}]\ ]\ =\ 0 \quad (A.2)$$

it is also zero. One can verify that one ends with a collection of 48 terms, all of which can be gathered into groups (up to indices permutations) of the form

$$\omega_{ik}\ \omega_{pj}\ \xi_{ql}\ +\ \omega_{pj}\ \omega_{ki}\ \xi_{lq}\ =\ 0 \quad (A.3)$$

eq-(A.3) is zero because $\omega_{ik}+\omega_{ki}=0$ and $\xi_{ql}=\xi_{lq}$, etc...

$$(-1)^1\ [\ \xi_i,\ \{\xi_j,\ \xi_k\ \}\ ]\ +(-1)^1\ [\ \xi_j,\ \{\xi_k,\ \xi_i\ \}\ ]\ +(-1)^1\ [\ \xi_k,\ \{\xi_i,\ \xi_j\}\ ]\ =$$

$$-\ 2\ [\ \xi_i,\ \xi_{jk}\ ]\ -2\ [\ \xi_j,\ \xi_{ki}\ ]\ -2\ [\ \xi_k,\ \xi_{ij}\ ]\ =$$

$$-2\ (\ \omega_{ij}\ \xi_k\ +\ \omega_{ik}\ \xi_j\ +\ \omega_{jk}\ \xi_i\ +\ \omega_{ji}\ \xi_k\ +\ \omega_{ki}\ \xi_j\ +\ \omega_{kj}\ \xi_i\ )\ =\ 0 \quad (A.4)$$

Similarly one obtains

$$(-1)^0\ [\ \xi_m,\ [\ \xi_{ij},\ \xi_{kl}\ ]\ +(-1)^0\ [\ \xi_{ij},\ [\ \xi_{kl},\ \xi_m\ ]\ +(-1)^0\ [\ \xi_{kl},\ [\ \xi_m,\ \xi_{ij}\ ]\ =\ 0 \quad (A.5)$$

The graded Jacobi relation among $\xi_{ij}, \gamma_a, \xi_{pq}$ is

$$(-1)^0[\ \xi_{ij},\ [\gamma_a,\ \xi_{pq}]\ ]\ +(-1)^0\ [\ \gamma_a,\ [\ \xi_{pq},\ \xi_{ij}]\ ]\ +(-1)^0\ [\ \xi_{pq},\ [\xi_{ij},\ \gamma_a]\ ]\ =0 \quad (A.6)$$

because $[\gamma_a, \xi_{pq}]=0, ...$ and $[\xi_{pq}, \xi_{ij}]=\omega_{pi}\ \xi_{qj}+.....$ The same result occurs with

$$(-1)^0[\ \xi_{ij},\ [\gamma_{ab},\ \xi_{pq}]\ ]\ +(-1)^0\ [\ \gamma_{ab},\ [\xi_{pq},\ \xi_{ij}]\ ]\ +(-1)^0\ [\ \xi_{pq},\ [\xi_{ij},\ \gamma_{ab}]\ ]\ =0 \quad (A.7)$$

because $[\gamma_{ab}, \xi_{pq}]=0, ...$

The graded Jacobi relation among $\xi_{ij}, \gamma_a, \gamma_b$ is

$$(-1)^0[\ \xi_{ij},\ [\gamma_a,\ \gamma_b]\ ]\ +(-1)^0\ [\ \gamma_a,\ [\ \gamma_b,\ \xi_{ij}]\ ]\ +(-1)^0\ [\ \gamma_b,\ [\xi_{ij},\ \gamma_a]\ ]\ =0 \quad (A.8)$$

etc....

In the case that γ_a is chosen to have on *odd* grade, eq-(A.8) is replaced by

$$(-1)^0[\ \xi_{ij},\ \{\gamma_a,\ \gamma_b\}\ ]+(-1)^0\ \{\ \gamma_a,\ [\ \gamma_b,\ \xi_{ij}]\ \}+(-1)^1\ \{\ \gamma_b,\ [\xi_{ij},\ \gamma_a]\ \}\ =\ [\ \xi_{ij},\ 2g_{ab}\ \mathbf{1}\ ]\ =\ 0 \quad (A.9)$$

since $[\ \gamma_b,\ \xi_{ij}]=0,\ ...$

Let us look at the graded Jacobi identities involving $\gamma_s, \gamma_{mn}, \gamma_r$ when the grade of γ_s, γ_r is taken to be odd, and the grade of γ_{mn} is even

