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Investigations on the Theory of Riemann Zeta Function III: A Simple Proof for the Lindelöf Hypothesis

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ABSTRACT

We create new formulas for proving Lindelof Hypothesis from Zeta Function.

1. INTRODUCTION

In [1], we encounter that Lindelöf, in his paper [2], showed that the function$\mu\left(\frac{1}{2}\right)$ is decreasing and convex. This led him to conjecture that$\mu\left(\frac{1}{2}\right)=0$, and consequently that

$$(1.1)\qquad \zeta\left(\frac{1}{2}+it\right)\leq t^{\epsilon},$$

Whatever$\epsilon>0$.

In this paper, we will demonstrate that

$$(1.2)\zeta\left(\frac{1}{2}+it\right)\leq t^{\epsilon},$$

Whatsoever$\epsilon>0$ and any $t\in\mathbb{R}_{\geq 0}$.

2. PRELIMINARES

In [3] we have a convergent series representation for $\zeta(s,q)$, defined when $q>-1$ and any complex $s\neq 1$, which was given by Helmut Hasse, in 1930 [4]:

$$(2.1)\zeta(s,q)=\frac{1}{s-1}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(q+k)^{1-s}.$$

This series converges uniformly on compact subsets of the s-plane to an entire function. The inner sum may be understood to be the n*th* forward difference of q^{1-s}; i.e.,

$$(2.2)\Delta^n q^{1-s}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(q+k)^{1-s},$$

WhereΔ denotes the forward difference operator. As soon, we may write

$$(2.3)\zeta(s,q)=\frac{1}{s-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\Delta^n q^{1-s}$$

$$=\frac{1}{s-1}\frac{\log(1+\Delta)}{\Delta}q^{1-s}.$$

In [5], we see that the complex exponentiation satisfies

(2.4) $(a+bi)^{c+di}=(a^2+b^2)^{(c+id)/2}e^{i(c+id)\arg(a+ib)}$,

Where $\arg(z)$ denotes the complex argument. We explicitly written in terms of real and imaginary parts, as follows

$$(2.5)\ (a+bi)^{c+di}=(a^2+b^2)^{c/2}$$
$$\times\left\{\cos\left[c\cdot\arg(a+ib)+\frac{1}{2}d\log(a^2+b^2)\right]+i\sin\left[c\cdot\arg(a+ib)+\frac{1}{2}d\log(a^2+b^2)\right]\right\}.$$

THEOREM 1. *Let* $\mathrm{Re}(s)>0$ *and* $s\neq 1$, *then*

$$(2.6)\ \zeta(s)=\frac{2^s}{2^s-1}+\frac{\zeta\left(s,\frac{3}{2}\right)}{2^s-1},$$

Where $\zeta(s)$ *is the Riemann zeta function and* $\zeta(s,a)$ *is the Hurwitz zeta function.*

Proof. See [6]. □

3. LEMMAS AND THEOREMS

LEMMA 1. *For* $t\in\mathbb{R}_{\geq 0}$, *then*

$$(3.1)\ \zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)=-\frac{2t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$-\frac{4t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$+\frac{2i}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$-\frac{4it}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right],$$

Where $\zeta(s,a)$ *is the Hurwitz zeta function.*

Proof: Let $s=\frac{1}{2}+it$ and $q=\frac{3}{2}$ in (2.1)

$$(3.2)\ \zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)=\frac{2}{-1+2it}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}$$
$$=\frac{2}{-1+2it}\times\left(\frac{-1-2it}{-1-2it}\right)\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}$$
$$=\frac{-2-4it}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}.$$

On the other hand, we evaluate, using (2.5), that

$$(3.3)\ \left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}=\left(\frac{2k+3}{2}\right)^{1/2}\times$$

$$\times\left\{\cos\left[\frac{1}{2}\cdot\arg\left(\frac{2k+3}{2}\right)-t\log\left(\frac{2k+3}{2}\right)\right]+i\sin\left[\frac{1}{2}\cdot\arg\left(\frac{2k+3}{2}\right)-t\log\left(\frac{2k+3}{2}\right)\right]\right\}.$$

