Investigations on the Theory of Riemann Zeta Function III: A Simple Proof for a Restricted Lindelöf Hypothesis

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Day unto day uttereth speech, and night unto night sheweth knowledge.
Psalms 19:2

ABSTRACT

We use the contradiction method for prove a restricted Lindelöf hypothesis.

1. INTRODUCTION

In [1], we encounter that Lindelöf, in his paper [2], showed that the function $\mu\left(\frac{1}{2}\right)$ is decreasing and convex. This led him to conjecture that $\mu\left(\frac{1}{2}\right) = 0$, and consequently that

(1.1) $$\zeta\left(\frac{1}{2} + it\right) \leq t^{\epsilon},$$

whatever $\epsilon > 0$.

In this paper, we will demonstrate that

(1.2) $$\zeta\left(\frac{1}{2} + it\right) \leq t^{\epsilon},$$

whatsoever $\epsilon > 0$ and any $t \in \mathbb{R}_{\geq 0.30611645227149686\ldots}$.

2. PRELIMINARES

In [3] we have a convergent series representation for $\zeta(s, q)$, defined when $q > -1$ and any complex $s \neq 1$, which was given by Helmut Hasse, in 1930 [4]:

(2.1) $$\zeta(s, q) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (q+k)^{1-s}.$$

This series converges uniformly on compact subsets of the s-plane to an entire function. The inner sum may be understood to be the n*th* forward difference of q^{1-s}; i.e.,

(2.2) $$\Delta^n q^{1-s} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (q+k)^{1-s},$$

where Δ denotes the forward difference operator. As soon, we may write

(2.3) $$\zeta(s, q) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \Delta^n q^{1-s}$$

$$= \frac{1}{s-1} \frac{\log(1+\Delta)}{\Delta} q^{1-s}.$$

In [5], we see that the complex exponentiation satisfies

(2.4) $$(a+bi)^{c+di} = (a^2+b^2)^{(c+id)/2} e^{i(c+id)\arg(a+ib)},$$

where $\arg(z)$ denotes the complex argument. We explicitly written in terms of real and imaginary parts, as follows

(2.5) $(a+bi)^{c+di} = (a^2+b^2)^{c/2}$

$$\times\left\{\cos\left[c\cdot\arg(a+ib)+\frac{1}{2}d\log(a^2+b^2)\right]+i\sin\left[c\cdot\arg(a+ib)+\frac{1}{2}d\log(a^2+b^2)\right]\right\}.$$

THEOREM 1. *Let* $\mathrm{Re}(s) > 0$ *and* $s \neq 1$, *then*

$$\zeta(s) = \frac{2^s}{2^s-1} + \frac{\zeta\left(s,\frac{3}{2}\right)}{2^s-1}, \tag{2.6}$$

where $\zeta(s)$ *is the Riemann zeta function and* $\zeta(s,a)$*is the Hurwitz zeta function.*

Proof. See [6]. □

3. LEMMAS AND THEOREMS

LEMMA 1. *For* $t \in \mathbb{R}_{\geq 0}$, *then*

$$(3.1)\ \zeta\left(\frac{1}{2}+it,\frac{3}{2}\right) = -\frac{2t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$-\frac{4t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$+\frac{2i}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$-\frac{4it}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right],$$

where $\zeta(s,a)$ *is the Hurwitz zeta function.*

Proof. Let $s = \frac{1}{2}+it$ and $q = \frac{3}{2}$ in (2.1)

$$\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right) = \frac{2}{-1+2it}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it} \tag{3.2}$$

$$= \frac{2}{-1+2it}\times\left(\frac{-1-2it}{-1-2it}\right)\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}$$

$$= \frac{-2-4it}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}.$$

On the other hand, we evaluate, using (2.5), that

$$\left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it} = \left(\frac{2k+3}{2}\right)^{1/2}\times \tag{3.3}$$

$$\times\left\{\cos\left[\frac{1}{2}\cdot\arg\left(\frac{2k+3}{2}\right)-t\log\left(\frac{2k+3}{2}\right)\right]+i\sin\left[\frac{1}{2}\cdot\arg\left(\frac{2k+3}{2}\right)-t\log\left(\frac{2k+3}{2}\right)\right]\right\}.$$