$$(-1)^1\ \{\ \gamma_s,\ [\gamma_{mn},\ \gamma_r]\ \}\ +(-1)^0\ [\ \gamma_{mn},\ \{\gamma_r,\ \gamma_s\}\ ]\ +(-1)^0\ \{\ \gamma_r\ [\gamma_s,\ \gamma_{mn}]\ \}\ =$$

$$4\ g_{mr}\ g_{sn}\ -4\ g_{nr}\ g_{sm}\ +4\ g_{sm}\ g_{rn}\ -4\ g_{sn}\ g_{rm}\ =\ 0 \quad (A.10)$$

Compare (A.10) when the grade of $\gamma_s, \gamma_r, \gamma_{mn}$ are all chosen to be even

$$[\ \gamma_s,\ [\gamma_{mn},\ \gamma_r]\ ]\ +\ [\ \gamma_{mn},\ [\gamma_r,\ \gamma_s]\ ]\ +\ [\ \gamma_r\ [\gamma_s,\ \gamma_{mn}]\ ]\ =$$

$$-4\ g_{mr}\ \gamma_{sn}\ +4\ g_{nr}\ \gamma_{sm}\ +4\ g_{sm}\ \gamma_{rn}\ -4\ g_{sn}\ \gamma_{rm}\ -4\ g_{mr}\ \gamma_{ns}\ +$$
$$4\ g_{nr}\ \gamma_{ms}\ +4\ g_{sm}\ \gamma_{nr}\ -4\ g_{sn}\ \gamma_{mr}\ =\ 0 \quad (A.11)$$

due to the antisymmetry of the bivector generators $\gamma_{mr} = -\gamma_{rm}, ...$

The (anti) commutators of the $\gamma_a, \gamma_{a_1a_2}, ..., \gamma_{a_1a_2.....a_m}$ generators can be found in pages 543-545 of the monograph [23]. To check that the graded Jacobi identities are satisfied for the remaining of the infinite number of generators $\xi_{i_1i_2,j_1}, \xi_{i_1i_2,j_1j_2}, ...$ would require a computer algebra package. They are satisfied for the graded commutators involving ξ_i, ξ_{jk} as shown in eqs-(A.1-A.5). Therefore, there is true algebraic closure in the description of the $\mathbf{A}_\mu$ field, and its field strength $\mathbf{F}_{\mu\nu}$ components defined in eqs-(15-22) and which allowed us to proceed with the remaining analysis and results in sections $\mathbf{1}, \mathbf{2}$.

APPENDIX B : Invariance of the Lagrangian

We shall derive the invariance of the Lagrangian (33) under the graded gauge transformations (39-45). For simplicity we shall not include the spacetime indices. Let us begin with the variation

$$\frac{1}{4}\ \delta(F^{ij}\ F^{kl})\ (\omega_{ik}\ \omega_{jl} + \omega_{il}\ \omega_{jk})\ =\ \frac{1}{2}\ \delta(F^{ij}\ F^{kl})\ \omega_{il}\ \omega_{jk} \quad (B.1)$$

where $F^{ij} = F^{ji}, ...$ It contains two types of terms. One of them is

$$X_1\ =\ -2\ F^{jm}\ \omega_{mn}\ \lambda^{ni}\ \omega_{il}\ F^{lk}\ \omega_{kj} \quad (B.2)$$

Due to the symmetry $\lambda^{ni} = \lambda^{in}$, eq-(B.2) is the same as

$$X_1\ =\ -2\ F^{jm}\ \omega_{mn}\ \lambda^{in}\ \omega_{il}\ F^{lk}\ \omega_{kj} \quad (B.3)$$

upon rearrangement of terms and indices, eq-(B.3) can be rewritten as

$$X_1\ =\ -2\ \omega_{il}\ \lambda^{in}\ \omega_{mn}\ F^{jm}\ \omega_{kj}\ F^{lk}\ =\ 2\ \omega_{li}\ \lambda^{in}\ \omega_{nm}\ F^{mj}\ \omega_{jk}\ F^{lk} \quad (B.4)$$

the overall change of sign in (B.4) is due to 3 changes of sign due to the antisymmetry of the omega tensors. A cyclic permutation of (B.4) and after using $F^{lk} = F^{kl}$ yields

$$X_1\ =\ 2\ F^{kl}\ \omega_{li}\ \lambda^{in}\ \omega_{nm}\ F^{mj}\ \omega_{jk}\ =\ -\ X_1 \quad (B.5)$$

therefore, after equating eq-(B.3) with eq-(B.5) one arrives at $X_1 = -X_1 \ \Rightarrow\ X_1 = 0$. The reason that eq-(B.3) is the same as eq-(B.5) is because both have the same index contraction structure; i.e. the same trace. A relabeling of indices reveals that they are the same.

