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On Andrica's Conjecture, Cramér's Conjecture, gaps Between Primes and Jacobi Theta Functions I

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ABSTRACT. The main objective of this paper is to develop upper and lower bound for the Andrica conjecture, gaps between primes, using Jacobi elliptic functions.

1. INTRODUCTION

In [1, p. 34] Richard K. Guy posted that DorinAndrica conjectures that, for all natural n, we have

(1) $\sqrt{p_{n+1}}-\sqrt{p_n}<1,$

consequently, dividing both sides of the equation (1) by $\sqrt{p_n}$, we have

(2) $\sqrt{\frac{p_{n+1}}{p_n}}-\frac{1}{\sqrt{p_n}}<1.$

We will use the following notation for gaps between primes:

$g(p_n):=p_{n+1}-p_n,$

that is related to Cramér's conjecture, which states

$$\limsup_{n\to\infty}\frac{p_{n+1}-p_n}{(\log p_n)^2}=1,$$

and the Rosser's theorem [2], which states that p_n is larger than $n\log n$. This can be improved by the following pair of bounds:

(3) $\log n+\log\log n-1<\frac{p_n}{n}<\log n+\log\log n,$

for $n>6.$

2. THEOREMS

THEOREM 1. *Let* $k:=\frac{p_n}{p_{n+1}}$ *to be a k modulus and* $n\geq 6$, *then*

$$\left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n+n\log\log n-n}<\sqrt{p_{n+1}}-\sqrt{p_n}<\left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n+n\log\log n},$$

where θ_2 *and* θ_3 *are Jacobi theta functions.*

Proof. Firstly, we consider the sequence of prime numbers

(4) $$2<3<5<7<11<13<17<19<\cdots p_{n-2}<p_{n-1}<p_n<p_{n+1}.$$

Second, we note that

(5) $$0<\frac{2}{3}<1, 0<\frac{3}{5}<1, 0<\frac{5}{7}<1, 0<\frac{7}{11}<1, 0<\frac{11}{13}<1, 0<\frac{13}{17}<1, 0<\frac{17}{19}<1, \ldots,$$
$$0<\frac{p_{n-2}}{p_{n-1}}<1, 0<\frac{p_{n-1}}{p_n}<1, 0<\frac{p_n}{p_{n+1}}<1.$$

Then, we define that

(6) $$k:=k_{n,n+1}=\frac{p_n}{p_{n+1}}\Leftrightarrow p_{n+1}=\frac{p_n}{k},$$

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where$k_{n,n+1}$ is the k modulus.

Substituting (6) in the left-hand side of (1), we find

$$(7)\sqrt{p_{n+1}}-\sqrt{p_n}=\frac{\sqrt{p_n}}{k^{1/2}}-\sqrt{p_n}=\sqrt{p_n}\left(\frac{1-k^{1/2}}{k^{1/2}}\right).$$

In [3, p. 83], we knew that

$$(8)k^{1/2}=\frac{\theta_2}{\theta_3}=\frac{\theta_2(0)}{\theta_3(0)}=\frac{\theta_2(0|\tau)}{\theta_3(0|\tau)},$$

whereτ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

We set (8) in (7)

$$(9)\sqrt{p_{n+1}}-\sqrt{p_n}=\sqrt{p_n}\left(\frac{1-\frac{\theta_2}{\theta_3}}{\frac{\theta_2}{\theta_3}}\right)=\sqrt{p_n}\left(\frac{\frac{\theta_3-\theta_2}{\theta_3}}{\frac{\theta_2}{\theta_3}}\right)=\sqrt{p_n}\left(\frac{\theta_3-\theta_2}{\theta_2}\right).$$

From (3) and(9), weconcludethat

$$\left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n+n\log\log n-n}<\sqrt{p_{n+1}}-\sqrt{p_n}<\left(\frac{\theta_3-\theta_2}{\theta_2}\right)\sqrt{n\log n+n\log\log n}.\square$$

COROLLARY 1. *Let* $k:=\frac{p_n}{p_{n+1}}$ *to be a k modulus, then Andrica's conjecture is equivalent to*

$$\sqrt{p_n}<\frac{\theta_2}{\theta_3-\theta_2}.$$

Proof. Dividing both members of (9) by$\sqrt{p_n}$, we have

$$(10)\ \sqrt{\frac{p_{n+1}}{p_n}}-\left(\frac{\theta_3-\theta_2}{\theta_2}\right)=1.$$

