Some formulas and pattern

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Abstract

Some formulas and pattern.

Definition of zeta function

Let

$$\zeta_A^B(x) := \sum_{n=A}^{B} \frac{1}{n^x} \tag{1}$$

where $A, B \in \mathbb{Z}$ and $x \in \mathbb{C}$.

Consider further $a_n := 1, 2, 4, 11, 31, 83, 227, \ldots$ (OEIS-A002387) which is based on $\zeta_1^\infty(1)$.

A zeta function based construction of e, γ and π

Based on the zeta function ζ and the sequence a_n,
the constants γ, e and $\tau := 2\pi$ can be constructed as follows:

$$e = \lim_{n\to\infty} \frac{a_{n+1}}{a_{n+0}}$$

$$\gamma = \lim_{n\to\infty} \zeta_1^{\lfloor (\frac{a_{n+1}}{a_{n+0}})^n \rfloor}(1) - n$$

$$\tau = \min(\{ x \in \mathbb{R}^+ : \lim_{n\to\infty} \frac{\zeta_{a_{n+0}}^{a_{n+1}}(1+ix)}{\zeta_{a_{n+1}}^{a_{n+2}}(1+ix)} \neq (\frac{a_{n+1}}{a_{n+0}})^{ix} \})$$

Note that even though numerical evidence indicates that the sequence a_n coincides with $[e^{n-\gamma}]$, this relationship is unproven and is not exploited in the above construction of e, γ and τ.

Euler's identity

The previous construction of π is based on the formula:

$$\lim_{n\to\infty} \frac{\zeta_{a_{n+0}}^{a_{n+1}}(x)}{\zeta_{a_{n+1}}^{a_{n+2}}(x)} = e^{x-1},$$

which holds for all $x \in \mathbb{C}$ except those x of the form $2ki\pi + 1$ with $k \in \mathbb{Z}\backslash\{0\}$.

Euler's identity $e^{i\pi} = -1$ appears as special case of above formula for $x = 1i\pi + 1$.

x	$cos(x) + i \cdot sin(x)$		x	$\lim \frac{\zeta_{a_{n+0}}^{a_{n+1}}(x)}{\zeta_{a_{n+1}}^{a_{n+2}}(x)}$
5π	$e^{5i\pi}$		$5i\pi + 1$	$e^{5i\pi}$
4π	$e^{4i\pi}$		$4i\pi + 1$	$\sim$
3π	$e^{3i\pi}$		$3i\pi + 1$	$e^{3i\pi}$
2π	$e^{2i\pi}$		$2i\pi + 1$	$\sim$
1π	$e^{i\pi}$		$1i\pi + 1$	$e^{i\pi}$
0π	1		$0i\pi + 1$	1
-1π	$e^{-i\pi}$		$-1i\pi + 1$	$e^{-i\pi}$
-1π	$e^{-2i\pi}$		$-2i\pi + 1$	$\sim$
-3π	$e^{-3i\pi}$		$-3i\pi + 1$	$e^{-3i\pi}$
-4π	$e^{-4i\pi}$		$-4i\pi + 1$	$\sim$
-5π	$e^{-5i\pi}$		$-5i\pi + 1$	$e^{-5i\pi}$

The wave symbol ($\sim$) indicates that the limes does not exist

An alternative zeta function (in esp. harmonic) based construction of π is given by:

$$\pi = \lim_{n\to\infty} \quad n \cdot \Im(1 - (\frac{1}{\gamma - H_n})^{\frac{1}{n}})$$

or

$$\pi = \lim_{n\to\infty} n \cdot \Im((\gamma - H_n)^{\frac{1}{n}})$$

where H_n is the n-th harmonic number. This construction arises from the indeterminate forms 0^0 and $1/0$ (see Section 4).

It can be concluded, that γ, e and π can be constructed exclusively from the harmonic series, considering the sequence a_n. In contrast to that, it is unknown if $a_n \stackrel{?}{=} [e^{n-\gamma}]$ can be constructed from those constants. This question might be of interest in an evolutionary context.

1 Some notes on the fraction of harmonic numbers

Definitions

$$A(n) := \sum_{i=1}^{n} \frac{1}{i} - log(n)$$

$$B(n) := \sum_{i=1}^{n} \frac{1}{i} - \lfloor \sum_{i=1}^{n} \frac{1}{i} \rfloor$$

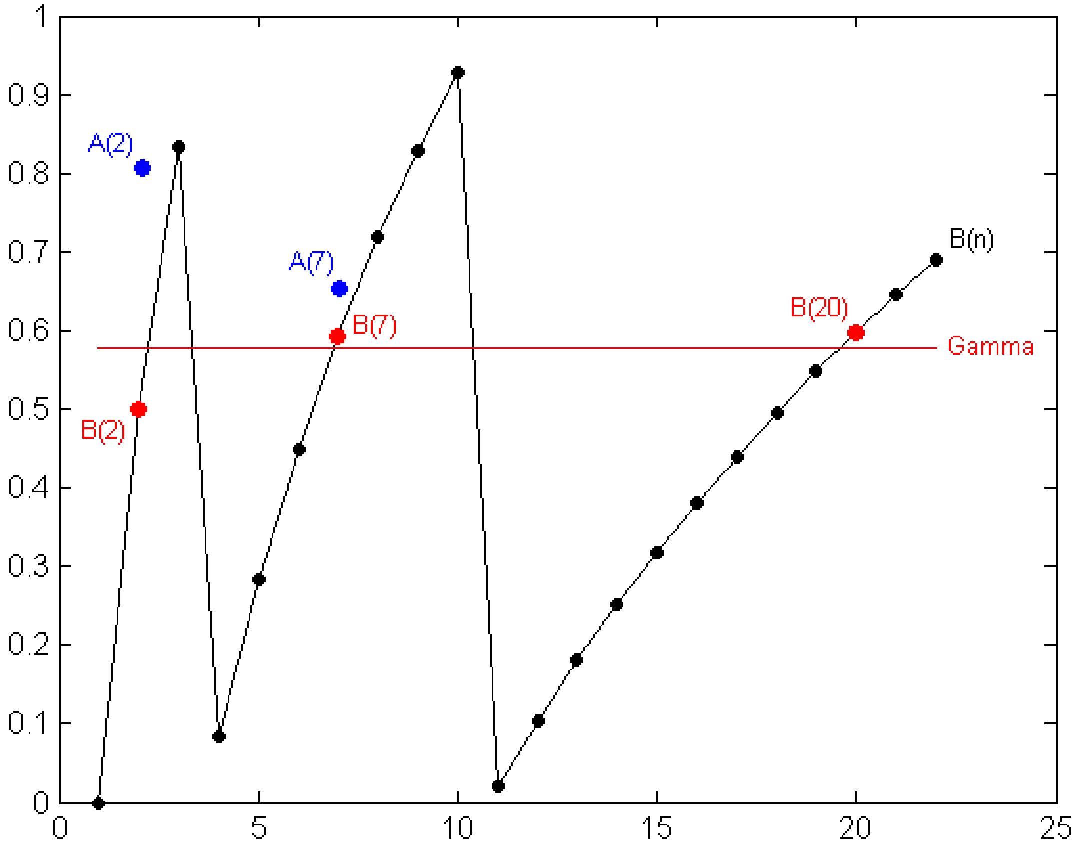
$$C(n) := log(n) - \lfloor log(n) \rfloor$$

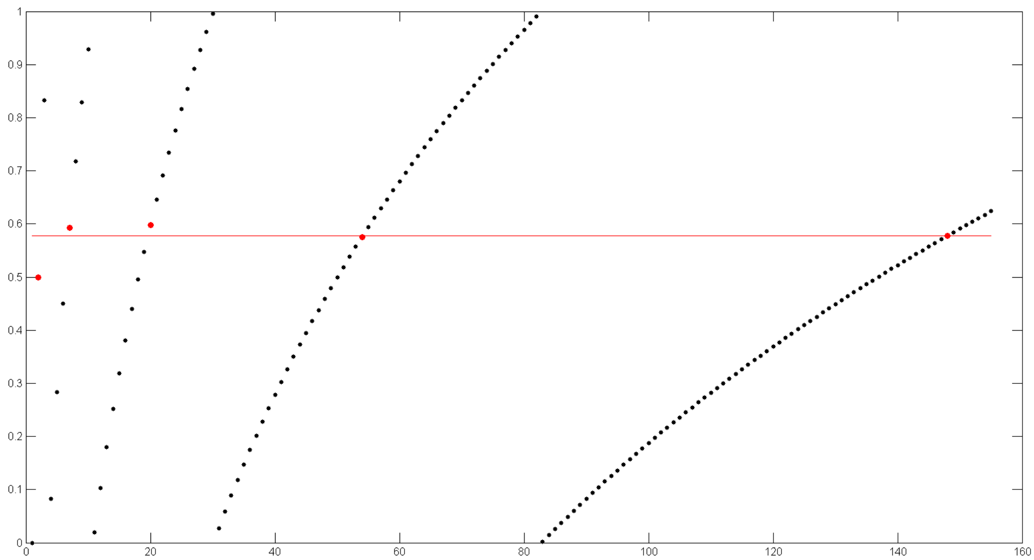
$$\gamma := \lim_{n\to\infty} A(n)$$

In every period of oscillation of B there are n, so that $B(n)$ is closer to γ than $A(n)$

$$\exists\, n \in \mathbb{N}: \quad B(n) - \gamma < A(n) - \gamma$$

n	$B(n)$	$B(n) - \gamma$	$<$	$A(n) - \gamma$	$A(n)$
2	0.50000000000000	-0.07721566490153	$<$	0.22963715453852	0.80685281944005
7	0.59285714285714	0.01564147795561	$<$	0.06973132890030	0.64694699380183
20	0.59773965714368	0.02052399224215	$<$	0.02479171868816	0.60200738358969
54	0.57543039374427	-0.00178527115727	$<$	0.00923068227846	0.58644634717999
148	0.57780251258124	0.00058684767971	$<$	0.00337457391559	0.58059023881713
403	0.57739240852976	0.00017674362823	$<$	0.00124018168154	0.57845584658308
1096	0.57709426741491	-0.00012139748662	$<$	0.00045613500542	0.57767179990695
2980	0.57706202025190	-0.00015364464963	$<$	0.00016777585092	0.57738344075245
8103	0.57726701163418	0.00005134673265	$<$	0.00006170427193	0.57727736917346
22026	0.57721721789959	0.00000155299805	$<$	0.00002270027317	0.57723836517470
59874	0.57722164886082	0.00000598395928	$<$	0.00000835084692	0.57722401574845
162754	0.57721387436125	-0.00000179054028	$<$	0.00000307211796	0.57721873701949
442413	0.57721590899706	0.00000024409553	$<$	0.00000113016521	0.57721679506674


A(2)
A(7)
B(n)
B(20)
B(7)
Gamma
B(2)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
5
10
15
20
25



The periods of oscillation of B encode $e \approx 2.71828$

$$\forall \ a, \ b \ \sim \ Figure \ 1,2,3 \quad \longrightarrow \quad \frac{a}{b} \ \approx \ e$$

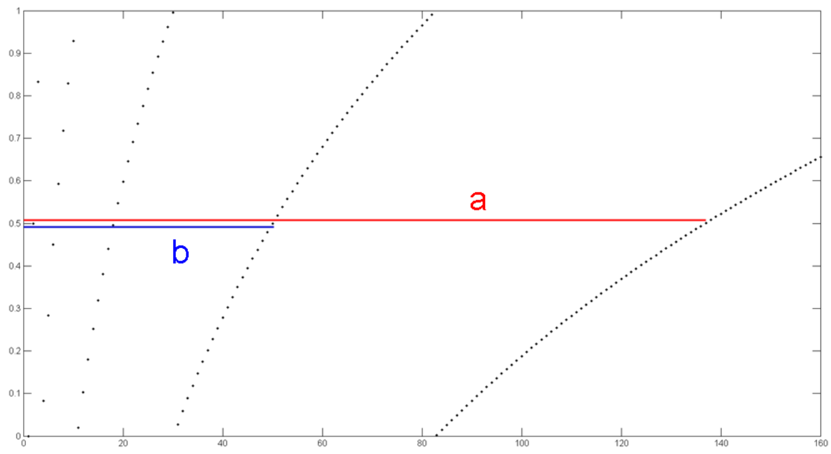


Figure 1:

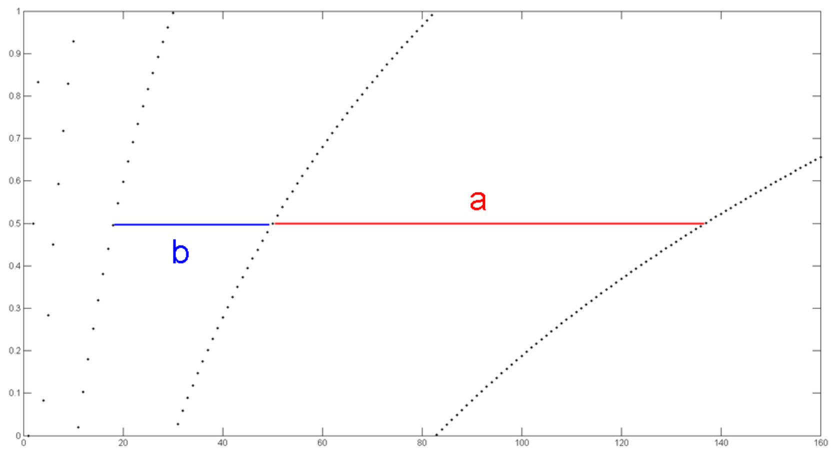


Figure 2:

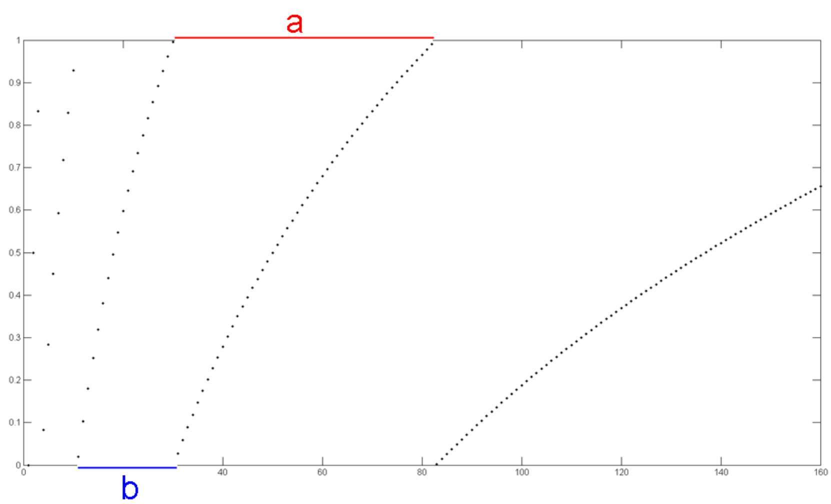


Figure 3:

Relationship between $B(n)$, $C(n)$ and γ

$$B(n) \approx D(n) := \begin{cases} C(n) + \gamma - 1 & ,\text{if } B(n) < \gamma \\ C(n) + \gamma & ,\text{if } B(n) > \gamma \end{cases}$$

n	$B(n)$	$\approx$	$D(n)$
1	0.000000000000000		-0.422784335098467
2	0.500000000000000	$\approx$	0.270362845461478
3	0.833333333333333	$\approx$	0.675827953569642
4	0.083333333333333	$\approx$	-0.036489973978577
5	0.283333333333333	$\approx$	0.186653577335633
6	0.450000000000000	$\approx$	0.368975134129588
7	0.592857142857143		1.523125813956846
8	0.717857142857143	$\approx$	0.656657206581369
9	0.828968253968254	$\approx$	0.774440242237752
10	0.928968253968254	$\approx$	0.879800757895579
20	0.597739657143682		1.572947938455524
100	0.187377517639621	$\approx$	0.182385850889625
148	0.577802512581241		1.574427938665648
1000	0.485470860550343	$\approx$	0.484970943883670
10000	0.787606036044348	$\approx$	0.787556036877716
100000	0.090146129863335	$\approx$	0.090141129871761
1000000	0.392726722864989	$\approx$	0.392726222865806
10000000	0.695311365857272	$\approx$	0.695311315859853

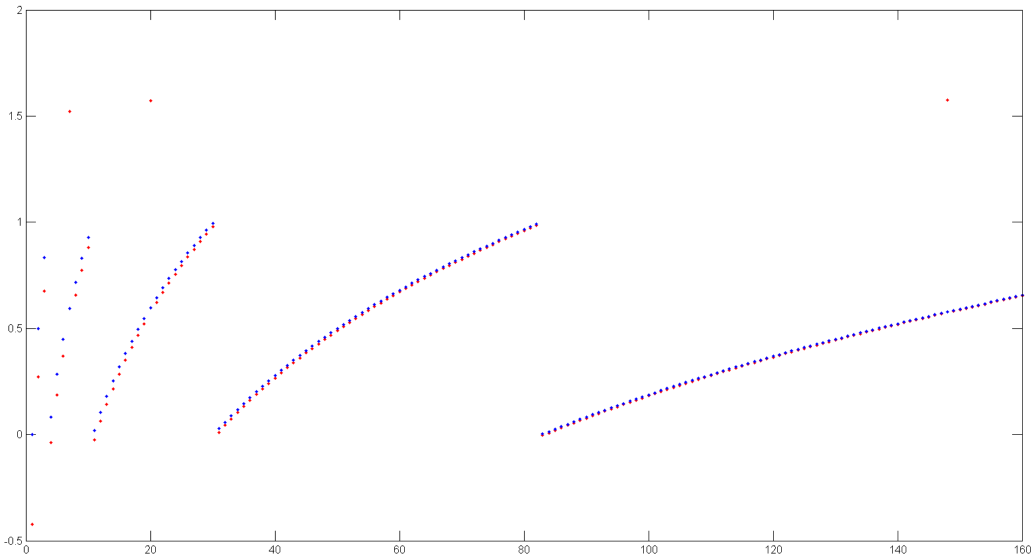
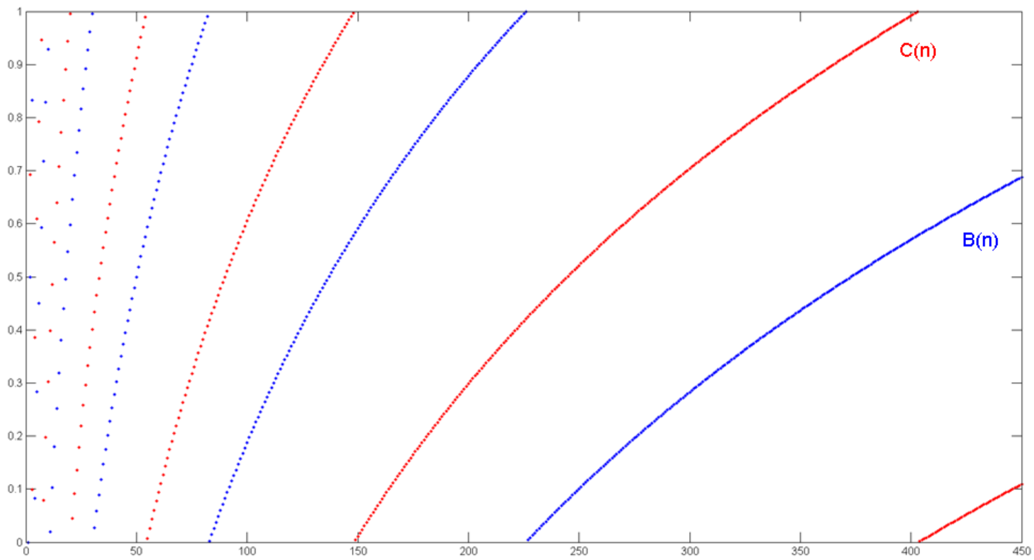
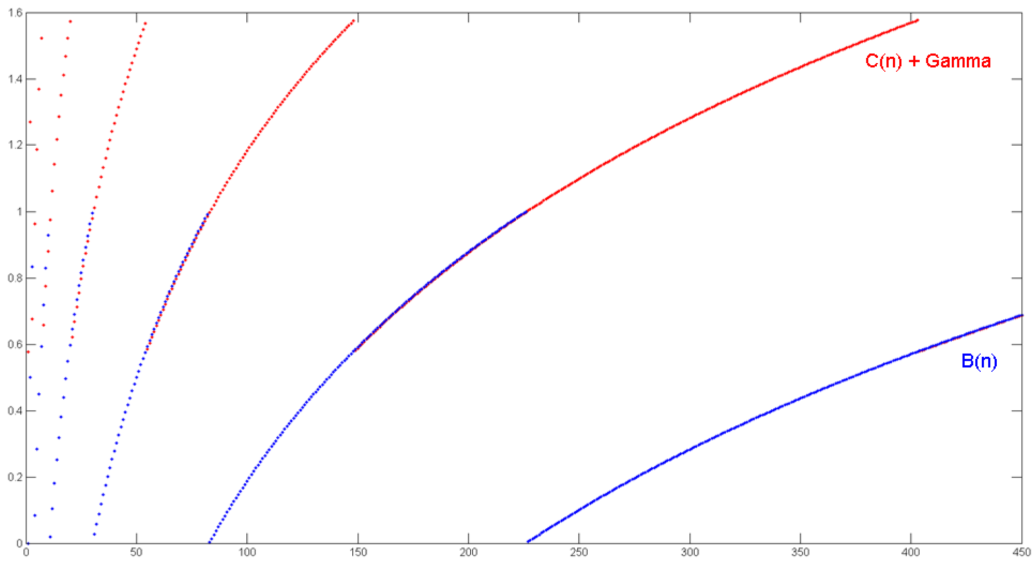
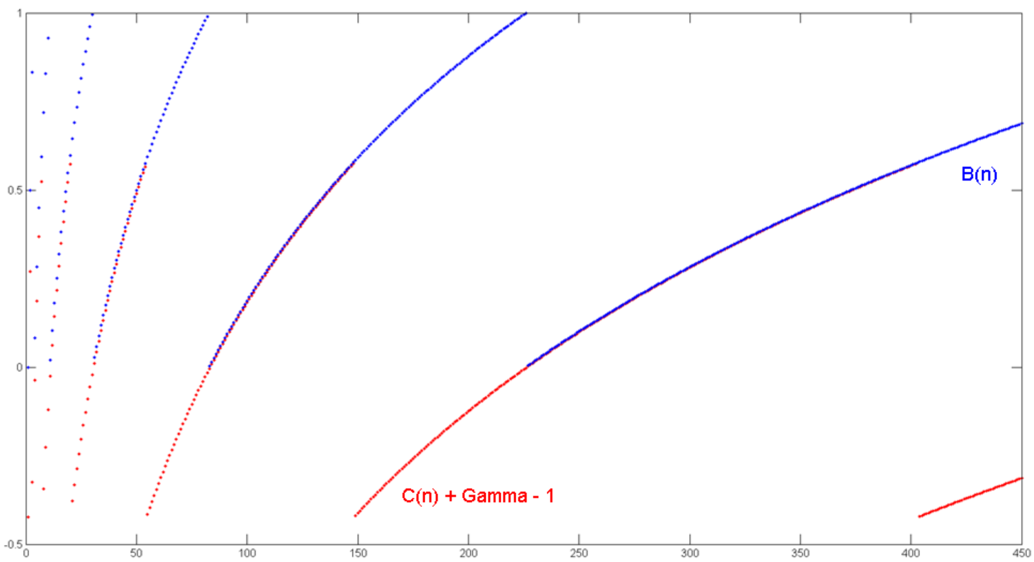


Figure 4: Plot of $B(n)$ (blue) and $F(n)$ (red)


C(n)
B(n)


C(n) + Gamma
B(n)


B(n)
C(n) + Gamma - 1

Max/Min Relationships

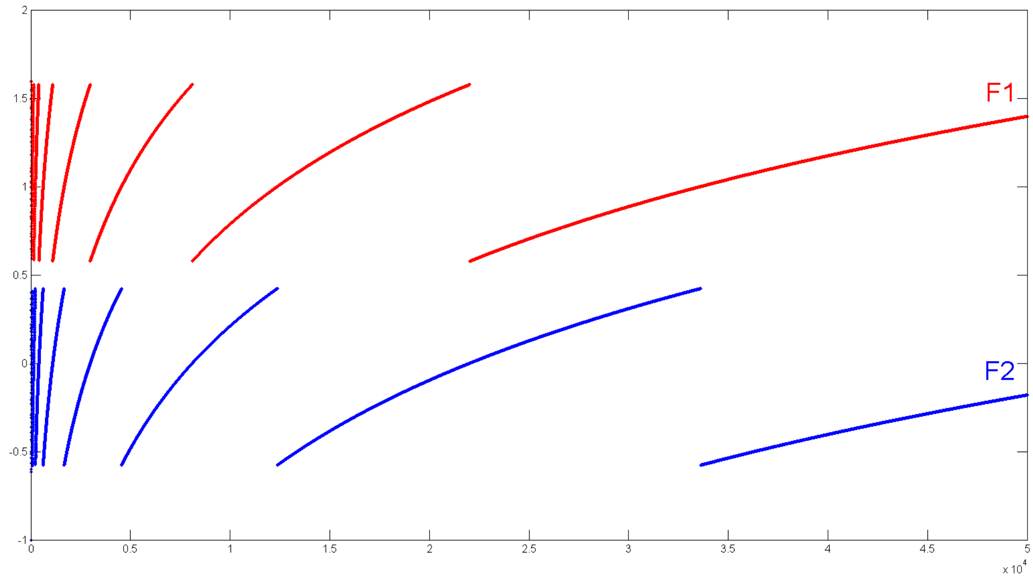
$$F_1(n) \;:=\; \sum_{i=1}^{n}\frac{1}{i} - \lfloor log(n)\rfloor \qquad (\in \mathbb{Q})$$

$$F_2(n) \;:=\; log(n) - \lfloor\sum_{i=1}^{n}\frac{1}{i}\rfloor \qquad (\in \mathbb{R})$$

Formula for γ:

$$\lim_{n\to\infty} \quad min\{\ F_1\ \} \;=\; \gamma$$

$$\lim_{n\to\infty} \quad max\{\ F_2\ \} \;=\; 1-\gamma$$



n	$min\{\ F_1\ \}$	$max\{\ F_2\ \}$	$min\{\ F_1\ \} + max\{\ F_2\ \}$
1	1.000000000000000	-1.000000000000000	0.000000000000000
2	1.000000000000000	-0.306852819440055	0.693147180559945
3	0.833333333333333	0.098612288668110	0.931945622001443
4	0.833333333333333	0.098612288668110	0.931945622001443
5	0.833333333333333	0.098612288668110	0.931945622001443
6	0.833333333333333	0.098612288668110	0.931945622001443
7	0.833333333333333	0.098612288668110	0.931945622001443
8	0.717857142857143	0.098612288668110	0.816469431525252
9	0.717857142857143	0.197224577336220	0.915081720193362
10	0.717857142857143	0.302585092994046	1.020442235851188
100	0.593612211926086	0.406719247264253	1.000331459190339
1000	0.579867656054511	0.421622267806518	1.001489923861029
10000	0.577390407488084	0.422662707570003	1.000053115058087
100000	0.577238350322196	0.422770971743748	1.000009322065944

2 Some notes on the Euler-Mascheroni constant

Formula for γ

$$\gamma = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} - log(n)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - log(\lfloor e^n \rfloor)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n$$

Note that the closer $e^n - \lfloor e^n \rfloor$ gets to $\frac{1}{2}$, the better is the approximation by the above formula. Therefore the subsequnece $n = 1, 2, 4, 5, 6, 10, 16, 21, 85, 115, ...$ (OEIS-A080053) gives a strictly monotonic approximation of γ by the above formula.