Since $k=0,1,2,3,\ldots$, then $\arg\left(\frac{2k+3}{2}\right)=0$; we set this in (3.3)

$$\text{(3.4)}\qquad \left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}=\left(\frac{2k+3}{2}\right)^{1/2}\times\left\{\cos\left[-t\log\left(\frac{2k+3}{2}\right)\right]+i\sin\left[-t\log\left(\frac{2k+3}{2}\right)\right]\right\}$$

$$=\left(\frac{2k+3}{2}\right)^{1/2}\times\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]-i\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}.$$

Substituting (3.4) in (3.2), we encounter

$$\text{(3.5)}\qquad \zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)=\left(\frac{-2-4it}{4t^2+1}\right)\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\times$$

$$\times\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]-i\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}$$

$$=\left(\frac{-2-4it}{4t^2+1}\right)\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$+\left(\frac{-4t+2i}{4t^2+1}\right)\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$=-\frac{2t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$-\frac{4it}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$-\frac{4t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$+\frac{2i}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$=-\frac{2t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$-\frac{4t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$+\frac{2i}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$-\frac{4it}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right].\ \square$$

THEOREM 1. *For* $\epsilon > 0$ *and any* $t \in \mathbb{R}_{\geq 0}$, *then*

$$\zeta\left(\frac{1}{2}+it\right) \leq t^{\epsilon}. \tag{3.6}$$

Proof. Hereinafter, we will use the *reductio ad absurdum* to prove (3.6).

Step 1. We assume, by hypothesis, that

$$\zeta\left(\frac{1}{2}+it\right) > t^{\epsilon}, \tag{3.7}$$

whatsoever $\epsilon > 0$ and any $t \in \mathbb{R}_{\geq 0}$. Let $s = \frac{1}{2} + it$ in (2.6)

$$\zeta\left(\frac{1}{2}+it\right) = \frac{2^{\frac{1}{2}+it}}{2^{\frac{1}{2}+it}-1} + \frac{\zeta\left(\frac{1}{2}+it, \frac{3}{2}\right)}{2^{\frac{1}{2}+it}-1}. \tag{3.8}$$

Substituting the right-hand side of (3.8) in (3.7), we obtain

$$\frac{2^{\frac{1}{2}+it}}{2^{\frac{1}{2}+it}-1} + \frac{\zeta\left(\frac{1}{2}+it, \frac{3}{2}\right)}{2^{\frac{1}{2}+it}-1} > t^{\epsilon} \Rightarrow 2^{it} + \frac{t^{\epsilon}}{\sqrt{2}} + \frac{\zeta\left(\frac{1}{2}+it, \frac{3}{2}\right)}{\sqrt{2}} > 2^{it}t^{\epsilon}. \tag{3.9}$$

Step 2. We defined

$$C(t) := \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos\left[t \log\left(\frac{2k+3}{2}\right)\right] \tag{3.10}$$

and

$$S(t) := \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \sin\left[t \log\left(\frac{2k+3}{2}\right)\right] \tag{3.11}$$

using this in (3.1)

$$\begin{aligned} \zeta\left(\frac{1}{2}+it, \frac{3}{2}\right) &= -\frac{2t}{4t^2+1}C(t) - \frac{4t}{4t^2+1}S(t) + \frac{2i}{4t^2+1}S(t) - \frac{4it}{4t^2+1}C(t) \\ &= -\frac{2t}{4t^2+1} \cdot [C(t) + 2S(t)] + \frac{2i}{4t^2+1}[S(t) - 2tC(t)]. \end{aligned} \tag{3.12}$$