Comparing (2) with (10) and after some algebraic manipulation, we find

$$(11)\frac{\theta_3-\theta_2}{\theta_2}<\frac{1}{\sqrt{p_n}},$$

therefrom,

$$\sqrt{p_n}<\frac{\theta_2}{\theta_3-\theta_2}.\square$$

THEOREM 2. *Let* $k:=\frac{p_n}{p_{n+1}}$ *to be a k modulus, then*

$$\left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n+n\log\log n-n)<g(p_n)<\left(\frac{\theta_3^2-\theta_2^2}{\theta_2^2}\right)(n\log n+n\log\log n),$$

whereθ_2 *and* θ_3 *are Jacobi theta functions.*

Proof. We define that

$$(12)\qquad k:=k_{n,n+1}=\frac{p_n}{p_{n+1}}\Leftrightarrow p_{n+1}=\frac{p_n}{k},$$

where$k_{n,n+1}$ is the k modulus.

We consider that

$$(13)\qquad g(p_n):=p_{n+1}-p_n=\frac{p_n}{(k^{1/2})^2}-p_n=p_n\left[\frac{1-(k^{1/2})^2}{(k^{1/2})^2}\right].$$

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In [2, p. 83], we knew that

$$(14) k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)},$$

where τ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

We set (14) in (13)

$$(15) \qquad g(p_n) = p_{n+1} - p_n = p_n \left(\frac{1 - \frac{\theta_2^2}{\theta_3^2}}{\frac{\theta_2^2}{\theta_3^2}} \right) = p_n \left(\frac{\frac{\theta_3^2 - \theta_2^2}{\theta_3^2}}{\frac{\theta_2}{\theta_3^2}} \right) = p_n \left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right).$$

From (3) and (15), we conclude that

$$\left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right) (n \log n + n \log \log n - n) < g(p_n) < \left(\frac{\theta_3^2 - \theta_2^2}{\theta_2^2} \right) (n \log n + n \log \log n). \square$$

THEOREM 3. *Let* $k := \frac{p_n}{p_{n+1}}$ *to be a k modulus, then*

$$\left(\frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n - n)^2 < p_{n+1}^2 - p_n^2 < \left(\frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n)^2,$$

where θ_2 *and* θ_3 *are Jacobi theta functions.*

Proof. We define that

$$(16) \qquad k := k_{n,n+1} = \frac{p_n}{p_{n+1}} \Leftrightarrow p_{n+1} = \frac{p_n}{k},$$

where $k_{n,n+1}$ is the k modulus.

We consider that

$$(17) p_{n+1}^2 - p_n^2 = \frac{p_n^2}{(k^{1/2})^4} - p_n^2 = p_n^2 \left[\frac{1 - (k^{1/2})^4}{(k^{1/2})^4} \right].$$

In [2, p. 83], we knew that

$$(18) k^{1/2} = \frac{\theta_2}{\theta_3} = \frac{\theta_2(0)}{\theta_3(0)} = \frac{\theta_2(0|\tau)}{\theta_3(0|\tau)},$$

where τ is the parameter and $\theta_2(z|\tau)$ and $\theta_3(z|\tau)$ are Jacobi theta functions.

We set (18) in (17)

$$(19) p_{n+1}^2 - p_n^2 = p_n^2 \left(\frac{1 - \frac{\theta_2^4}{\theta_3^4}}{\frac{\theta_2^4}{\theta_3^4}} \right) = p_n^2 \left(\frac{\frac{\theta_3^4 - \theta_2^4}{\theta_3^4}}{\frac{\theta_2^4}{\theta_3^4}} \right) = p_n^2 \left(\frac{\theta_3^4 - \theta_2^4}{\theta_2^4} \right) = p_n^2 \left(\frac{\theta_4^4}{\theta_2^4} \right),$$

to see [3, p. 84] which states $\theta_2^4 + \theta_4^4 = \theta_3^4$, the Jacobi identity.

From (3) and (19), we conclude that

$$\left(\frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n - n)^2 < p_{n+1}^2 - p_n^2 < \left(\frac{\theta_4^4}{\theta_2^4} \right) (n \log n + n \log \log n)^2. \square$$

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