Since $k=0,1,2,3,\ldots$, then $\arg\left(\frac{2k+3}{2}\right)=0$; we set this in (3.3)

$$(3.4)\ \left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}=\left(\frac{2k+3}{2}\right)^{1/2}\times\left\{\cos\left[-t\log\left(\frac{2k+3}{2}\right)\right]+i\sin\left[-t\log\left(\frac{2k+3}{2}\right)\right]\right\}$$
$$=\left(\frac{2k+3}{2}\right)^{1/2}\times\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]-i\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}.$$

Substituting (3.4) in (3.2), we encounter

$$(3.5)\ \zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)=\left(\frac{-2-4it}{4t^2+1}\right)\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\times$$
$$\times\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]-i\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}$$
$$=\left(\frac{-2-4it}{4t^2+1}\right)\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$+\left(\frac{-4t+2i}{4t^2+1}\right)\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$=-\frac{2t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$-\frac{4it}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$-\frac{4t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$+\frac{2i}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$=-\frac{2t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$-\frac{4t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$+\frac{2i}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$-\frac{4it}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right].\ \square$$

THEOREM 1.*For* $\epsilon > 0$ *and any* $t \in \mathbb{R}_{\geq 0}$, *then*

$$\zeta\left(\frac{1}{2}+it\right) \leq t^{\epsilon}. \tag{3.6}$$

Proof: Hereinafter, we will use the *reduction ad absurdum* to prove (3.6).
Step1. We assume, by hypothesis, that

$$\zeta\left(\frac{1}{2}+it\right) > t^{\epsilon}, \tag{3.7}$$

Whatsoever $\epsilon > 0$ and any $t \in \mathbb{R}_{\geq 0}$.Let $s = \frac{1}{2}+it$ in (2.6)

$$\zeta\left(\frac{1}{2}+it\right) = \frac{2^{\frac{1}{2}+it}}{2^{\frac{1}{2}+it}-1} + \frac{\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)}{2^{\frac{1}{2}+it}-1}. \tag{3.8}$$

Substituting the right-hand side of (3.8) in (3.7), we obtain

$$\frac{2^{\frac{1}{2}+it}}{2^{\frac{1}{2}+it}-1} + \frac{\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)}{2^{\frac{1}{2}+it}-1} > t^{\epsilon} \Rightarrow 2^{it} + \frac{t^{\epsilon}}{\sqrt{2}} + \frac{\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)}{\sqrt{2}} > 2^{it}t^{\epsilon}. \tag{3.9}$$

Step 2. We defined

$$C(t) := \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right] \tag{3.10}$$

and

$$S(t) := \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right] \tag{3.11}$$

using this in (3.1)

$$\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right) = -\frac{2t}{4t^2+1}C(t) - \frac{4t}{4t^2+1}S(t) + \frac{2i}{4t^2+1}S(t) - \frac{4it}{4t^2+1}C(t) \tag{3.12}$$

$$= -\frac{2t}{4t^2+1}\cdot[C(t)+2S(t)] + \frac{2i}{4t^2+1}[S(t)-2tC(t)].$$

Step 3. We use (2.5) for evaluate 2^{it}, as follows

$$2^{it} = \cos(t\log 2) + i\sin(t\log 2). \tag{3.13}$$

Step 4. From (3.9), (3.12) and (3.13), we obtain

$$\cos(t\log 2) + i\sin(t\log 2) + \frac{t^{\epsilon}}{\sqrt{2}} - \frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)] + \frac{\sqrt{2}i}{4t^2+1}[S(t)-2tC(t)] \tag{3.14}$$
$$> \cos(t\log 2)\cdot t^{\epsilon} + i\sin(t\log 2)\cdot t^{\epsilon},$$

so

$$\cos(t\log 2) + \frac{t^{\epsilon}}{\sqrt{2}} - \frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)] + i\sin(t\log 2) + \frac{\sqrt{2}i}{4t^2+1}[S(t)-2tC(t)] \tag{3.15}$$
$$> \cos(t\log 2)\cdot t^{\epsilon} + i\sin(t\log 2)\cdot t^{\epsilon},$$