The second type of terms in eq-(B.1) is

$$2\ (\ F^i\ \lambda^j\ +\ F^j\ \lambda^i\ )\ F^{kl}\ \frac{1}{2}\ \omega_{ik}\ \omega_{jl}\ =\ -\ 2\ F^i\ \omega_{ik}\ F^{kl}\ \omega_{lj}\ \lambda^j \quad (B.6)$$

These terms (B.6) are not zero but will cancel out with those *mixed* terms stemming from the variation $\frac{1}{2}\ \delta(F^i\ F^j)\omega_{ij}$. These mixed terms stemming from the latter variation are given by

$$\frac{1}{2}\ \omega_{ij}\ (\ 2\ F^{im}\ \omega_{mn}\ \lambda^n\ F^j\ +2\ F^i\ F^{jm}\ \omega_{mn}\ \lambda^n\ )\ =$$
$$2\ F^{im}\ \omega_{mn}\ \lambda^n\ F^j\ \omega_{ij} \quad (B.7)$$

due to the Grassmannian *odd* nature of F^j and λ^n. One must change signs as follows $F^j\ \lambda^n = -\lambda^n\ F^j$. Hence, after reversing their ordering in (B.7) one obtains a change of sign

$$-\ 2\ F^{im}\ \omega_{mn}\ F^j\ \lambda^n\ \omega_{ij} \quad (B.8)$$

rearranging terms and indices in (B.8) one gets

$$2\ F^j\ \omega_{ji}\ F^{im}\ \omega_{mn}\ \lambda^n \quad (B.9)$$

due to a sign change $\omega_{ji} = -\omega_{ij}$. Therefore, upon combining (B.9) with (B.6) one has a cancellation of the mixed terms

$$2\ F^j\ \omega_{ji}\ F^{im}\ \omega_{mn}\ \lambda^n\ \ -\ 2\ F^i\ \omega_{ik}\ F^{kl}\ \omega_{lj}\ \lambda^j\ =\ 0 \quad (B.10)$$

since after an index relabeling one can see that the first and second terms in (B.10) are equal but with opposite signs. These terms have the same index contraction structure.

The terms in $\frac{1}{2}\ \delta(F^i\ F^j)\omega_{ij}$ which are not mixed are

$$X_2\ =\ 2\ \frac{1}{2}\ 2\ F^m\ \omega_{mn}\ \lambda^{ni}\ \omega_{ij}\ F^j \quad (B.11)$$

due to the Grassmannian odd nature of F^j and F^m, after reversing their order in (B.11) one obtains a change of sign

$$X_2\ =\ -\ 2\ F^j\ \omega_{mn}\ \lambda^{ni}\ \omega_{ij}\ F^m\ =\ -\ 2\ F^j\ (-\omega_{ji})\ \lambda^{in}\ (-\omega_{nm})\ F^m\ =$$
$$-2\ F^j\ \omega_{ji}\ \lambda^{in}\ \omega_{nm}\ F^m\ =\ -\ X_2 \quad (B.12)$$

hence from the equality of (B.11) and (B.12) (both have the same index contraction structure) one arrives at $X_2 = -X_2 \ \Rightarrow\ X_2 = 0$. To sum up, after dropping the spacetime indices for convenience we have seen that the gauge variations of the quadratic terms involving F^i, F^{jk} are exactly zero.

Following a similar procedure, one can show that the variation of the remaining terms $F^aF_a - 2F^{ab}F_{ab} + F^2$, dropping the spacetime indices, is also zero. The variation of $F_{\mu\nu}F^{\mu\nu}$ is trivially zero since $\delta F_{\mu\nu} = 0$.

To conclude, the gauge variation of the quadratic Lagrangian (33) is zero and consequently it is invariant under the graded ortho-symplectic Clifford-valued gauge transformations. Ordinary graded gauge theories can be found in [25], for example.

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