Step 3. We use (2.5) for evaluate 2^{it}, as follows

$$2^{it} = \cos(t\log 2) + i\sin(t\log 2). \tag{3.13}$$

Step 4. From (3.9), (3.12) and (3.13), we obtain

$$\begin{aligned} &\cos(t\log 2) + i\sin(t\log 2) + \frac{t^{\epsilon}}{\sqrt{2}} - \frac{\sqrt{2}t}{4t^2+1} \cdot [C(t) + 2S(t)] \\ &+ \frac{\sqrt{2}i}{4t^2+1}[S(t) - 2tC(t)] > \cos(t\log 2) \cdot t^{\epsilon} + i\sin(t\log 2) \cdot t^{\epsilon}, \end{aligned} \tag{3.14}$$

so

$$(3.15)\quad \cos(t\log 2)+\frac{t^{\epsilon}}{\sqrt{2}}-\frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)]+i\sin(t\log 2)$$
$$+\frac{\sqrt{2}i}{4t^2+1}[S(t)-2tC(t)]>\cos(t\log 2)\cdot t^{\epsilon}+i\sin(t\log 2)\cdot t^{\epsilon},$$

Step 5. We compare the real and imaginary part separately of (3.15). Therefore, for the real part, we find

$$(3.16)\qquad \cos(t\log 2)+\frac{t^{\epsilon}}{\sqrt{2}}-\frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)]>\cos(t\log 2)\cdot t^{\epsilon}$$
$$\Rightarrow \cos(t\log 2)+\frac{t^{\epsilon}}{\sqrt{2}}>\cos(t\log 2)\cdot t^{\epsilon}+\frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)]$$
$$\Rightarrow \frac{\cos(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>\cos(t\log 2)+\frac{\sqrt{2}t}{(4t^2+1)t^{\epsilon}}\cdot[C(t)+2S(t)].$$

and, for the imaginary part, we encounter

$$(3.17)\qquad \sin(t\log 2)+\frac{\sqrt{2}}{4t^2+1}[S(t)-2tC(t)]>\sin(t\log 2)\cdot t^{\epsilon}$$
$$\Rightarrow \sin(t\log 2)+\frac{\sqrt{2}}{4t^2+1}S(t)>\sin(t\log 2)\cdot t^{\epsilon}+\frac{2t\sqrt{2}}{4t^2+1}C(t).$$

Step 6. Real part. We divide the inequality (3.16) by $t^{2+\epsilon}$

$$(3.18)\qquad \frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}+\frac{1}{t^{2+\epsilon}\sqrt{2}}>\frac{\cos(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot[C(t)+2S(t)].$$

We evaluate the limit when $t\to+\infty$ of (3.18)

$$(3.19)\quad \lim_{t\to+\infty}\left[\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}+\frac{1}{t^{2+\epsilon}\sqrt{2}}\right]>\lim_{t\to+\infty}\left\{\frac{\cos(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot[C(t)+2S(t)]\right\},$$
$$\lim_{t\to+\infty}\left[\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}\right]+\lim_{t\to+\infty}\left(\frac{1}{t^{2+\epsilon}\sqrt{2}}\right)$$
$$>\lim_{t\to+\infty}\left[\frac{\cos(t\log 2)}{t^{2+\epsilon}}\right]+\lim_{t\to+\infty}\left\{\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot[C(t)+2S(t)]\right\}.$$

Note 1: We calculate, for any $\epsilon>0$ and $k=0,1,2,3,\ldots$: 1.º) when $t\to+\infty$, then $\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}\to 0$; 2.º) when $t\to+\infty$, then $\frac{1}{t^{2+\epsilon}\sqrt{2}}\to 0$; 3.º) when $t\to+\infty$, then $\frac{\cos(t\log 2)}{t^{2+\epsilon}}\to 0$; 4.º) and when $t\to+\infty$, then $\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]+2\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}\to 0$. So, our hypothesis is false, because $0\ngtr 0$; but, $0=0$. We conclude that our hypothesis for the real part is false.

Step 7. Imaginary part. We divide the inequality (3.17) by $t^{2+\epsilon}$

$$(3.20)\qquad \frac{\sin(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}S(t)>\frac{\sin(t\log 2)\cdot t^{\epsilon}}{t^{2+\epsilon}}+\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}C(t)$$
$$\Rightarrow \frac{\sin(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}S(t)>\frac{\sin(t\log 2)}{t^2}+\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}C(t).$$

We evaluate the limit when $t \to +\infty$ of (3.20)

(3.21) $\lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}S(t)\right] > \lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^{2}}+\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}C(t)\right],$

$$\lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^{2+\epsilon}}\right]+\lim_{t\to+\infty}\left[\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}S(t)\right] > \lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^{2}}\right]+\lim_{t\to+\infty}\left[\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}C(t)\right].$$

