Riemann's R-function and the distribution of primes

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Riemann's R-function is shown to alternately under- and over-estimate the number of primes in the intervals defined by the Fibonacci numbers, specifically from the interval $[55, 89]$ to the interval $[317\,811, 514\,229]$.

The author earlier explored the distribution of primes, where the ratio of the number of primes in the successive intervals defined by the Fibonacci numbers was found to alternate high, low, high, low, etc. from the interval $[55, 89]$ to the interval $[121\,393, 196\,418]$ [1]. Here Riemann's R-function is shown to alternately under- and over-estimate the number of primes from the interval $[55, 89]$ to the interval $[317\,811, 514\,229]$.

Below, the values on the right-hand-side reveal whether Riemann's R-function either under- or over-estimates the number of primes in an interval defined by consecutive Fibonacci numbers, where $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, ...etc., and where $R(x)$ equals `RiemannR[x]` in *Mathematica*. The function $\pi(x)$ gives the number of primes less than or equal to x.

$$\frac{R(F_4) - R(F_3)}{\pi(F_4) - \pi(F_3)} = \frac{R(3) - R(2)}{\pi(3) - \pi(2)} \approx 0.4636676 \tag{1}$$

$$\frac{R(F_5) - R(F_4)}{\pi(F_5) - \pi(F_4)} = \frac{R(5) - R(3)}{\pi(5) - \pi(3)} \approx 0.8164928 \tag{2}$$

$$\frac{R(F_6) - R(F_5)}{\pi(F_6) - \pi(F_5)} = \frac{R(8) - R(5)}{\pi(8) - \pi(5)} \approx 1.0800166 \tag{3}$$

$$\frac{R(F_7) - R(F_6)}{\pi(F_7) - \pi(F_6)} = \frac{R(13) - R(8)}{\pi(13) - \pi(8)} \approx 0.8014656 \tag{4}$$

$$\frac{R(F_8) - R(F_7)}{\pi(F_8) - \pi(F_7)} = \frac{R(21) - R(13)}{\pi(21) - \pi(13)} \approx 1.1490585 \tag{5}$$

$$\frac{R(F_9) - R(F_8)}{\pi(F_9) - \pi(F_8)} = \frac{R(34) - R(21)}{\pi(34) - \pi(21)} \approx 1.1231519 \tag{6}$$

$$\frac{R(F_{10}) - R(F_9)}{\pi(F_{10}) - \pi(F_9)} = \frac{R(55) - R(34)}{\pi(55) - \pi(34)} \approx 0.9881781 \tag{7}$$

$$\frac{R(F_{11}) - R(F_{10})}{\pi(F_{11}) - \pi(F_{10})} = \frac{R(89) - R(55)}{\pi(89) - \pi(55)} \approx 0.9129715 \tag{8}$$

$$\frac{R(F_{12}) - R(F_{11})}{\pi(F_{12}) - \pi(F_{11})} = \frac{R(144) - R(89)}{\pi(144) - \pi(89)} \approx 1.0844899 \tag{9}$$

$$\frac{R(F_{13}) - R(F_{12})}{\pi(F_{13}) - \pi(F_{12})} = \frac{R(233) - R(144)}{\pi(233) - \pi(144)} \approx 0.9522431 \tag{10}$$

$$\frac{R(F_{14}) - R(F_{13})}{\pi(F_{14}) - \pi(F_{13})} = \frac{R(377) - R(233)}{\pi(377) - \pi(233)} \approx 1.0552858 \tag{11}$$

$$\frac{R(F_{15}) - R(F_{14})}{\pi(F_{15}) - \pi(F_{14})} = \frac{R(610) - R(377)}{\pi(610) - \pi(377)} \approx 0.9877612 \tag{12}$$

$$\frac{R(F_{16}) - R(F_{15})}{\pi(F_{16}) - \pi(F_{15})} = \frac{R(987) - R(610)}{\pi(987) - \pi(610)} \approx 1.0044540 \tag{13}$$

$$\frac{R(F_{17}) - R(F_{16})}{\pi(F_{17}) - \pi(F_{16})} = \frac{R(1597) - R(987)}{\pi(1597) - \pi(987)} \approx 0.9859970 \tag{14}$$

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$$\frac{R(F_{18})-R(F_{17})}{\pi(F_{18})-\pi(F_{17})} \approx 1.0205233 \tag{15}$$

$$\frac{R(F_{19})-R(F_{18})}{\pi(F_{19})-\pi(F_{18})} \approx 0.9836314 \tag{16}$$

$$\frac{R(F_{20})-R(F_{19})}{\pi(F_{20})-\pi(F_{19})} \approx 1.0039880 \tag{17}$$

$$\frac{R(F_{21})-R(F_{20})}{\pi(F_{21})-\pi(F_{20})} \approx 0.9993921 \tag{18}$$

$$\frac{R(F_{22})-R(F_{21})}{\pi(F_{22})-\pi(F_{21})} \approx 1.0004330 \tag{19}$$

$$\frac{R(F_{23})-R(F_{22})}{\pi(F_{23})-\pi(F_{22})} \approx 0.9982875 \tag{20}$$

$$\frac{R(F_{24})-R(F_{23})}{\pi(F_{24})-\pi(F_{23})} \approx 1.0032416 \tag{21}$$

$$\frac{R(F_{25})-R(F_{24})}{\pi(F_{25})-\pi(F_{24})} \approx 0.9987094 \tag{22}$$

$$\frac{R(F_{26})-R(F_{25})}{\pi(F_{26})-\pi(F_{25})} \approx 1.0003750 \tag{23}$$