Pattern

$$\sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n - \gamma < \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - log(\lfloor e^n \rfloor) - \gamma$$

and

$$|\sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n - \gamma| < |\sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - log(\lfloor e^n \rfloor) - \gamma|$$

Definition

$$A(n) \;:=\; \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - log(\lfloor e^n \rfloor)$$

$$B(n) \;:=\; \sum_{i=1}^{\lfloor e^n \rfloor} \frac{1}{i} - n$$

$$T_A(n) \;:=\; \frac{1}{2}\,\frac{n - log(\lfloor e^n \rfloor)}{e^n - \lfloor e^n \rfloor}$$

$$T_B(n) \;:=\; \frac{\frac{1}{2} + \lfloor e^n \rfloor - e^n}{\lfloor e^n \rfloor}$$

Pattern

$$A(n) - \gamma \;\approx\; T_A(n)$$

$$B(n) - \gamma \;\approx\; T_B(n)$$

n	$A(n) - \gamma$	$\approx$	$T_A(n)$	$B(n) - \gamma$	$\approx$	$T_B(n)$
1	0.229637154538522	≈	0.213601964634381	-0.077215664901533	≈	-0.109140914229523
2	0.069731328900296	≈	0.069514205140797	0.015641477955610	≈	0.015849128724193
3	0.024791718688158	≈	0.024946691364183	0.020523992242149	≈	0.020723153840617
4	0.009230682278460	≈	0.009208353109854	-0.001785271157266	≈	-0.001817593206375
5	0.003374573915593	≈	0.003373671569258	0.000586847679708	≈	0.000586762820428
6	0.001240181681544	≈	0.001240035204982	0.000176743628226	≈	0.000176691085025
7	0.000456135005418	≈	0.000456072655816	-0.000121397486621	≈	-0.000121494916477
8	0.000167775850920	≈	0.000167758271537	-0.000153644649630	≈	-0.000153686926754
9	0.000061704271926	≈	0.000061705221596	0.000051346732651	≈	0.000051347948243
10	0.000022700273166	≈	0.000022700204904	0.000001552998054	≈	0.000001552946213
11	0.000008350846916	≈	0.000008350860275	0.000005983959284	≈	0.000005983979727
12	0.000003072117962	≈	0.000003072113647	-0.000001790540282	≈	-0.000001790548951
13	0.000001130165210	≈	0.000001130165204	0.000000244095527	≈	0.000000244095629
14	0.000000415763583	≈	0.000000415764407	0.000000179472383	≈	0.000000179473229
15	0.000000152950261	≈	0.000000152951169	0.000000039010172	≈	0.000000039011082
16	0.000000056265555	≈	0.000000056267590	-0.000000002309892	≈	-0.000000002307857
17	0.000000020695265	≈	0.000000020699688	-0.000000010502283	≈	-0.000000010497860
18	0.000000007612997	≈	0.000000007614984	0.000000005521458	≈	0.000000005523449
19	0.000000002802197	≈	0.000000002801399	-0.000000002594347	≈	-0.000000002595144
20	0.000000001031574	≈	0.000000001030576	0.000000000186934	≈	0.000000000185936

Generalization

$$n = \lfloor\, e^{\frac{v}{w}} \,\rfloor$$

$$v \;:=\; \lfloor\, log(n+1)\cdot(n+1) \,\rfloor$$

$$w \;:=\; n+1$$

n	v/w	$\lfloor\, e^{\frac{v}{w}} \,\rfloor$
1	1 / 2 = 0.50000	1
2	3 / 3 = 1.00000	2
3	5 / 4 = 1.25000	3
4	8 / 5 = 1.60000	4
5	10 / 6 = 1.66667	5
6	13 / 7 = 1.85714	6
7	16 / 8 = 2.00000	7
8	19 / 9 = 2.11111	8
9	23 / 10 = 2.30000	9
10	26 / 11 = 2.36364	10

Definition

$$\tilde{A}(n) \;:=\; \sum_{i=1}^{n} \frac{1}{i} - log(n)$$

$$\tilde{B}(n) \;:=\; \sum_{i=1}^{n} \frac{1}{i} - \frac{v}{w}$$

$$\tilde{T}_A(n) \;:=\; \frac{1}{2}\,\frac{\frac{v}{w} - log(n)}{e^{\frac{v}{w}} - n}$$

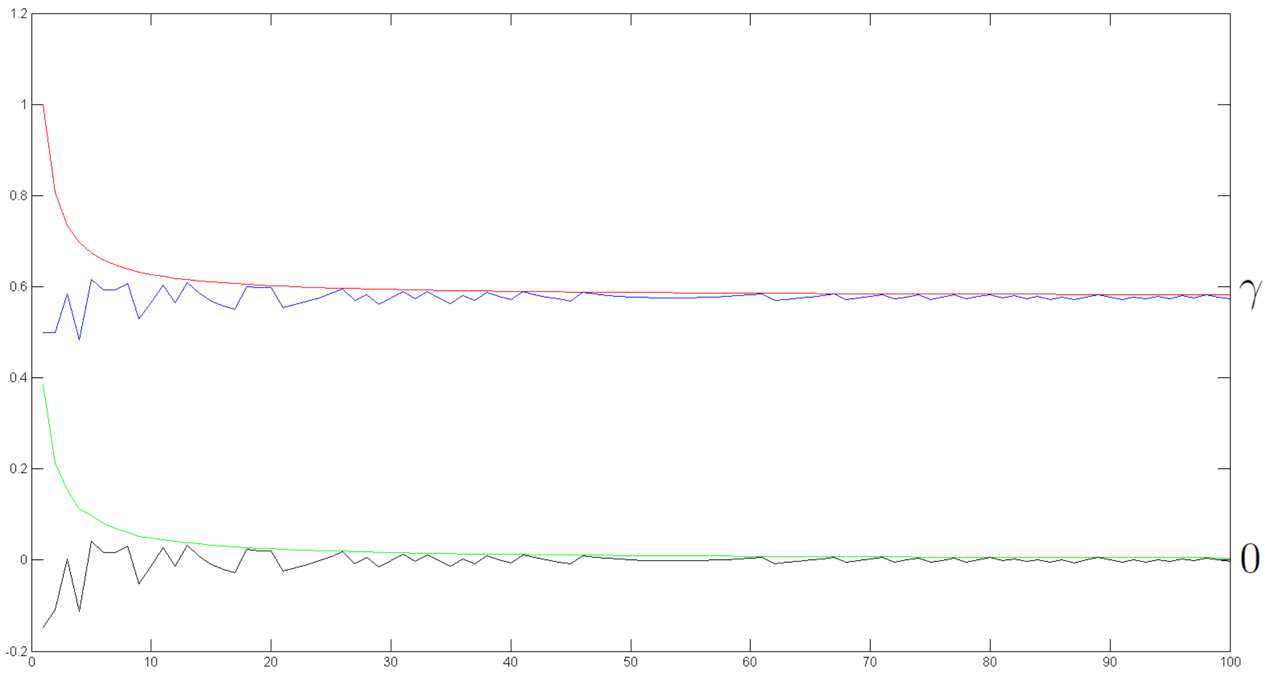
$$\tilde{T}_B(n) \;:=\; \frac{\frac{1}{2} + n - e^{\frac{v}{w}}}{n}$$

Pattern (generalized)

$$\tilde{A}(n) - \gamma \;\approx\; \tilde{T}_A(n)$$

$$\tilde{B}(n) - \gamma \;\approx\; \tilde{T}_B(n)$$

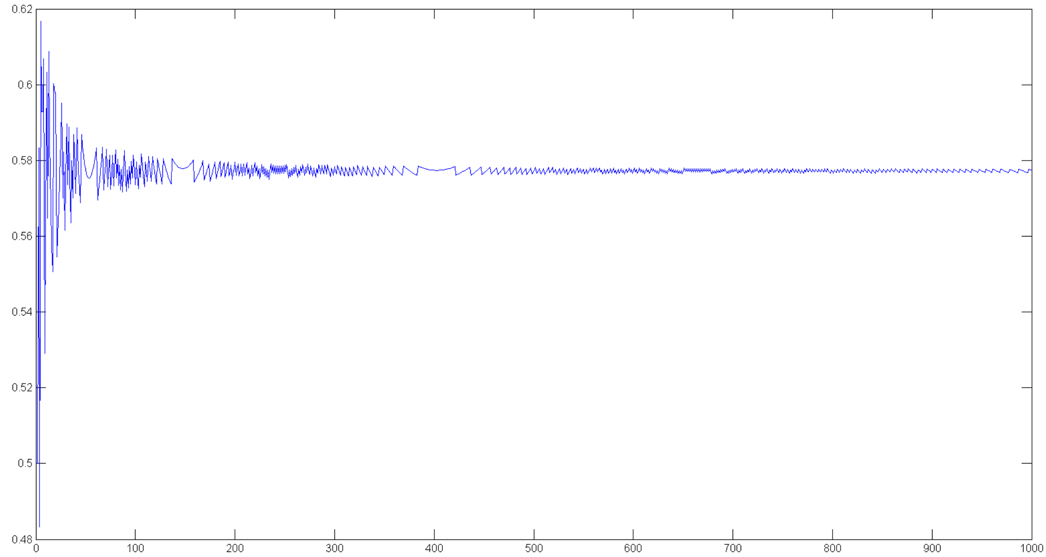
n	$\tilde{A}(n)-\gamma$	$\approx$	$\tilde{T}_A(n)$	$\tilde{B}(n)-\gamma$	$\approx$	$\tilde{T}_B(n)$	$\frac{v}{w}$
1	0.42278434	$\approx$	0.38537352	**-0.07721566**	$\approx$	-0.14872127	0.5000
2	0.22963715	$\approx$	0.21360196	**-0.07721566**	$\approx$	-0.10914091	**1.0000**
3	0.15750538	$\approx$	0.15436921	0.00611767	$\approx$	0.00321901	1.2500
4	0.11982331	$\approx$	0.11211877	-0.09388233	$\approx$	-0.11325811	1.6000
5	0.09667976	$\approx$	0.09716585	0.03945100	$\approx$	0.04110199	1.6667
6	0.08102487	$\approx$	0.08063871	**0.01564148**	$\approx$	0.01576509	1.8571
7	0.06973133	$\approx$	0.06951421	**0.01564148**	$\approx$	0.01584913	**2.0000**
8	0.06119994	$\approx$	0.06151555	0.02953037	$\approx$	0.03032361	2.1111
9	0.05452801	$\approx$	0.05274958	-0.04824741	$\approx$	-0.05268694	2.3000
10	0.04916750	$\approx$	0.04848925	-0.01188377	$\approx$	-0.01295341	2.3636
11	0.04476641	$\approx$	0.04502926	0.02599501	$\approx$	0.02650586	2.4167
12	0.04108836	$\approx$	0.04056090	-0.01246653	$\approx$	-0.01334823	2.5385
13	0.03796873	$\approx$	0.03833707	0.03148952	$\approx$	0.03196129	2.5714
14	0.03528933	$\approx$	0.03522353	0.00767999	$\approx$	0.00772028	2.6667
15	0.03296313	$\approx$	0.03263906	-0.00898667	$\approx$	-0.00950879	2.7500
16	0.03092461	$\approx$	0.03046081	-0.02001608	$\approx$	-0.02101048	2.8235
17	0.02912351	$\approx$	0.02860060	-0.02655203	$\approx$	-0.02784283	2.8889
18	0.02752066	$\approx$	0.02771720	0.02315557	$\approx$	0.02340315	2.8947
19	0.02608501	$\approx$	0.02624269	**0.02052399**	$\approx$	0.02073928	2.9500
20	0.02479172	$\approx$	0.02494669	**0.02052399**	$\approx$	0.02072315	**3.0000**

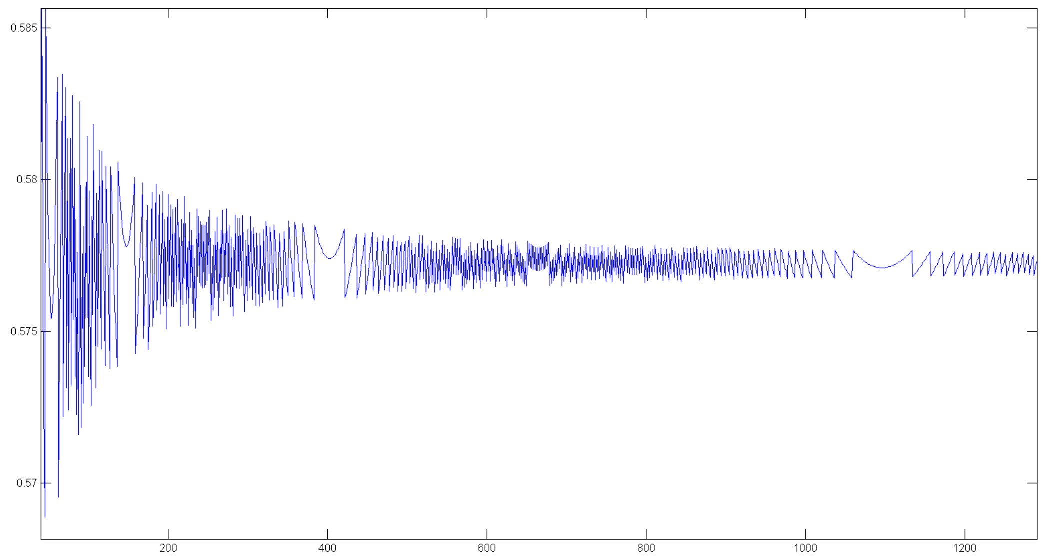


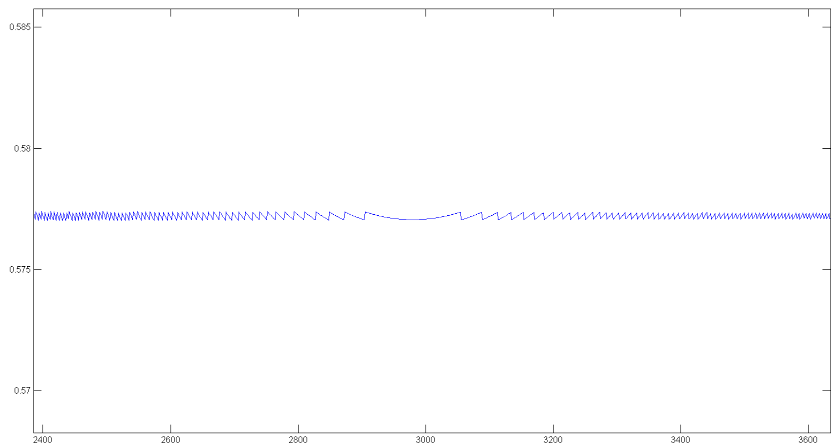
$\tilde{A}(n)$ = red, $\tilde{B}(n)$ = blue, $\tilde{T}_A(n)$ = green, $\tilde{T}_B(n)$ = black $\quad (n = 1, ..., 100)$

Note: $\tilde{T}_A(n)$ and $\tilde{T}_B(n)$ appears to be some kind of copy of $\tilde{A}(n)$ and $\tilde{B}(n)$

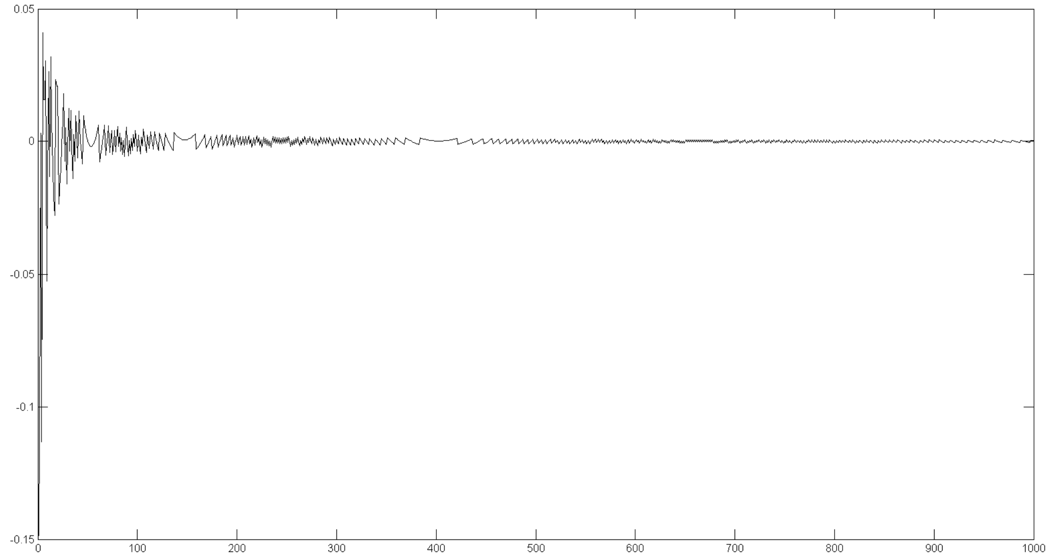
Plots of $\tilde{B}(n)$ (Note: $\tilde{B}(n)$ converges to γ)