Step 5. We compare the real and imaginary part separately of (3.15). Therefore, for the real part, we find

$$\cos(t\log 2) + \frac{t^{\epsilon}}{\sqrt{2}} - \frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)] > \cos(t\log 2)\cdot t^{\epsilon} \tag{3.16}$$
$$\Rightarrow \cos(t\log 2) + \frac{t^{\epsilon}}{\sqrt{2}} > \cos(t\log 2)\cdot t^{\epsilon} + \frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)]$$

$$\Rightarrow \frac{\cos(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>\cos(t\log 2)+\frac{\sqrt{2}t}{(4t^2+1)t^{\epsilon}}\cdot[C(t)+2S(t)].$$

and, for the imaginary part, we encounter

$$(3.17)\ \sin(t\log 2)+\frac{\sqrt{2}}{4t^2+1}[S(t)-2tC(t)]>\sin(t\log 2)\cdot t^{\epsilon}$$

$$\Rightarrow \sin(t\log 2)+\frac{\sqrt{2}}{4t^2+1}S(t)>\sin(t\log 2)\cdot t^{\epsilon}+\frac{2t\sqrt{2}}{4t^2+1}C(t).$$

Step 6. Real part. We divide the inequality (3.16) by $t^{2+\epsilon}$

$$(3.18)\ \frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}+\frac{1}{t^{2+\epsilon}\sqrt{2}}>\frac{\cos(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot[C(t)+2S(t)].$$

We evaluate the limit when $t\to+\infty$ of (3.18)

$$(3.19)\ \lim_{t\to+\infty}\left[\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}+\frac{1}{t^{2+\epsilon}\sqrt{2}}\right]>\lim_{t\to+\infty}\left\{\frac{\cos(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot[C(t)+2S(t)]\right\},$$

$$\lim_{t\to+\infty}\left[\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}\right]+\lim_{t\to+\infty}\left(\frac{1}{t^{2+\epsilon}\sqrt{2}}\right)>\lim_{t\to+\infty}\left[\frac{\cos(t\log 2)}{t^{2+\epsilon}}\right]+\lim_{t\to+\infty}\left\{\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot[C(t)+2S(t)]\right\}.$$

Note 1: We calculate, for any $\epsilon>0$ and $k=0,1,2,3,\ldots$: 1.º) when $t\to+\infty$, then $\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}\to 0$; 2.º) when $t\to+\infty$, then $\frac{1}{t^{2+\epsilon}\sqrt{2}}\to 0$;3.º) when $t\to+\infty$, then $\frac{\cos(t\log 2)}{t^{2+\epsilon}}\to 0$;4.º) and when $t\to+\infty$, then $\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]+2\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}\to 0$.So, our hypothesis is false, because $0\ngtr 0$; but, $0=0$.

We evaluate the limit when $t\to 0^+$ of (3.18)

$$(3.20)\ \lim_{t\to 0^+}\left[\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}+\frac{1}{t^{2+\epsilon}\sqrt{2}}\right]>\lim_{t\to 0^+}\left\{\frac{\cos(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot[C(t)+2S(t)]\right\},$$

$$\lim_{t\to 0^+}\left[\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}\right]+\lim_{t\to 0^+}\left(\frac{1}{t^{2+\epsilon}\sqrt{2}}\right)>\lim_{t\to 0^+}\left[\frac{\cos(t\log 2)}{t^{2+\epsilon}}\right]+\lim_{t\to 0^+}\left\{\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot[C(t)+2S(t)]\right\}.$$

Note 2: We calculate, for any $\epsilon>0$ and $k=0,1,2,3,\ldots$: 1.º) when $t\to 0^+$, then $\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}\to+\infty$; 2.º) when $t\to 0^+$, then $\frac{1}{t^{2+\epsilon}\sqrt{2}}\to+\infty$; 3.º) when $t\to 0^+$, then $\frac{\cos(t\log 2)}{t^{2+\epsilon}}\to+\infty$;4.º) and when $t\to 0^+$, then $\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]+2\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}\to+\infty$. So, we hypothesis is false, because $\infty+\infty\ngtr\infty+\infty$; but, $\infty+\infty=\infty+\infty$.

Conclusion 1: we conclude, from Note 1 and Note 2, that our hypothesis for the real part is false.

