Using probabilities of Prime Number Theorem to prove Goldbach's conjecture

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Abstract

This article will define the range in which there is at least one pair of odd primes that when added produce specific even number. It will prove that every even number in that range can be written as a sum of at least one pair of odd primes. Additionally, this paper will prove trivial "soft" versions of Goldbach's conjecture that follow from the "hard" version.

1   Introduction

In 1749, more than 260 years ago, Prussian mathematician Christian Goldbach in his letter to Leonhard Euler proposed an idea today known as Goldbach's conjecture. His conjecture states:

> *Every even integer greater than 2 can be written as the sum of two primes.*

At the time, 1 was considered a prime number, a notion that's changed over the years. Today 1 is not counted amongst prime numbers and stands on his own, neither prime nor composite. In order to compensate for that and in order to generalize it a modern version has been stated as:

> *Every integer greater than 5 can be written as the sum of three primes.*

It is obvious that 6 is the smallest number possible to write, as 6=2+2+2, two being the smallest prime accepted today. The strong Goldbach conjecture implies the conjecture that all odd numbers greater than 7 are the sum

of three odd primes, which is known today variously as the "weak" Goldbach conjecture As one of the oldest unsolved mathematical problems it has received a lot of attention in the last years. Tomás Oliveira e Silva et al. have checked every number $n \leq 10^{17}$ using distributed computing and found no exception. Others, like Donald Knuth, have stated that perhaps Goldbach's conjecture is unprovable, basing their statements on implications of Gödel's incompleteness theorem. Indeed, there are conjectures that can't be proven, but I strongly believe I can prove that Goldbach's is not one of them.

2 Theorems

This section will discuss theorems required to prove Goldbach's conjecture in modern form, both soft and hard.

Basic Goldbach's Every number $a \in \mathbb{Z}$ can be written as a sum of two numbers $m, n \geq 1 \in \mathbb{N}$ so that

$$a = 2q = (2m + 1) + (2n - 1)$$

$$2q = 2(m + n)$$

$$q = m + n$$

where $q \in \mathbb{Z}$ and $q \geq 1$.

If q is an even number, there are $\frac{q}{2} + 1$ unique combinations of m and n for which $m + n = q$ is true. Likewise, if q is odd, there are $\frac{q+1}{2}$ unique combinations.

Let's define two sets, $\mathbf{M}$ and $\mathbf{N}$.

$$\mathbf{M} = \left\{ 0, 1, 2, \ldots, \left\lceil \frac{q}{2} \right\rceil \right\}$$

$$\mathbf{N} = \left\{ q, q - 1, q - 2, \ldots, \left\lfloor \frac{q}{2} \right\rfloor \right\}$$

so that for every z the following expression is true:

$$\mathbf{M}_z + \mathbf{N}_z = q$$

Prime Number Theorem   Prime Number Theorem is a theorem related to distribution of primes. It defines function $\pi(x)$ as number of primes from 0 to x.

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$$

$$\pi(x) \sim \frac{x}{\ln x}$$

From that we can show that as $x \to \infty$ probability that x is a prime is

$$P(x_p) = \frac{1}{\ln x}$$

Estimating probability of prime pairs   First we must estimate the probability that a random $\mathbf{M}_z$ is prime.

$$P(\mathbf{M}_p) = \frac{1}{\ln(q/2)}$$

We are using $q/2$ because as q gets larger and larger its parity doesn't matter. First equation describes length of $\mathbf{M}_z$ when q is odd and second when q is even.

$$\lim_{q\to\infty} \frac{q+1}{2} = \frac{q}{2}; \qquad \lim_{q\to\infty} \frac{q}{2} + 1 = \frac{q}{2}$$

Probability that $\mathbf{N}_p$ is prime is somewhat more complex:

$$P(\mathbf{N}_p) = \frac{1}{\frac{q}{\ln q} - \frac{q/2}{\ln(q/2)}} = \frac{2\ln(q/2)\ln(q)}{q\ln(q/4)}$$

Therefore, probability that both $\mathbf{M}_p$ and $\mathbf{N}_p$ are prime is

$$P(\mathbf{M}_p, \mathbf{N}_p) = P(\mathbf{M}_p) \cdot P(\mathbf{N}_p)$$

$$P(\mathbf{M}_p, \mathbf{N}_p) = \frac{1}{\ln(q/2)} \cdot \frac{2\ln(q/2)\ln(q)}{q\ln(q/4)}$$

$$P(\mathbf{M}_p, \mathbf{N}_p) = \frac{2\ln(q)}{q\ln(q/4)}$$

Number of prime pairs for which $m + n = q$ is valid is $q/2$ times the probability that both are primes.

$$N = \frac{q}{2} \cdot P(\mathbf{M}_p, \mathbf{N}_p)$$

$$N = \frac{q}{2} \cdot \frac{2\ln(q)}{q\ln(q/4)}$$

$$N = \frac{\ln(q)}{\ln(q/4)}$$

For every q that has $N \geq 1$ there is at least one pair of primes that when added give q.

$$N \geq 1$$

$$\frac{\ln(q)}{\ln(q/4)} \geq 1$$

Solving for q results in

$$q > 4$$

$q = 4$ is problematic, as for $q = 4$ $\frac{\ln(q)}{\ln(q/4)}$ seems undetermined. However, taking a limit using L'Hopital's rule gives us that indeed as we approach $q = 4$ number of combinations grows larger and larger until it reaches 1 when q is very close to 4.

$$\lim_{x \to 4} \frac{\ln(q)}{\ln(q/4)} = \frac{(\ln(q))'}{(\ln(q) - \ln(4))'} = \frac{\frac{1}{q}}{\frac{1}{q}} = 1$$

Since the function rises around $q = 4$ we can safely assume that at $q = 4$ number of combinations is at least one. Thus,

$$q \geq 4$$

Further work is required to establish a more formal proof for q around 4.

Therefore it follows that for every $a \geq 2q$, that is $a \geq 8$ there is at least one pair of primes $\mathbf{M}_p$ and $\mathbf{N}_p$ that give a. However, as this holds for *every even number larger or equal to 4* and not just $a = 2q$, as $q \geq 4$ it follows that 4 can also be written as a sum of two primes. Namely, Goldbach's conjecture is valid for every $n \geq 4$.

Interesting, but not relevant, is the fact that number of combinations of $\mathbf{M}_p$ and $\mathbf{M}_p$ can be closely approximated with $f(x) = \frac{x}{\ln^2 3x/4}$. Further investigation in this is required. Conclusion: there is at least one way you can write any number $n \geq 6$ as a sum of two primes.

Proving for a sum of three primes   Every even number can be written as a sum of p_1 and p_2 and 2, where $p_1 + p_2 = (n - 2)$. Thus, every number $n - 2 \geq 4 \to n \geq 6$ can be written as a sum of two primes and 2. Likewise, subtracting a prime $p_n > 2$ from an odd number yields an even number which can be written as $p_1 + p_2 = (n - p_n)$. This property holds for every odd number $n - 3 \geq 4 \to n \geq 7$.

3 Conclusion

This paper proves that Goldbach's conjecture is valid for all $n \in \mathbb{N} \geq 4$. Certainly, this proof is only a proof of concept and will require further work to add mathematical rigor to it. For further precision, this proof could be re-done using $Li(x)$ function which is more precise than $\pi(x)$ and other interesting results obtained. Problem with including 4 amongst numbers that can be produced by adding two primes is very bothering and further work is yet again required. Unless enormous mistakes are found in this version, further versions will address all issues, known and unknown to the author, that are present here.