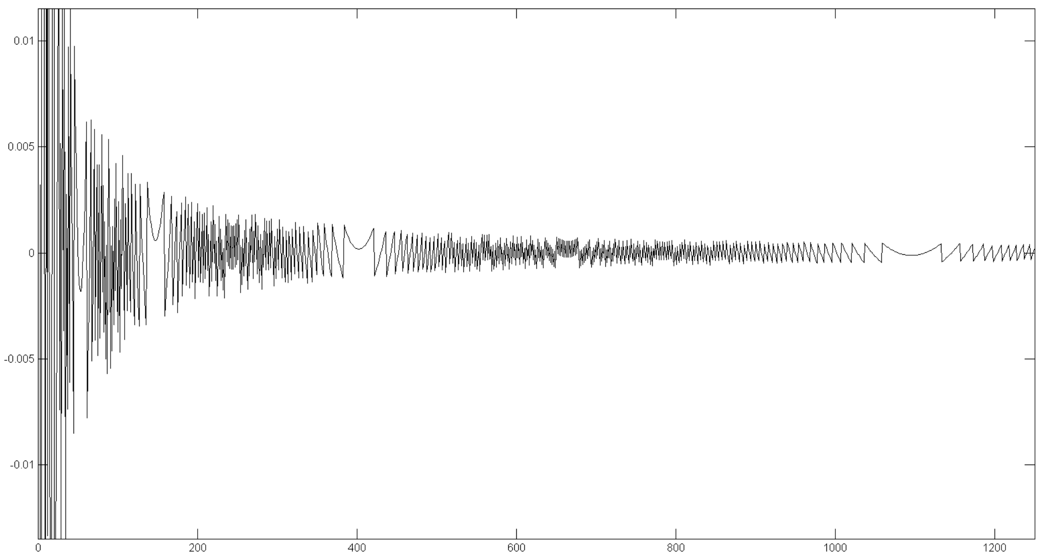


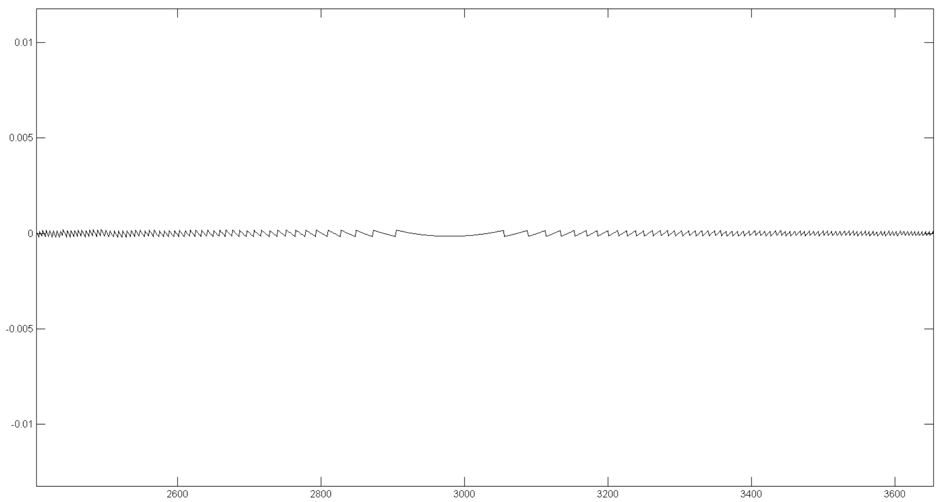




Plots of $\tilde{T}_B(n)$ (Note: $\tilde{T}_B(n)$ converges to zero)







$\tilde{T}_A(n)$ and $\tilde{T}_B(n)$ can be used as correction terms for $\tilde{A}(n)$ and $\tilde{B}(n)$

n	$\lvert\tilde{A}(n)-\gamma\rvert$	$\lvert\tilde{A}(n)-\tilde{T}_A(n)-\gamma\rvert$	$\lvert\tilde{B}(n)-\gamma\rvert$	$\lvert\tilde{B}(n)-\tilde{T}_B(n)-\gamma\rvert$	$\frac{v}{w}$
1	4.22784335e-001	3.74108145e-002	**7.72156649e-002**	7.15056058e-002	0.5000
2	2.29637155e-001	1.60351899e-002	**7.72156649e-002**	3.19252493e-002	**1.0000**
3	1.57505380e-001	3.13616835e-003	6.11766843e-003	2.89865425e-003	1.2500
4	1.19823307e-001	7.70454125e-003	9.38823316e-002	1.93757745e-002	1.6000
5	9.66797560e-002	4.86097554e-004	3.94510018e-002	1.65098814e-003	1.6667
6	8.10248659e-002	3.86155069e-004	**1.56414780e-002**	1.23616682e-004	1.8571
7	6.97313289e-002	2.17123759e-004	**1.56414780e-002**	2.07650769e-004	**2.0000**
8	6.11999363e-002	3.15613351e-004	2.95303668e-002	7.93246836e-004	2.1111
9	5.45280117e-002	1.77843584e-003	4.82474109e-002	4.43952849e-003	2.3000
10	4.91674961e-002	6.78248563e-004	1.18837746e-002	1.06963652e-003	2.3636
11	4.47664072e-002	2.62850398e-004	2.59950133e-002	5.10848073e-004	2.4167
12	4.10883635e-002	5.27465407e-004	1.24665252e-002	8.81706728e-004	2.5385
13	3.79687328e-002	3.68339974e-004	3.14895188e-002	4.71770177e-004	2.5714
14	3.52893320e-002	6.58015589e-005	7.67999499e-003	4.02839237e-005	2.6667
15	3.29631272e-002	3.24069057e-004	8.98667167e-003	5.22120607e-004	2.7500
16	3.09246061e-002	4.63796961e-004	2.00160834e-002	9.94397926e-004	2.8235
17	2.91235137e-002	5.22909915e-004	2.65520311e-002	1.29080051e-003	2.8889
18	2.75206554e-002	1.96540316e-004	2.31555712e-002	2.47581522e-004	2.8947
19	2.60850131e-002	1.57672889e-004	**2.05239922e-002**	2.15285219e-004	2.9500
20	2.47917187e-002	1.54972676e-004	**2.05239922e-002**	1.99161598e-004	**3.0000**
21	2.36206021e-002	3.59031254e-004	2.27660510e-002	9.03769035e-004	3.0909
22	2.25551320e-002	2.72560480e-004	1.68371973e-002	6.14026037e-004	3.1304
23	2.15816303e-002	1.79570471e-004	9.59082048e-003	3.33448745e-004	3.1667
24	2.06886825e-002	8.31189405e-005	1.25748715e-003	9.79377315e-005	3.2000
25	1.98666880e-002	1.46137118e-005	7.97328208e-003	6.23042354e-005	3.2308
26	1.91075133e-002	1.12078093e-004	1.79447921e-002	1.22579716e-004	3.2593
27	1.84042223e-002	1.21653371e-004	7.18748308e-003	2.15982971e-004	3.3214
28	1.77508639e-002	6.18995297e-006	5.12778786e-003	2.62716515e-005	3.3448
29	1.71423027e-002	1.81319188e-004	1.55618673e-002	4.41582596e-004	3.4000
30	1.65740844e-002	5.82719273e-005	1.58337269e-003	7.32665906e-005	3.4194
31	1.60423260e-002	5.83937978e-005	1.25295305e-002	8.05291112e-005	3.4375
32	1.55436277e-002	6.74691464e-005	3.56895431e-003	1.02442321e-004	3.4848
33	1.50749994e-002	5.00733127e-005	1.15825608e-002	7.04101107e-005	3.5000
34	1.46338009e-002	4.88837369e-005	1.86281731e-003	6.47391048e-005	3.5429
35	1.42176926e-002	1.30940800e-004	1.37675792e-002	3.27244657e-004	3.5833
36	1.38245934e-002	1.24768478e-005	2.74893730e-003	2.73330197e-006	3.5946
37	1.34526463e-002	7.82536329e-005	7.20838845e-003	1.54049523e-004	3.6316
38	1.31001887e-002	3.50908745e-005	9.66070737e-003	5.17842662e-005	3.6410
39	1.27657279e-002	1.83980842e-005	1.32737403e-003	1.08831938e-005	3.6750
40	1.24479199e-002	6.28009318e-005	5.98969914e-003	1.18942284e-004	3.7073

A Further Generalization

For all $k \in \mathbb{Z}$, let the set $\mathbb{X}_k$ be defined as

$$\mathbb{X}_k := \{n \in \mathbb{Z} |\ n > -k\}$$

Consider the following family of functions

$$\Gamma := \{\hat{A}_k, \hat{B}_k : k \in \mathbb{Z}\}$$

mapping from $\mathbb{X}_k$ to $\mathbb{R}$ and $\mathbb{Q}$ respectively

$$\hat{A}_k : \mathbb{X}_k \to \mathbb{R}$$

$$\hat{B}_k : \mathbb{X}_k \to \mathbb{Q}$$

where $\hat{A}_k$ and $\hat{B}_k$ are defined as

$$\hat{A}_k(n) \ := \begin{cases} -\sum_{i=1}^{-n} \frac{1}{i} & - \, log(n+k) \quad ,\text{if} \ \ n \ < \ 0 \\ & - \, log(n+k) \quad ,\text{if} \ \ n \ = \ 0 \\ \sum_{i=1}^{n} \frac{1}{i} & - \, log(n+k) \quad ,\text{if} \ \ n \ > \ 0 \end{cases}$$

$$\hat{B}_k(n) \ := \begin{cases} -\sum_{i=1}^{-n} \frac{1}{i} & - \, \frac{\lfloor (n+k) \ log(n+k) \rfloor}{n+k} \quad ,\text{if} \ \ n \ < \ 0 \\ & - \, \frac{\lfloor (n+k) \ log(n+k) \rfloor}{n+k} \quad ,\text{if} \ \ n \ = \ 0 \\ \sum_{i=1}^{n} \frac{1}{i} & - \, \frac{\lfloor (n+k) \ log(n+k) \rfloor}{n+k} \quad ,\text{if} \ \ n \ > \ 0 \end{cases}$$

Then

$$\tilde{A}(n) \ = \ \sum_{i=1}^{n} \frac{1}{i} - log(n)$$

and

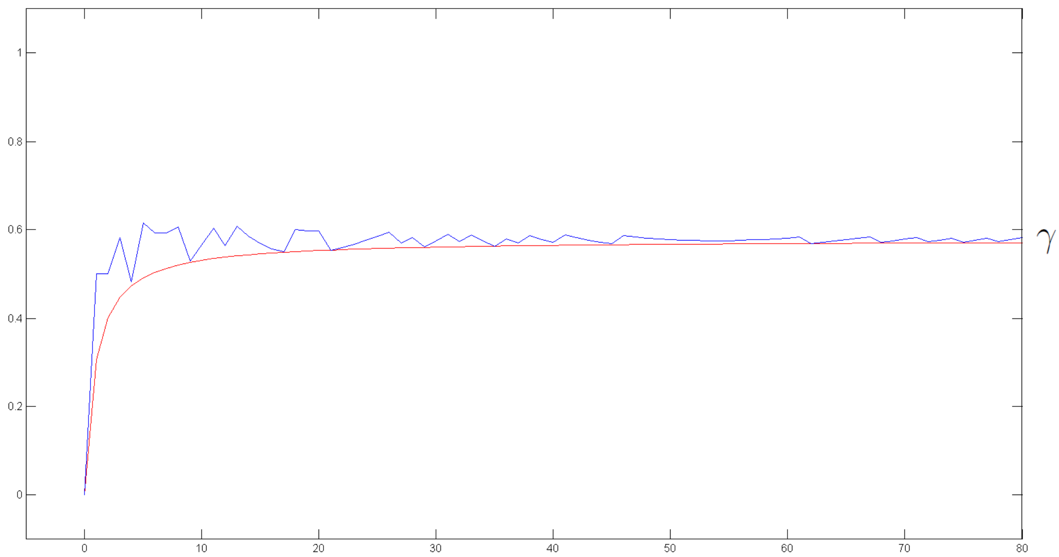
$$\tilde{B}(n) \ = \ \sum_{i=1}^{n} \frac{1}{i} - \frac{v}{w}$$

are members of Γ, because

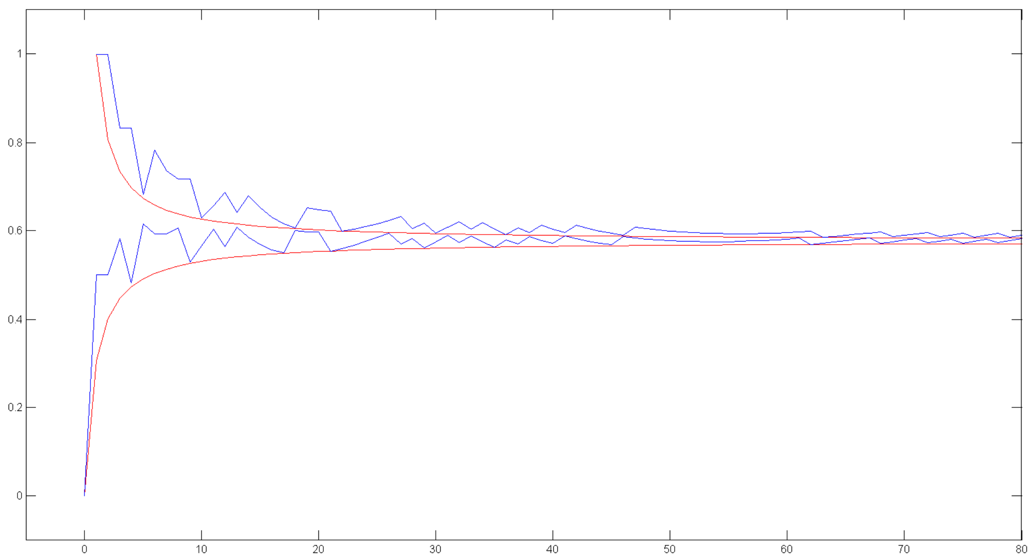
$$\hat{A}_0(n) \ = \ \tilde{A}(n) \qquad\qquad (\mathbb{X}_0 = 1, 2, 3, ...)$$

$$\hat{B}_1(n) \ = \ \tilde{B}(n) \qquad\qquad (\mathbb{X}_1 = 0, 1, 2, 3, ...)$$

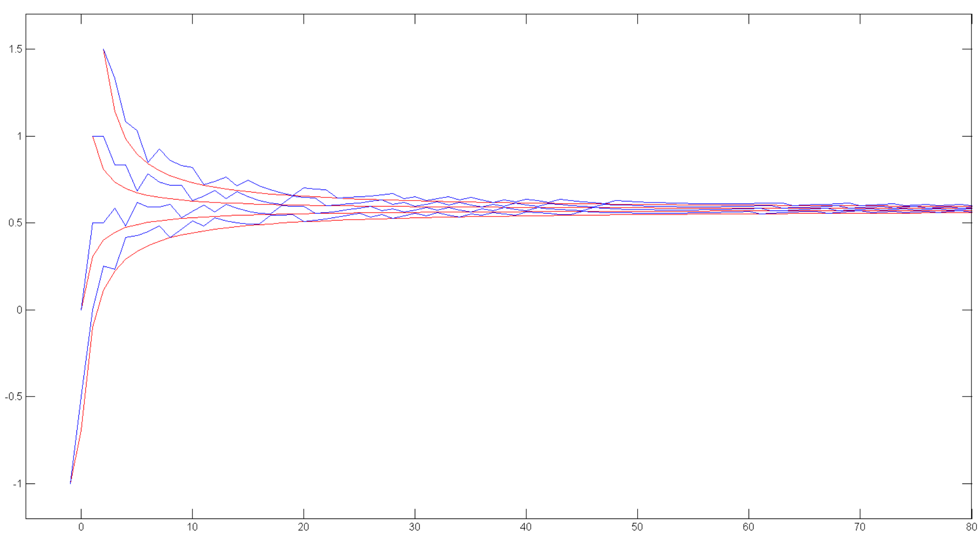
Plot of $\hat{A}_1(n)$ (red) and $\hat{B}_1(n)$ (blue)