Step 7. Imaginary part. We divide the inequality (3.17) by $t^{2+\epsilon}$

$$(3.21)\ \frac{\sin(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}S(t)>\frac{\sin(t\log 2)\cdot t^{\epsilon}}{t^{2+\epsilon}}+\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}C(t)$$

$$\Rightarrow \frac{\sin(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}S(t)>\frac{\sin(t\log 2)}{t^2}+\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}C(t).$$

We evaluate the limit when $t\to+\infty$ of (3.21)

$$(3.22)\ \lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}}{(4t^2+1)\,t^{2+\epsilon}}S(t)\right]>\lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^2}+\frac{2t\sqrt{2}}{(4t^2+1)\,t^{2+\epsilon}}C(t)\right],$$

$$\lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^{2+\epsilon}}\right]+\lim_{t\to+\infty}\left[\frac{\sqrt{2}}{(4t^2+1)\,t^{2+\epsilon}}S(t)\right]>\lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^2}\right]+\lim_{t\to+\infty}\left[\frac{2t\sqrt{2}}{(4t^2+1)\,t^{2+\epsilon}}C(t)\right].$$

Note 3: We calculate, for any $\epsilon > 0$ and $k = 0,1,2,3,\ldots$: 1.º) when $t \to +\infty$, then $\frac{\sin(t \log 2)}{t^{2+\epsilon}} \to 0$; 2.º) when $t \to +\infty$, then $\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} \sin\left[t \log\left(\frac{2k+3}{2}\right)\right] \to 0$; 3.º) when $t \to +\infty$, then $\frac{\sin(t \log 2)}{t^2} \to 0$; 4.º) and when $t \to +\infty$, then $\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} \cos\left[t \log\left(\frac{2k+3}{2}\right)\right] \to 0$. So, our hypothesis is false, because $0 \not> 0$; but, $0 = 0$.

We evaluate the limit when $t \to 0^+$ of (3.18)

$$(3.23)\ \lim_{t \to 0^+}\left[\frac{\sin(t \log 2)}{t^{2+\epsilon}} + \frac{\sqrt{2}}{(4t^2+1)\,t^{2+\epsilon}} S(t)\right] > \lim_{t \to 0^+}\left[\frac{\sin(t \log 2)}{t^2} + \frac{2t\sqrt{2}}{(4t^2+1)\,t^{2+\epsilon}} C(t)\right],$$

$$\lim_{t \to 0^+}\left[\frac{\sin(t \log 2)}{t^{2+\epsilon}}\right] + \lim_{t \to 0^+}\left[\frac{\sqrt{2}}{(4t^2+1)\,t^{2+\epsilon}} S(t)\right] > \lim_{t \to 0^+}\left[\frac{\sin(t \log 2)}{t^2}\right] + \lim_{t \to 0^+}\left[\frac{2t\sqrt{2}}{(4t^2+1)\,t^{2+\epsilon}} C(t)\right].$$

Note 4: We calculate, for any $\epsilon > 0$ and $k = 0,1,2,3,\ldots$: 1.º) when $t \to 0^+$, then $\frac{\sin(t \log 2)}{t^{2+\epsilon}} \to \infty$; 2.º) when $t \to 0^+$, then $\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} \sin\left[t \log\left(\frac{2k+3}{2}\right)\right] \to \infty$; 3.º) when $t \to 0^+$, then $\frac{\sin(t \log 2)}{t^2} \to \infty$; 4.º) and when $t \to 0^+$, then $\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} \cos\left[t \log\left(\frac{2k+3}{2}\right)\right] \to \infty$. So, we hypothesis is false, because $\infty + \infty \not> \infty + \infty$; but, $\infty + \infty = \infty + \infty$.

Conclusion 2: we conclude, from Note 3 and Note 4, that our hypothesis for the imaginary part is false.

Step 8. We evaluate any particular limit of (3.7), it follows that

$$\lim_{t \to N^+} \zeta\left(\frac{1}{2} + it\right) > \lim_{t \to N^+} t^{\epsilon} \geq N^0 = 1,$$

for$N \in \mathbb{R}^+$; we consider $N = 1$, and we obtain

$$\zeta\left(\frac{1}{2} + i\right) > 1.$$

Conclusion 3: numerically speaking, our hypothesis is:

$0.143936427077\ldots - 0.722099743532\ldots i > 1,$

This is false; because, for real part:$0.143936427077\ldots < 1$; and, for imaginary part, $-0.722099743532\ldots i < 0 \cdot i$.

Step 9. Thus, from Conclusion 1, 2 and 3, we show that$\zeta\left(\frac{1}{2} + it\right) \leq t^{\epsilon}$, for whatsoever$\epsilon > 0$ and any $t \in \mathbb{R}_{\geq 0}$.□

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