Note 2: We calculate, for any $\epsilon > 0$ and $k = 0,1,2,3,\ldots$: 1.º) when $t \to +\infty$, then $\frac{\sin(t\log 2)}{t^{2+\epsilon}} \to 0$; 2.º) when $t \to +\infty$, then $\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right] \to 0$; 3.º) when $t \to +\infty$, then $\frac{\sin(t\log 2)}{t^2} \to 0$; 4.º) and when $t \to +\infty$, then $\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right] \to 0$. So, our hypothesis is false, because $0 \ngtr 0$; but, $0 = 0$. We conclude that our hypothesis for the imaginary part is false.

Step 8. We evaluate the function $\max_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}$. For $k = 0$, then $\max_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{3}{2}\right)\right]\right\} = 1$, when $t = 0$; for $k = 1$, then $\max_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{5}{2}\right)\right]\right\} = 1$, when $t = 0$; for $k = 2$, then $\max_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{7}{2}\right)\right]\right\} = 1$, when $t = 0$; and so on. We easily deduzimos que $\max_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\} = 1$, when $t = 0$. So,

(3.22)
$$\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right] =: C(t)$$
$$< \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\max_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}$$
$$= \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2},$$

where $\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}$ tends to 1.0095 ... very slowly. Simplifying, we find

(3.23) $$1.0095\ldots > C(t)$$

On the other hand, continuando with the evaluate for the function $\min_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}$. For $k = 0$, then $\min_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{3}{2}\right)\right]\right\} = \cos(\pi) = -1$, when $t = \frac{\pi}{\log 3-\log 2} = 7.74812083893\ldots$; for $k = 1$, then $\min_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{5}{2}\right)\right]\right\} = \cos(\pi) = -1$, when $t = \frac{\pi}{\log 5-\log 2} = 3.42859809044\ldots$; for $k = 2$, then $\min_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{7}{2}\right)\right]\right\} = \cos(\pi) = -1$, when $t = \frac{\pi}{\log 7-\log 2} = 2.50773109726\ldots$; and so on. We easily deduzimos que $\min_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\} = \cos(\pi) = -1$, when $t = \frac{\pi}{\log(2k+3)-\log 2}$, for $k = 0,1,2,3,\ldots$. So,

$$(3.24)\quad \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]=:C(t)>$$
$$>\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\min_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}$$
$$=-\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2},$$

where $\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}$ tends to 1.0095 ... very slowly. Simplifying, we have

$$(3.25)\quad C(t)>-1.0095\ ....$$

From (3.23) and (3.25), it follows that

$$(3.26)\quad -1.0095\ ...<C(t)<1.0095\ ....$$

Step 10. From (3.16) and (3.17), we have the following system of inequalities

$$(3.27)\quad \begin{cases}\cos(t\log 2)+\dfrac{t^{\epsilon}}{\sqrt{2}}>\cos(t\log 2)\cdot t^{\epsilon}+\dfrac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)]\\ \sin(t\log 2)+\dfrac{\sqrt{2}S(t)}{4t^2+1}>\sin(t\log 2)\cdot t^{\epsilon}+2t\dfrac{\sqrt{2}C(t)}{4t^2+1}\end{cases}$$

Dividing both the members of (3.27) by t^{ϵ}, we encounter

$$(3.28)\quad \begin{cases}\dfrac{\cos(t\log 2)}{t^{\epsilon}}+\dfrac{1}{\sqrt{2}}>\cos(t\log 2)+\dfrac{\sqrt{2}t}{t^{\epsilon}(4t^2+1)}\cdot[C(t)+2S(t)]\\ \dfrac{\sin(t\log 2)}{t^{\epsilon}}+\dfrac{\sqrt{2}S(t)}{t^{\epsilon}(4t^2+1)}>\sin(t\log 2)+\dfrac{2\sqrt{2}tC(t)}{t^{\epsilon}(4t^2+1)}\end{cases}$$

From second inequality of (3.28), we find

$$(3.29)\quad \frac{\sqrt{2}S(t)}{t^{\epsilon}(4t^2+1)}>\sin(t\log 2)-\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{2\sqrt{2}tC(t)}{t^{\epsilon}(4t^2+1)}.$$