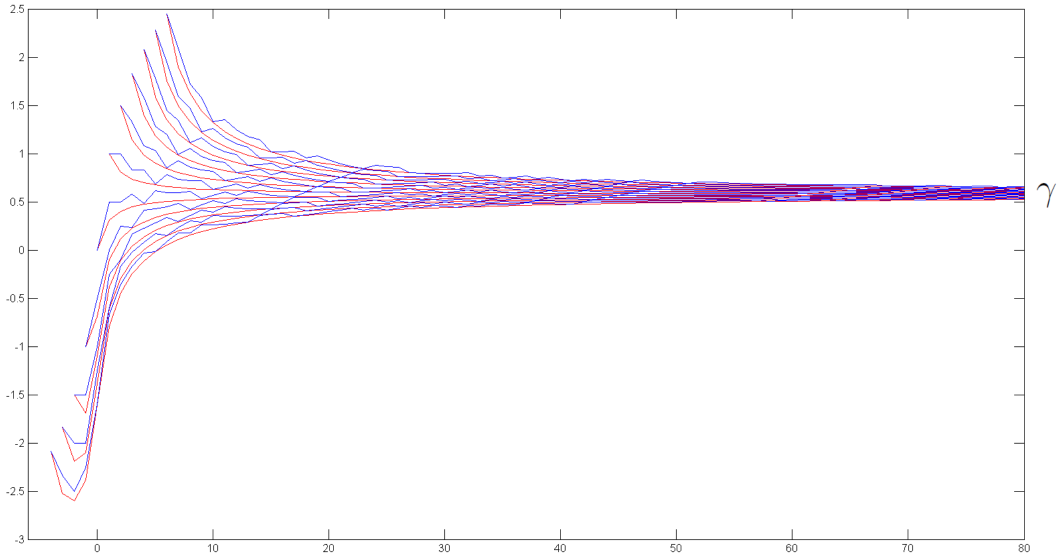
Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{0, 1\}$



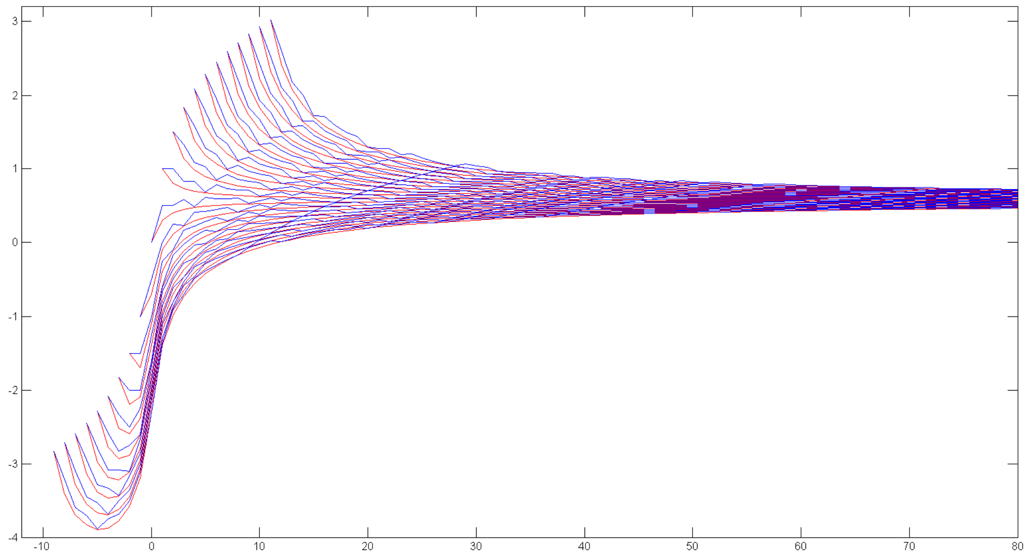
Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-1, 0, 1, 2\}$



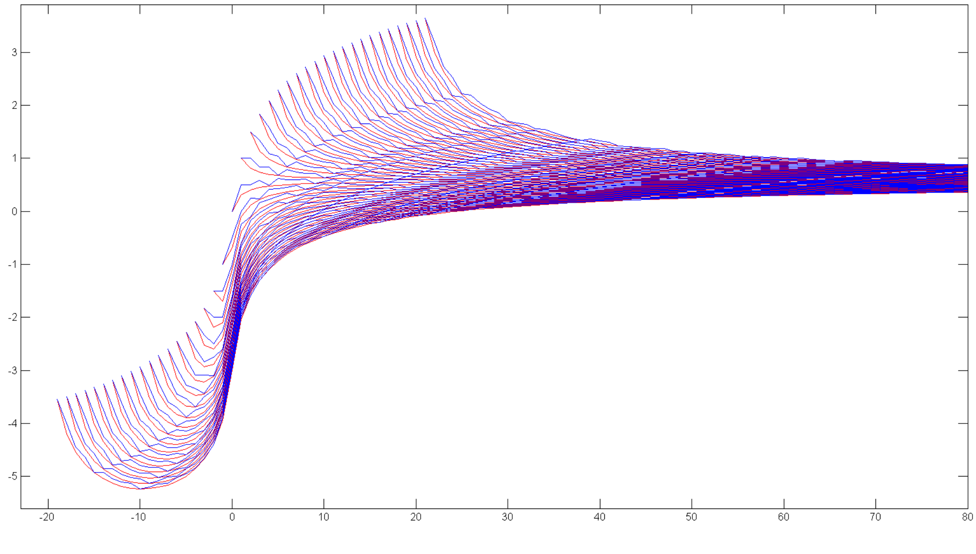
Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-5, -4, ..., 4, 5\}$



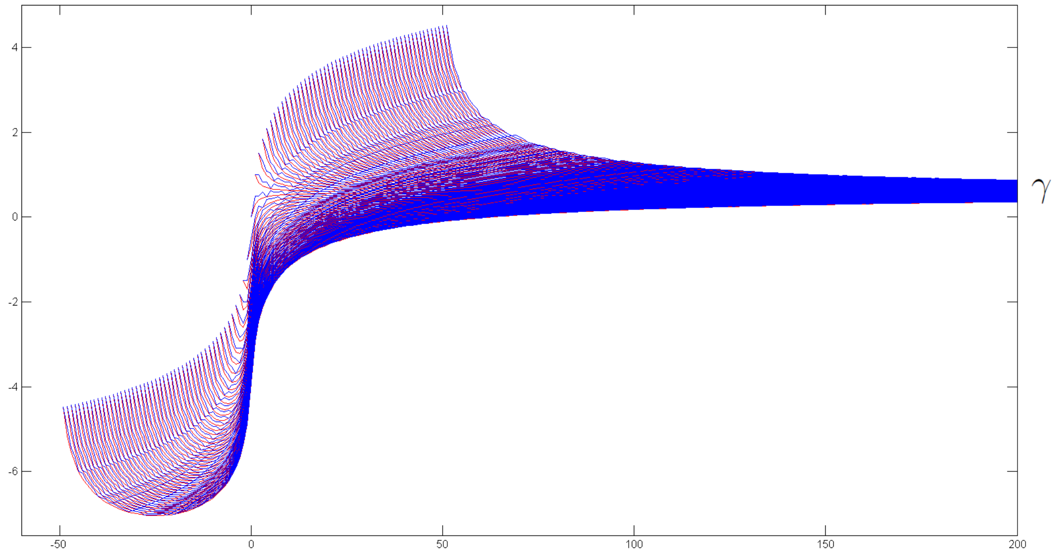
Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-10, ..., 10\}$



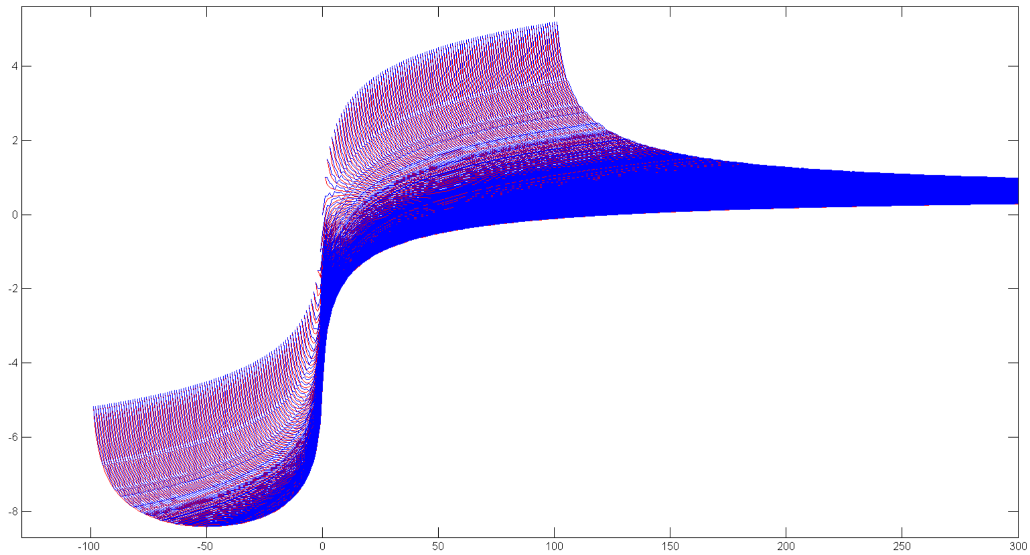
Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-20, ..., 20\}$



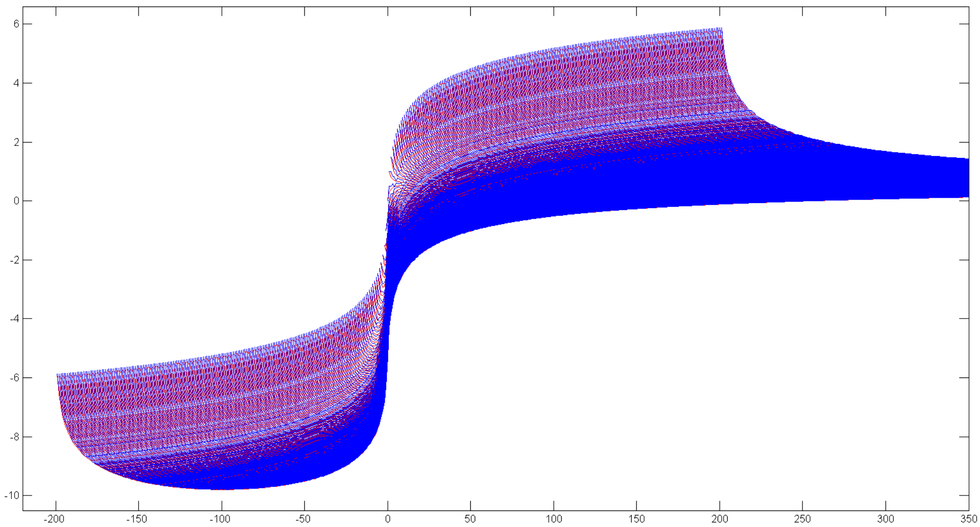
Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-50, ..., 50\}$

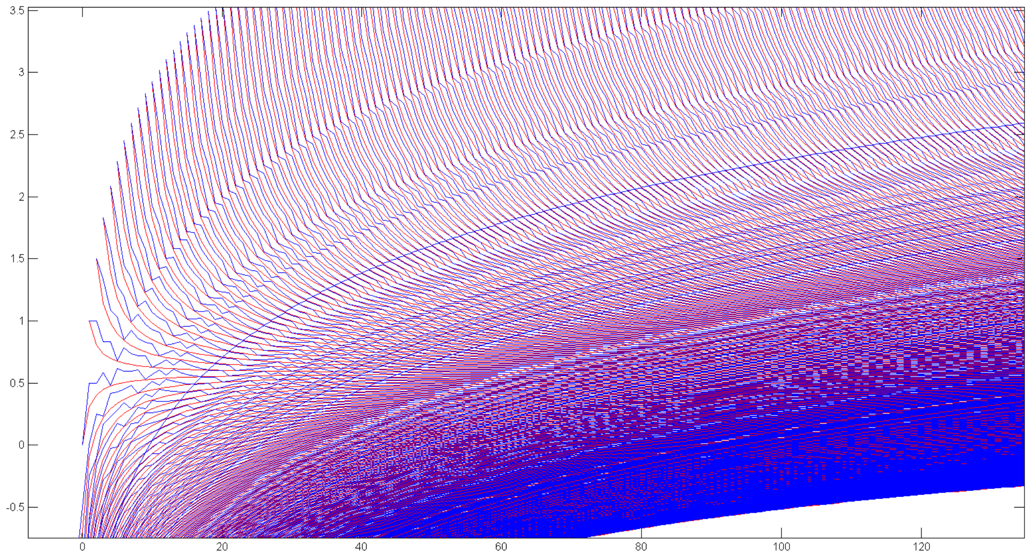


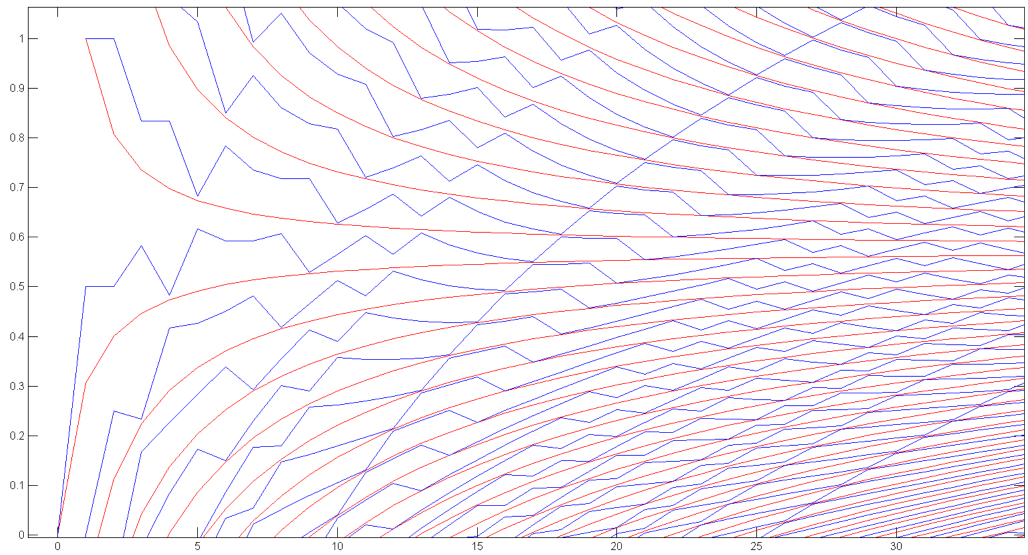
Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-100, ..., 100\}$