$$\frac{R(F_{27})-R(F_{26})}{\pi(F_{27})-\pi(F_{26})} \approx 0.9997837 \tag{24}$$

$$\frac{R(F_{28})-R(F_{27})}{\pi(F_{28})-\pi(F_{27})} \approx 1.0004894 \tag{25}$$

$$\frac{R(F_{29})-R(F_{28})}{\pi(F_{29})-\pi(F_{28})} \approx 0.9995964 \tag{26}$$

$$\frac{R(F_{30})-R(F_{29})}{\pi(F_{30})-\pi(F_{29})} \approx 0.9992383 \tag{27}$$

$$\frac{R(F_{31})-R(F_{30})}{\pi(F_{31})-\pi(F_{30})} \approx 1.0005718 \tag{28}$$

$$\frac{R(F_{32})-R(F_{31})}{\pi(F_{32})-\pi(F_{31})} \approx 0.9993671 \tag{29}$$

$$\frac{R(F_{33})-R(F_{32})}{\pi(F_{33})-\pi(F_{32})} \approx 1.0006990 \tag{30}$$

$$\frac{R(F_{34})-R(F_{33})}{\pi(F_{34})-\pi(F_{33})} \approx 0.9999341 \tag{31}$$

$$\frac{R(F_{35})-R(F_{34})}{\pi(F_{35})-\pi(F_{34})} \approx 0.9998355 \tag{32}$$

$$\frac{R(F_{36})-R(F_{35})}{\pi(F_{36})-\pi(F_{35})} \approx 1.0000960 \tag{33}$$

$$\frac{R(F_{37})-R(F_{36})}{\pi(F_{37})-\pi(F_{36})} \approx 1.0000344 \tag{34}$$

$$\frac{R(F_{38})-R(F_{37})}{\pi(F_{38})-\pi(F_{37})} \approx 0.9999823 \tag{35}$$

$$\frac{R(F_{39})-R(F_{38})}{\pi(F_{39})-\pi(F_{38})} \approx 1.0000619 \tag{36}$$

$$\frac{R(F_{40})-R(F_{39})}{\pi(F_{40})-\pi(F_{39})} \approx 0.9999415 \tag{37}$$

$$\frac{R(F_{41})-R(F_{40})}{\pi(F_{41})-\pi(F_{40})} \approx 1.0000607 \tag{38}$$

$$\frac{R(F_{42}) - R(F_{41})}{\pi(F_{42}) - \pi(F_{41})} \approx 0.9999453 \tag{39}$$

$$\frac{R(F_{43}) - R(F_{42})}{\pi(F_{43}) - \pi(F_{42})} \approx 1.0000317 \tag{40}$$

$$\frac{R(F_{44}) - R(F_{43})}{\pi(F_{44}) - \pi(F_{43})} \approx 0.9999653 \tag{41}$$

$$\frac{R(F_{45}) - R(F_{44})}{\pi(F_{45}) - \pi(F_{44})} \approx 1.0000053 \tag{42}$$

$$\frac{R(F_{46}) - R(F_{45})}{\pi(F_{46}) - \pi(F_{45})} \approx 0.9999977 \tag{43}$$

$$\frac{R(F_{47}) - R(F_{46})}{\pi(F_{47}) - \pi(F_{46})} \approx 1.0000050 \tag{44}$$

$$\frac{R(F_{48}) - R(F_{47})}{\pi(F_{48}) - \pi(F_{47})} \approx 0.9999997 \tag{45}$$

$$\frac{R(F_{49}) - R(F_{48})}{\pi(F_{49}) - \pi(F_{48})} \approx 1.0000142 \tag{46}$$

$$\frac{R(F_{50}) - R(F_{49})}{\pi(F_{50}) - \pi(F_{49})} \approx 0.9999886 \tag{47}$$

$$\frac{R(F_{51}) - R(F_{50})}{\pi(F_{51}) - \pi(F_{50})} \approx 1.0000059 \tag{48}$$

$$\frac{R(F_{52}) - R(F_{51})}{\pi(F_{52}) - \pi(F_{51})} \approx 0.9999961 \tag{49}$$

$$\frac{R(F_{53}) - R(F_{52})}{\pi(F_{53}) - \pi(F_{52})} \approx 0.9999981 \tag{50}$$

$$\frac{R(F_{54}) - R(F_{53})}{\pi(F_{54}) - \pi(F_{53})} \approx 1.0000027 \tag{51}$$

Above, beginning at Eq. (8) and ending at Eq. (26), Riemann's R-function alternately under- and over-estimates the number of primes, where it is the even-numbered equations that underestimate. In addition, beginning at Eq. (34) and ending at Eq. (49) the above pattern again manifests itself, though for these equations it is the odd-numbered equations that underestimate.

As noted by the author earlier with regard to different evidence [2], it is possible that a sequence dense in primes tends to be followed one less dense, which in turn is likely followed by one more dense, etc., where this tendency replicates itself at ever larger scales governed by the Fibonacci numbers. This is not entirely implausible. With one exception twin primes must be followed by three *non*-primes. This simple example of a sequence dense in primes necessarily followed by one less dense illustrates how variations in density might propagate along the number line. It only remains to consider that the Fibonacci numbers provide a natural scale along which prime density might fluctuate.

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[1] J. S. Markovitch, "On the distribution of prime numbers in the intervals defined by the Fibonacci numbers," (2010) `http://vixra.org/abs/1008.0036`.

[2] J. S. Markovitch, "On the Fibonacci numbers, the Koide formula, and the distribution of primes," (2012) `http://vixra.org/abs/1211.0022`.