From first inequality of (3.28) and the right-hand side of (3.29), we have

$$(3.30)\quad \frac{\cos(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>\cos(t\log 2)+\frac{\sqrt{2}t}{t^{\epsilon}(4t^2+1)}\cdot[C(t)+2S(t)]$$
$$=\cos(t\log 2)+t\cdot\left[\frac{\sqrt{2}C(t)}{t^{\epsilon}(4t^2+1)}+2\frac{\sqrt{2}S(t)}{t^{\epsilon}(4t^2+1)}\right]$$
$$>\cos(t\log 2)+t\cdot\left[\frac{\sqrt{2}C(t)}{t^{\epsilon}(4t^2+1)}+2\sin(t\log 2)-2\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{4\sqrt{2}tC(t)}{t^{\epsilon}(4t^2+1)}\right]$$
$$=\cos(t\log 2)+2t\sin(t\log 2)-2t\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{\sqrt{2}t(4t+1)C(t)}{t^{\epsilon}(4t^2+1)}.$$

Consequently,

$$(3.31)\quad \frac{\cos(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>\cos(t\log 2)+2t\sin(t\log 2)-2t\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{\sqrt{2}t(4t+1)C(t)}{t^{\epsilon}(4t^2+1)}.$$

Thus,

$$(3.32)\ \frac{t^{\epsilon}(4t^2+1)}{\sqrt{2}t(4t+1)}\left[\frac{\cos(t\log 2)}{t^{\epsilon}}-\cos(t\log 2)-2t\sin(t\log 2)+2t\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}\right]>C(t).$$

Step 10. We divide (3.32) by (3.23)

$$\frac{\cos(t\log 2)}{t^{\epsilon}}-\cos(t\log 2)-2t\sin(t\log 2)+2t\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>1.0095\ldots\frac{\sqrt{2}t(4t+1)}{t^{\epsilon}(4t^2+1)},$$

$$(3.33)\ \frac{\cos(t\log 2)}{t^{\epsilon}}+2t\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>\cos(t\log 2)+2t\sin(t\log 2)+1.0095\ldots\frac{\sqrt{2}t(4t+1)}{t^{\epsilon}(4t^2+1)}.$$

Multiplying (3.33) by t^{ϵ}, we find

$$(3.34)\quad \cos(t\log 2)+2t\sin(t\log 2)$$
$$>\left[\cos(t\log 2)+2t\sin(t\log 2)+1.0095\ldots\frac{\sqrt{2}t(4t+1)}{4t^2+1}-\frac{1}{\sqrt{2}}\right]t^{\epsilon}.$$

From (3.34), we deduce that

$$(3.35)\quad \cos(t\log 2)+2t\sin(t\log 2)$$
$$>\left[\cos(t\log 2)+2t\sin(t\log 2)+1.0095\ldots\frac{\sqrt{2}t(4t+1)}{4t^2+1}-\frac{1}{\sqrt{2}}\right]t^{\epsilon}$$
$$>\left[\cos(t\log 2)+2t\sin(t\log 2)+1.0095\ldots\frac{\sqrt{2}t(4t+1)}{4t^2+1}-\frac{1}{\sqrt{2}}\right]t^{0}$$
$$=\cos(t\log 2)+2t\sin(t\log 2)+1.0095\ldots\frac{\sqrt{2}t(4t+1)}{4t^2+1}-\frac{1}{\sqrt{2}}.$$

It is easy to see that: for $1.0095\ldots\frac{\sqrt{2}t(4t+1)}{4t^2+1}\geq\frac{1}{\sqrt{2}}$, the inequality (3.35) is false.

Note 3: For any $t\in\mathbb{R}^{+}$ and $t\geq 0.30611645227149686\ldots$, then the inequality (3.40) is false. So, our hypothesis is false.

Step 11. Thus, from Notes 1, 2 and 3, we show that $\zeta\left(\frac{1}{2}+it\right)\leq t^{\epsilon}$, for whatsoever $\epsilon>0$ and any $t\in\mathbb{R}_{\geq 0.30611645227149686\ldots}$. □

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