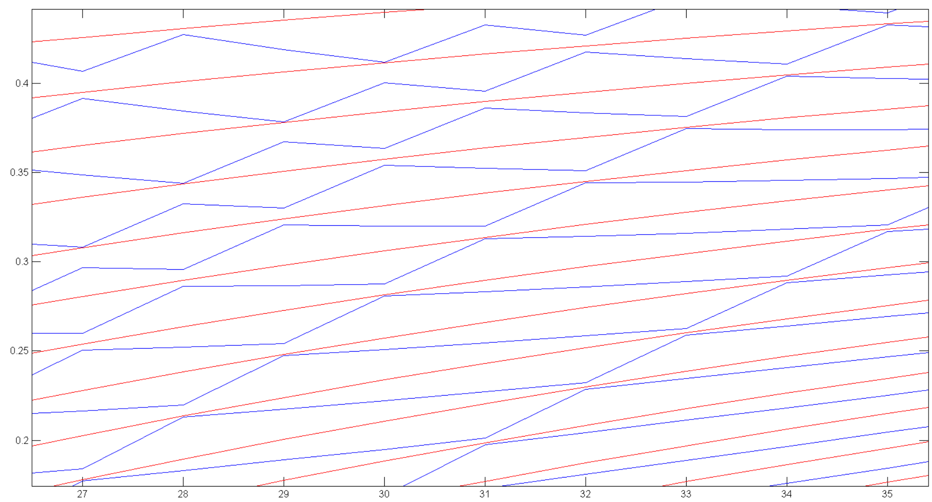


Plot of $\hat{A}_k(n)$ and $\hat{B}_k(n)$ for $k = \{-200, ..., 200\}$







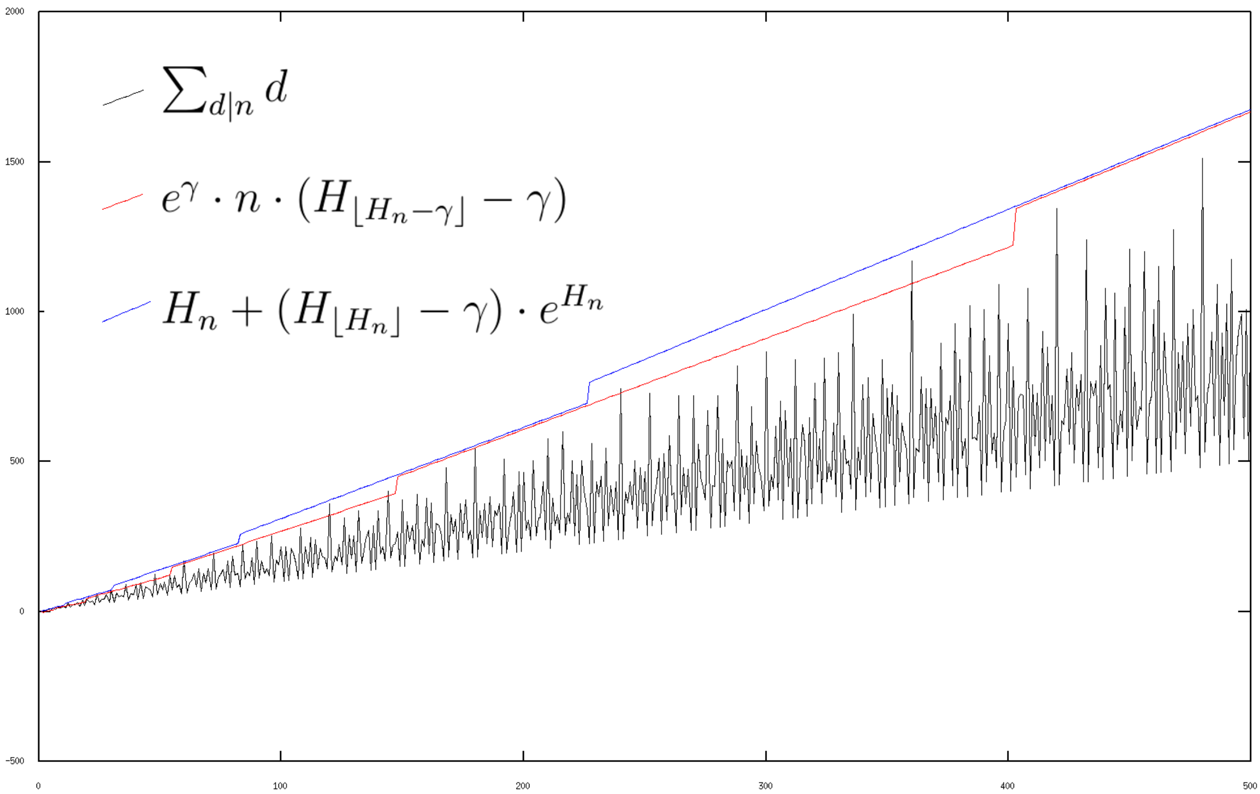


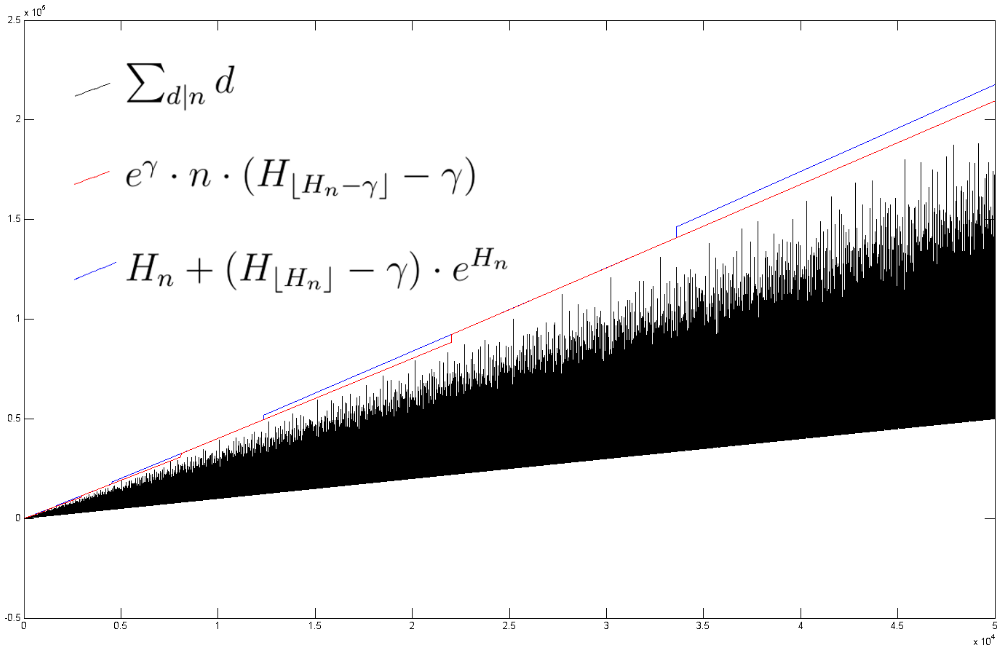
Pattern: $\hat{A}_k(n) < \hat{B}_k(n) < \hat{A}_{k-1}(n) \quad \forall\ k \in \mathbb{Z},\ n \in \mathbb{X}_{k-1}$
(In words: The red and blue lines never touch or cross each other)

3 Pattern related to divisor function

Substituting $log(x)$ by $H_{\lfloor x \rfloor} - \gamma$ in Lagarias criterion for the RH and Groenwall's theorem, gives the following pattern:

$$e^{\gamma} \cdot n \cdot (H_{\lfloor H_n - \gamma \rfloor} - \gamma) \quad < \quad H_n + (H_{\lfloor H_n \rfloor} - \gamma) \cdot e^{H_n}$$





Note that the "steps" of $H_n + (H_{\lfloor H_n \rfloor} - \gamma) \cdot e^{H_n}$ correspond with $a_n \stackrel{?}{=} [e^{n-\gamma}]$ (see Page 1).

4 A Note on the indeterminate forms 0^0 and $\frac{1}{0}$

While 0^0 is *"normally"* one, there is the following exception:

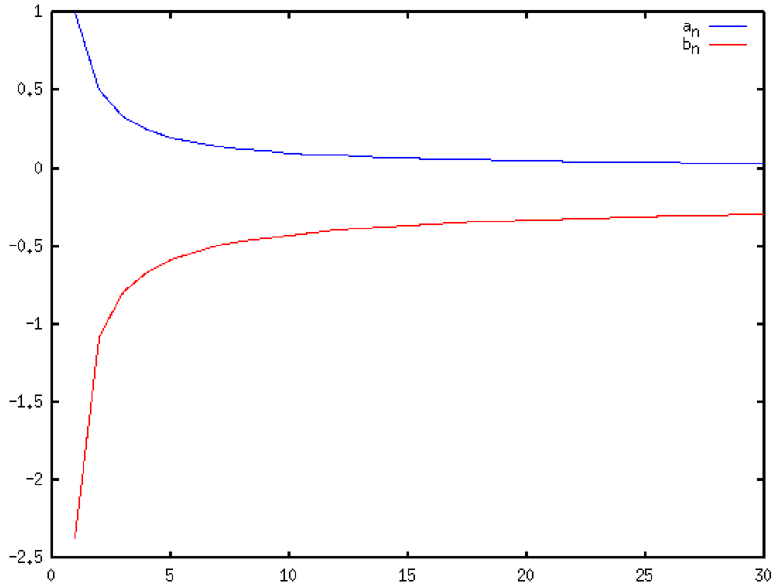
$$a_n := \frac{1}{n} \qquad (a_n \longrightarrow 0^+)$$
$$b_n := \frac{1}{\gamma - H_n} \qquad (b_n \longrightarrow 0^-)$$

then

$$\lim_{n\to\infty} \quad a_n^{b_n} \longrightarrow 0^0 \;=\; e$$
$$\lim_{n\to\infty} \quad b_n^{a_n} \longrightarrow 0^0 \;=\; 1 + 0\cdot\pi i$$

and

$$\lim_{n\to\infty} \quad \frac{b_n^{a_n}}{a_n} \longrightarrow \frac{1}{0} \;=\; \infty + \pi i$$



Matlab/Octave source code + Output:

```
function main
gamma = 0.5772156649;
n = 100000;
a = 1/n;
b = 1/(gamma-H(n));
printf('\n %10.9f'           ,a^b);
printf('\n %10.9f + %10.9f * i',b^a,imag(b^a));
printf('\n %10.9f + %10.9f * i',b^a/a,imag(b^a)/a);
% Harmonic Number
function [h] = H(n)
  h = 0;
  for i = 1 : n
      h = h + 1 / i;
  end_______________________________________
Output:
>> 2.718280648
>> 0.999975565 + 0.000031415 * i
>> 99997.556509714 + 3.141515890 * i
```

5 Harmonic approximation of the prime counting function

It can be observed, that the following approximation of $\pi(x)$ and $Li(x) = \int_2^x 1/\log(t)\,dt$ holds for small x (lower than ~2000, H_x is the x-th harmonic number):

$$\pi(x) \approx I_e(x) := \frac{x}{H_x - e\gamma}$$

$$Li(x) \approx I_\pi(x) := \frac{x}{H_x - \pi\gamma}$$

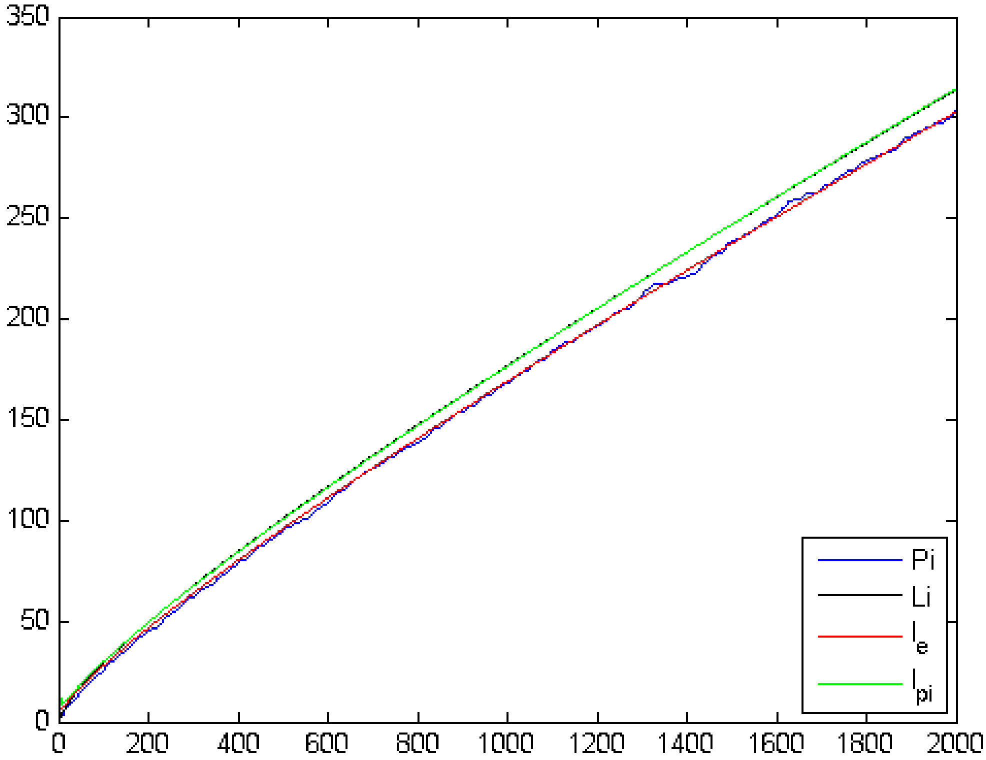


Figure 5: Observe how the red line fits the blue line and how the green line fits the black line.

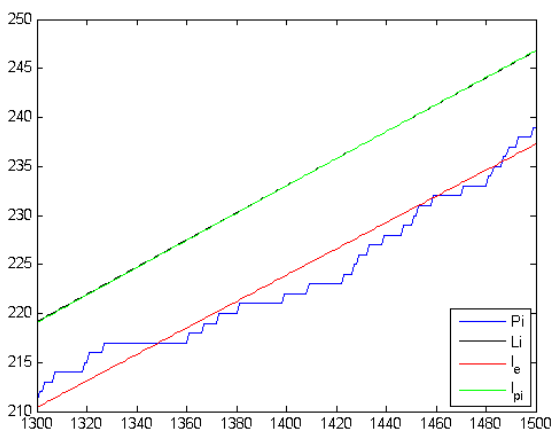


Figure 6: Close up for $x = 1300, ..., 1500$

However, for larger x this relationship doesn't hold any more. It appears that $\pi(x)$ *moves up*, while $Li(x)$ *moves down* in respect to $I_e(x)$ and $I_\pi(x)$ as their reference asymptotes.

