On the Theory of Triple Integrals and Trilinear Forms

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ABSTRACT

In this paper we hope to evaluate some triple integrals involving trilinear forms, defined appropriately below.

We begin with a definition. A **trilinear form** is a function $f(x, y, z)$ which is linear in each of its coordinates. In other words, $f(x_1 + x_2, y, z) = f(x_1, y, z) + f(x_2, y, z)$ and so on.

Now consider the following triple integral,

$$\int_a^b \int_c^d \int_n^m f(x_1 + x_2, y_1 + y_2, z_1 + z_2) dxdydz \tag{1}$$

where f is a trilinear form. Our main theorem is a formula for determining this integrals values.

Theorem Suppose we have an integral in the form of (1) where f is a trilinear form. Then the integral evaluates as seen below.

Proof.

Because f is trilinear, we have $f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = f(x_1, y_1, z_1) + f(x_2, y_1, z_1) + f(x_1, y_2, z_1) + f(x_1, y_1, z_2) + f(x_1, y_2, z_2) + f(x_2, y_1, z_2) + f(x_2, y_2, z_1) + f(x_2, y_2, z_2)$ (check this).

Now we apply the linearity of the integral to obtain

$$\int_a^b \int_c^d \int_n^m f(x_1 + x_2, y_1 + y_2, z_1 + z_2) dxdydz = \int_a^b \int_c^d \int_n^m f(x_1, y_1, z_1) + f(x_2, y_1, z_1) \tag{2}$$

$$+f(x_1, y_2, z_1) + f(x_1, y_1, z_2) + f(x_1, y_2, z_2) + f(x_2, y_1, z_2) + f(x_2, y_2, z_1) + f(x_2, y_2, z_2) dxdydz$$

$$= \int_a^b \int_c^d \int_n^m f(x_1, y_1, z_1) dxdydz + \int_a^b \int_c^d \int_n^m f(x_2, y_1, z_1) dxdydz \tag{3}$$

$$+ \int_a^b \int_c^d \int_n^m f(x_1, y_2, z_1) dxdydz + \tag{4}$$

$$\int_a^b \int_c^d \int_n^m f(x_1, y_1, z_2) dxdydz + \int_a^b \int_c^d \int_n^m f(x_1, y_2, z_2) dxdydz + \tag{5}$$

$$\int_a^b \int_c^d \int_n^m f(x_2, y_1, z_2) dxdydz + \int_a^b \int_c^d \int_n^m f(x_2, y_2, z_1) dxdydz + \int_a^b \int_c^d \int_n^m f(x_2, y_2, z_2) dxdydz \tag{6}$$

Now we may use normal triple integral techniques to evaluate each of the integrals. □

We note that in the proof of the theorem we required both the trilinearity of f and the linearity of the integeral. Without either the proof would fail, showing they are necessary and sufficient conditions for the theorem to hold. Also, by "normal triple integral techniques" we refer the reader to standard texts in advanced analysis (WARNING: the related material is incredibly difficult and requires Ph.D level knowledge in math).

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