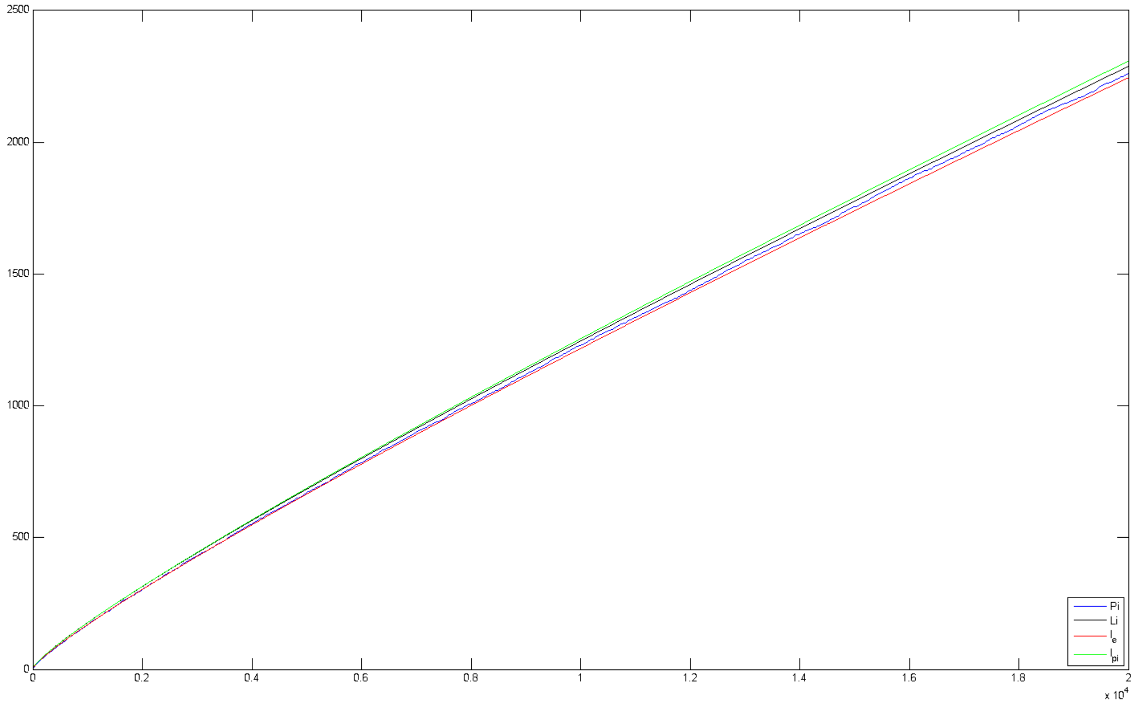


Figure 7: Observe how blue and black move to the inner region defined by the red and green

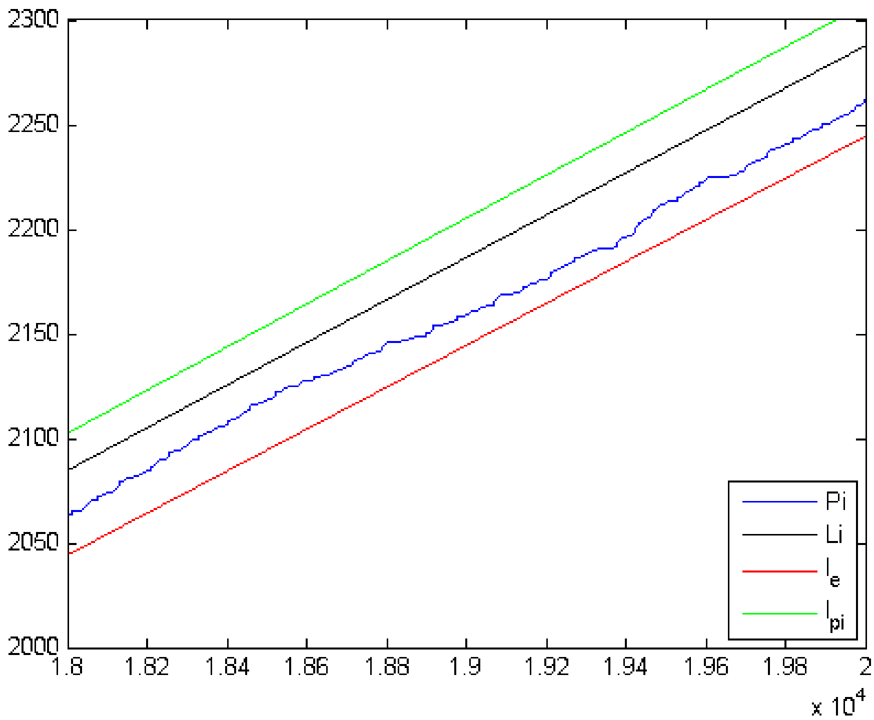


Figure 8: Close up for $x = 18000, ..., 20000$

Considering the idea that $I_e(x)$ and $I_\pi(x)$ act as reference asymptotes for $\pi(x)$, a general approximation formula $A_{\pi(x)}$ for $\pi(x)$ is constructed as follows:

$$A_{\pi(x)} \ := \ w_1 \cdot I_e(x) + w_2 \cdot I_\pi(x) - e$$

where w_1 and w_2 are weights for a convex combination of $I_e(x)$ and $I_{pi}(x)$. Their values are defined as follows:

$$\begin{aligned} w_1 &:= \frac{1}{H_x} + \gamma \\ w_2 &:= 1 - w_1 \end{aligned}$$

$A_{\pi(x)}$ has also an offset of e, as this value has shown to be numerically effective. The formula $A_{\pi(x)}$ is a heuristic attempt to create an approximation of $\pi(x)$ using $I_e(x)$ and $I_\pi(x)$ as reference asymptotes, whereas $A_{\pi(x)}$ *moves* to the inner region defined by $I_e(x)$ and $I_\pi(x)$. The beneath plots illustrate, how $A_{\pi(x)}$ *moves* to the inner region of $I_e(x)$ and $I_\pi(x)$, similar to $\pi(x)$.

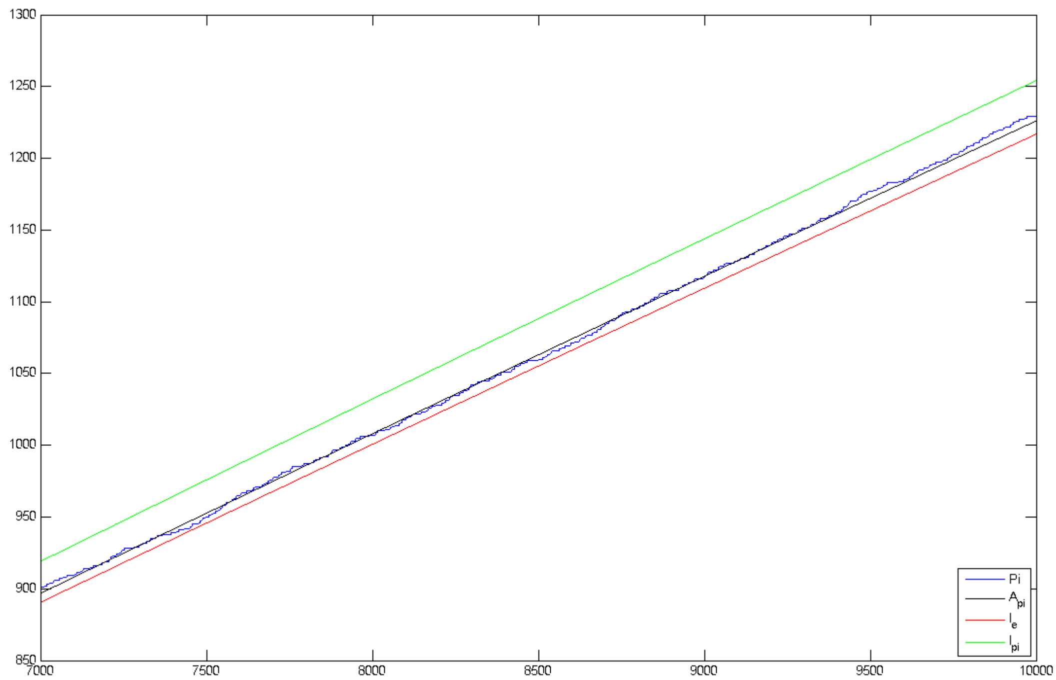


Figure 9: Plot for $x = 7000, ..., 10000$. Observe how $A_{\pi(x)}$ (black) fits $\pi(x)$ (blue)

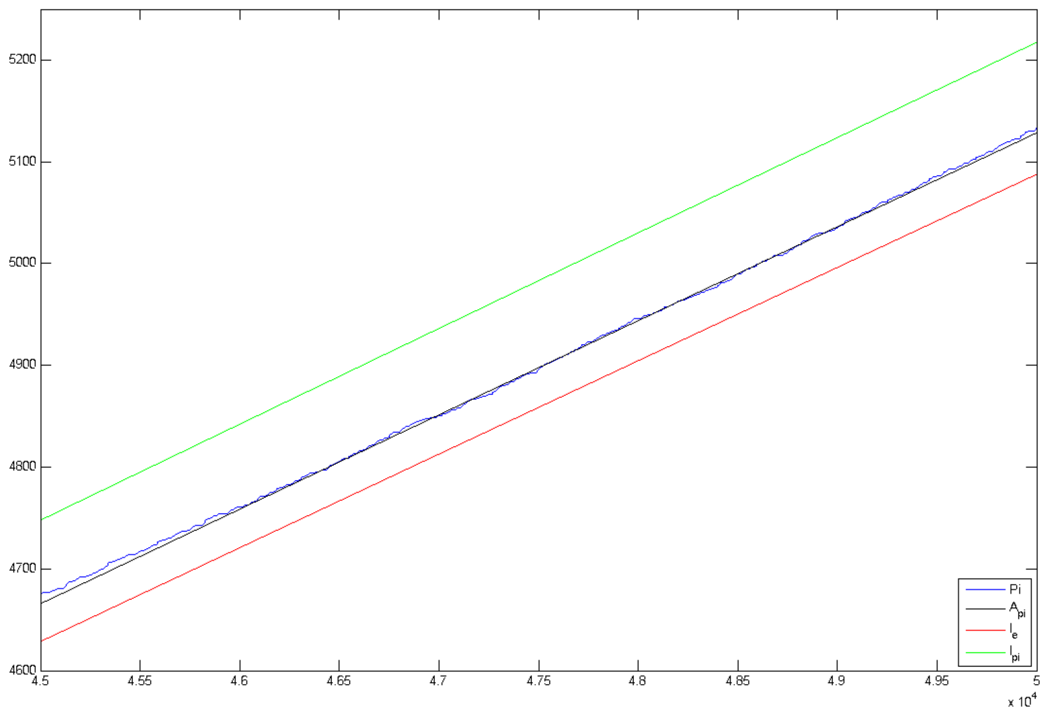


Figure 10: Plot for $x = 45000, ..., 50000$. Observe how $A_{\pi(x)}$ (black) fits $\pi(x)$ (blue)

It can be shown, that the average distance $|A_{\pi(x)}$-$\pi(x)|$ is about the same as $|Li(x) - \frac{1}{2}Li(\sqrt{x}) - \pi(x)|$ for n=4,...,100000. For lower values than 100000, the average distance by $A_{\pi(x)}$ is often better than the average distance of $|Li(x) - \frac{1}{2}Li(\sqrt{x}) - \pi(x)|$ (see plots beneath).

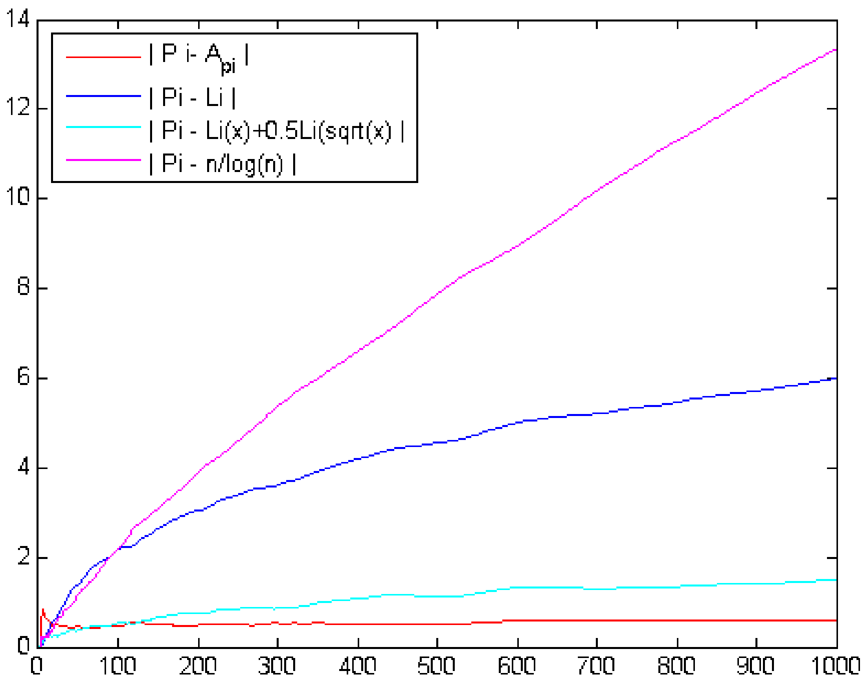


Figure 11: Comparison of average distance for n=4,...,1000

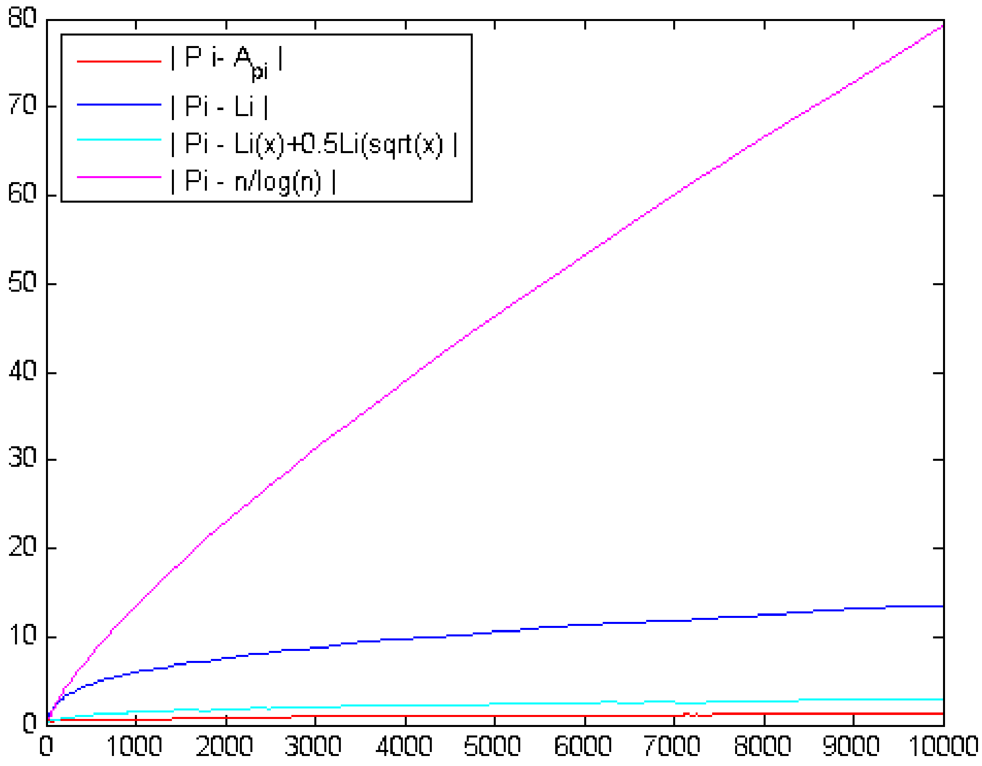


Figure 12: Comparison of average distance for n=4,...,10000

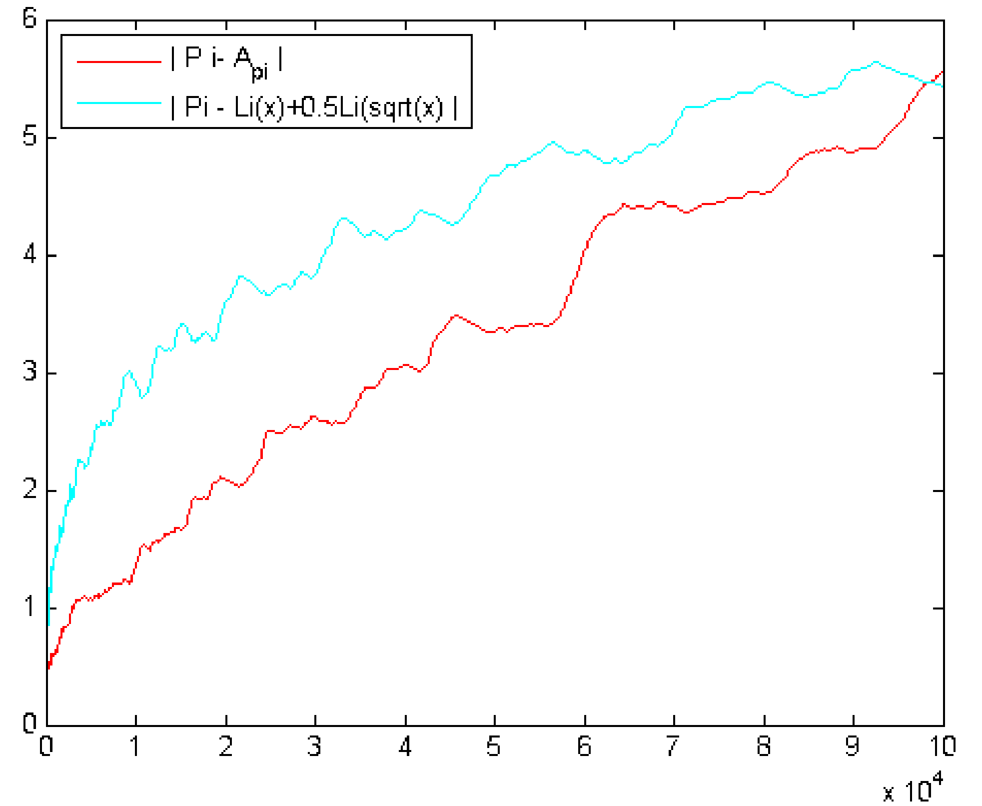


Figure 13: Comparison of average distance for n=4,...,100000

While in the presented heuristic $A_{\pi(x)}$ the weight w_1 works well for n up to some hundred thousands, it seems sub-optimal for larger n. The present weight w_1 will converge to γ for infinity, however, the following (near) optimal weights for w_1 can be calculated for larger n:

Table 1: Near optimal weight w_1 for larger n

n	w_1	$\pi(x) - (w_1 I_e + (1 - w_1) I_\pi)$
10^6	0.6590000000	-0.82127
10^7	0.6796000000	-0.01674
10^8	0.7077200000	-0.98752
10^9	0.7371800000	-0.05431
10^{10}	0.7624190000	-0.20676
10^{11}	0.7835495400	-0.93978
10^{12}	0.8007206500	-1.32021
10^{13}	0.8149598880	-0.81036
10^{14}	0.8269563556	-0.55664
10^{15}	0.8371916153	-1.98828
10^{16}	0.8460229137	-0.40625

As seen from above Table, γ does not appear to be the optimal weight choice for larger n. It might therefore be possible to construct better heuristics for larger n, exploiting this observation.

6 Another Gamma Approximation

Based on the previous formula

$$\gamma = \lim_{n\to\infty} H(n) - \frac{v}{w} \qquad \text{(See above for } v \text{ and } w\text{)}$$

and

$$log(x) = \lim_{n\to\infty} H(x \cdot n) - H(n)$$

we derive

$$\gamma = \lim_{n\to\infty} 2 \cdot H(n) - H(n \cdot (n+1))$$

7 Appendix: Source code to construct e,γ and π

Programming language: Matlab/Octave

```
function construct % This program numerically approximates 'e', 'gamma' and 'pi'
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  n    = 1;  % Initialize index 'n' for sequence 'a(n)'
  a(n) = 1;  % Define first element of 'a(n)' as 1
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  for k = 1:50000 % Generate sequence 'a(n)' which is OEIS-A004080 for 'n>=1'

    if  Zeta(1,k ,1) - floor(Zeta(1,k ,1)) < ...
        Zeta(1,k-1,1) - floor(Zeta(1,k-1,1)) % Check if 'k' is part of 'a(n)'

        n = n + 1; % Increase index 'n'
        a(n) = k; % Update sequence 'a(n)'

    end
  end
  a % Print sequence 'a(n)'
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  EULER = a(n) / a(n-1)                % Print approximation of 'e'
  GAMMA = Zeta(1,floor(EULER^n),1) - n % Print approximation of 'gamma'
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  TAU       = 0;       % Initialize 'tau' as zero
  x         = 0;       % Initialize 'x' as zero
  tolerance = 1;       % Tolerance for equality 'exp()=lim zeta()/zeta()'
  stepsize = 0.001;    % Stepsize to increase 'x' from zero to 'infinity'
  i         = sqrt(-1); % Define 'i' as imaginary unit sqrt(-1)
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  while (TAU == 0)  % Loop that increases 'x' and checks equality

    x = x + stepsize; % Increase 'x' by stepsize increment

    ratio = Zeta(a(n-2), a(n-1)-1, 1+i*x) / ...
            Zeta(a(n-1), a(n-0)-1, 1+i*x); % zeta()/zeta()

    % Check if equality 'exp()=lim zeta()/zeta()' fails
    if ( abs(ratio-EULER^(i*x)) > tolerance )

        TAU = x % If equality fails, 'tau' is found and printed

    end
  end
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  PI = TAU / 2 % Print 'pi' as half of 'tau'


% Zetafunction from 'A' to 'B' with input argument 'x'
function [z] = Zeta( A, B, x)
  z = 0;
  for n = A : B
      z = z + 1 / n^x;
  end
```

```
function main

N = 100
PRIMES = primes(N);
GAMMA  = 0.57721566490153286060;

for n=1:N
        a(n) = Pi(PRIMES,n);
        b(n) = Li(n);
    c(n) = Li_improved(n);

        Ie(n) = n / (H(n) - exp(1)*GAMMA);
        Ipi(n) = n / (H(n) -   pi*GAMMA);

    Api(n) = A(n);
end

plot(a)
hold on
hold on
plot(Api,'k')
plot(Ie,'r')
hold on
plot(Ipi,'g')
%axis([18000 20000 2000 2300]);
legend('Pi','A_{pi}','I_{e}','I_{pi}', 'Location','SouthEast');
hold off


%%%%%%%%%%%%%%
function y=A(n)
GAMMA  = 0.57721566490153286060;
HARMONIC = H(n);
Ie = n / (HARMONIC - exp(1)*GAMMA);
Ipi = n / (HARMONIC -   pi*GAMMA);
w1 = 1 / HARMONIC + GAMMA;
w2 = 1 - w1;
y = w1 * Ie + w2 * Ipi - exp(1);

%%%%%%%%%%%%%%
function y=H(n)
if(n<=100)
y = 0; for i=1:n; y = y + 1/i; end
else
GAMMA = 0.57721566490153286060;
y = log(n) + GAMMA + 1/(2*n) - 1/(12*n^2);
end
%%%%%%%%%%%%%%
function y=Pi(p,n)
%y=size(primes(n),2);
y = 0;
for i=1:size(p,2)
if(p(i) <= n)
y = y + 1;
```

```
end
end
%%%%%%%%%%%%%%
function y=Li(n)
F = @(x)1./log(x);
y = quad(F,2,n);
%%%%%%%%%%%%%%
function y=Li_improved(n)
F = @(x)1./log(x);
y = quad(F,2,n) - 0.5 * quad(F,2,sqrt(n));
```

```
function main

N = 100000
PRIMES = primes(N);
GAMMA = 0.57721566490153286060;

for n=4:N
       a(n) = Pi(PRIMES,n);
       b(n) = Li(n);
       c(n) = Li_improved(n);
       d(n) = n / log(n);
       e(n) = n / (H(n) - 3/2);     % Locker-Ernst 1959

       Ie(n) = n / (H(n) - exp(1)*GAMMA);
       Ipi(n) = n / (H(n) -   pi*GAMMA);

       Api(n) = A(n);

       dis_Api(n) = a(n) - Api(n);
       dis_Li(n) = a(n) - b(n);
       dis_LiX(n) = a(n) - c(n);
       dis_LOG(n) = a(n) - d(n);
       dis_LE(n) = a(n) - e(n);

       aver_Api(n) = sum(abs(dis_Api))/n;
       aver_Li(n) = sum(abs(dis_Li))/n;
       aver_LiX(n) = sum(abs(dis_LiX))/n;
       aver_LOG(n) = sum(abs(dis_LOG))/n;
       aver_LE (n) = sum(abs(dis_LE))/n;

       % Print average Distance
       fprintf('\n %i: Api: %f Li: %f LiX: %f LOG: %f LE:
          %f',n,aver_Api(n),aver_Li(n),aver_LiX(n),aver_LOG(n),aver_LE(n));
end

%%%%%%%%
figure()
plot(aver_Api,'r')
hold on
plot(aver_Li)
hold on
plot(aver_LiX,'c')
hold on
plot(aver_LOG,'m')
legend('| P i- A_{pi} |','| Pi - Li |','| Pi - Li(x)+0.5Li(sqrt(x) |','| Pi -
   n/log(n) |', 'Location','NorthWest');
hold off

%%%%%%%%
figure()
plot(a)
hold on
plot(Api,'k')
hold on
```

```
plot(Ie,'r')
hold on
plot(Ipi,'g')
%axis([18000 20000 2000 2300]);
legend('Pi','A_{pi}','I_{e}','I_{pi}', 'Location','SouthEast');
hold off



%%%%%%%%%%%%%%
function y=A(n)
GAMMA = 0.57721566490153286060;
HARMONIC = H(n);
Ie = n / (HARMONIC - exp(1)*GAMMA);
Ipi = n / (HARMONIC -  pi*GAMMA);
w1 = 1 / HARMONIC + GAMMA;
w2 = 1 - w1;
y = w1 * Ie + w2 * Ipi - exp(1);

%%%%%%%%%%%%%%
function y=H(n)
if(n<=100)
y = 0; for i=1:n; y = y + 1/i; end
else
GAMMA = 0.57721566490153286060;
y = log(n) + GAMMA + 1/(2*n) - 1/(12*n^2);
end
%%%%%%%%%%%%%%
function y=Pi(p,n)
%y=size(primes(n),2);
y = 0;
for i=1:size(p,2)
if(p(i) <= n)
y = y + 1;
end
end
%%%%%%%%%%%%%%
function y=Li(n)
F = @(x)1./log(x);
y = quad(F,2,n);
%%%%%%%%%%%%%%
function y=Li_improved(n)
F = @(x)1./log(x);
y = quad(F,2,n) - 0.5 * quad(F,2,sqrt(n));
```

```
function main

for i=6:16

  n = 10^i;

  % Solution
  if(n== 10^0); pix =                     0; end
  if(n== 10^1); pix =                     4; end
  if(n== 10^2); pix =                    25; end
  if(n== 10^3); pix =                   168; end
  if(n== 10^4); pix =                  1229; end
  if(n== 10^5); pix =                  9592; end
  if(n== 10^6); pix =                 78498; end
  if(n== 10^7); pix =                664579; end
  if(n== 10^8); pix =               5761455; end
  if(n== 10^9); pix =              50847534; end
  if(n==10^10); pix =             455052511; end
  if(n==10^11); pix =            4118054813; end
  if(n==10^12); pix =           37607912018; end
  if(n==10^13); pix =          346065536839; end
  if(n==10^14); pix =         3204941750802; end
  if(n==10^15); pix =        29844570422669; end
  if(n==10^16); pix =       279238341033925; end
  if(n==10^17); pix =      2623557157654233; end
  if(n==10^18); pix =     24739954287740860; end
  if(n==10^19); pix =    234057667276344607; end
  if(n==10^20); pix =   2220819602560918840; end
  if(n==10^21); pix =  21127269486018731928; end
  if(n==10^22); pix = 201467286689315906290; end
  if(n==10^23); pix = 1925320391606803968923; end

        % weight for convex combination
        if(n== 10^0); w1 = 1.75; end
        if(n== 10^1); w1 = 3.09; end
        if(n== 10^2); w1 = 2.35; end
        if(n== 10^3); w1 = 1.15; end
        if(n== 10^4); w1 = 0.68; end
        if(n== 10^5); w1 = 0.615; end
        if(n== 10^6); w1 = 0.659; end
        if(n== 10^7); w1 = 0.6796; end
        if(n== 10^8); w1 = 0.70772; end
        if(n== 10^9); w1 = 0.73718; end
        if(n==10^10); w1 = 0.762419; end
        if(n==10^11); w1 = 0.78354954; end
        if(n==10^12); w1 = 0.80072065; end
        if(n==10^13); w1 = 0.814959888; end
        if(n==10^14); w1 = 0.8269563556; end
        if(n==10^15); w1 = 0.83719161527; end
        if(n==10^16); w1 = 0.846022913731; end
        if(n==10^23); w1 = 0.8849435; end

   % Approximation
  dis = A(n,w1) - pix;
```

```
  % print distance)
  fprintf('\n $10^{%2i}$ & %15.10f & %10.5f \\\\',i,w1,dis)

 end

 %%%%%%%%%%%%%%%%%
function y = A(n,w1)
GAMMA = 0.5772156649015328;
H = log(n) + GAMMA;
Ie = n / (H - exp(1)*GAMMA);
Ipi = n / (H -   pi*GAMMA);
w2 = 1 - w1;
y = w1 * Ie + w2 * Ipi;
```
