Determinants in Geometric Algebra

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1 Definition

Let f be a linear map[1], of a real linear vector space $\mathbb{R}^n$ into itself, an endomorphism

$$f : \mathbf{a} \in \mathbb{R}^n \to \mathbf{a}' \in \mathbb{R}^n. \tag{1}$$

This map is extended by outermorphism (symbol $\underline{f}$) to act linearly on multivectors

$$\underline{f}(\mathbf{a}_1 \wedge \mathbf{a}_2 \ldots \wedge \mathbf{a}_k) = f(\mathbf{a}_1) \wedge f(\mathbf{a}_2) \ldots \wedge f(\mathbf{a}_k), \qquad k \leq n. \tag{2}$$

By definition $\underline{f}$ is grade-preserving and linear, mapping multivectors to multivectors. Examples are the reflections, rotations and translations described earlier. The outermorphism of a product of two linear maps fg is the product of the outermorphisms $\underline{f}\,\underline{g}$

$$\begin{aligned} f[g(\mathbf{a}_1)] \wedge f[g(\mathbf{a}_2)] \ldots \wedge f[g(\mathbf{a}_k)] &= \underline{f}[g(\mathbf{a}_1) \wedge g(\mathbf{a}_2) \ldots \wedge g(\mathbf{a}_k)] \\ &= \underline{f}[\underline{g}(\mathbf{a}_1 \wedge \mathbf{a}_2 \ldots \wedge \mathbf{a}_k)], \end{aligned} \tag{3}$$

with $k \leq n$. The square brackets can safely be omitted.

The n–grade pseudoscalars of a geometric algebra are unique up to a scalar factor. This can be used to define the determinant[2] of a linear map as

$$\det(f) = \underline{f}(I)I^{-1} = \underline{f}(I) * I^{-1}, \text{ and therefore } \underline{f}(I) = \det(f)I. \tag{4}$$

For an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ the unit pseudoscalar is $I = \mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n$ with inverse $I^{-1} = (-1)^q \mathbf{e}_n\mathbf{e}_{n-1} \ldots \mathbf{e}_1 = (-1)^q(-1)^{n(n-1)/2}I$, where q gives the number of basis vectors, that square to -1 (the linear space is then $\mathbb{R}^{p,q}$). According to Grassmann n-grade vectors represent oriented volume elements of dimension n. The determinant therefore shows how these volumes change under linear maps. Composing two linear maps gives the product of these volume factors

$$\underline{f}\,\underline{g}(I) = \underline{f}[\det(g)I] = \det(g)\underline{f}(I) = \det(g)\det(f)I. \tag{5}$$

Therefore

$$\det(fg) = \det(g)\det(f). \tag{6}$$

[1] The treatment in this section largely follows [1].

[2] The symbol $(*)$ means the (symmetric) scalar product of two multivectors, i.e. the scalar (0–grade) part of their geometric product.

2 Adjoint and Inverse Linear Maps

For every linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ exists[3] a unique adjoint linear map $\overline{f} : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\mathbf{b} * \overline{f}(\mathbf{a}) = \underline{f}(\mathbf{b}) * \mathbf{a}, \qquad \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^n. \tag{7}$$

The adjoint linear map extends again via outermorphism

$$\overline{f}(\mathbf{a}_1 \wedge \mathbf{a}_2 \ldots \wedge \mathbf{a}_k) = \overline{f}(\mathbf{a}_1) \wedge \overline{f}(\mathbf{a}_2) \ldots \wedge \overline{f}(\mathbf{a}_k), \qquad k \leq n. \tag{8}$$

In general we have for multivectors A, B that

$$B * \overline{f}(A) = \underline{f}(B) * A, \tag{9}$$

which can be applied to the defining[4] relationship[3] for the (right) contraction

$$(C \llcorner A) * B = C * (A \wedge B), \qquad \forall \text{ multivectors } A, B, C. \tag{10}$$

For simple grade c-vectors C and a-vectors A, the right contraction $(C \llcorner A)$ is a grade $c - a$ sub-space multivector of C perpendicular to A. We now get $\forall A, B, C$

$$\begin{aligned}
\overline{f}(C \llcorner A) * B &= (C \llcorner A) * \underline{f}(B) = C * (A \wedge \underline{f}(B)) \\
&= C * (\underline{f}(\underline{f}^{-1}(A)) \wedge \underline{f}(B)) = C * \underline{f}(\underline{f}^{-1}(A) \wedge B) \\
&= \overline{f}(C) * (\underline{f}^{-1}(A) \wedge B) = (\overline{f}(C) \llcorner \underline{f}^{-1}(A)) * B,
\end{aligned} \tag{11}$$

and therefore

$$\overline{f}(C \llcorner A) = \overline{f}(C) \llcorner \underline{f}^{-1}(A). \tag{12}$$

Similarly we obtain

$$\underline{f}(C \llcorner A) = \underline{f}(C) \llcorner \overline{f}^{-1}(A). \tag{13}$$

Reversion gives two more identities

$$\overline{f}(A \lrcorner C) = \underline{f}^{-1}(A) \lrcorner \overline{f}(C), \qquad \underline{f}(A \lrcorner C) = \overline{f}^{-1}(A) \lrcorner \underline{f}(C). \tag{14}$$

By substituting in $\overline{f}(C \llcorner A)$ the pseudoscalar I for C and left multiplying with the inverse I^{-1} we get a general formula for calculating the inverse of $\underline{f}$

$$\begin{aligned}
&I^{-1}\overline{f}(IA) = I^{-1}(\overline{f}(I) \llcorner \underline{f}^{-1}(A)) = I^{-1}\overline{f}(I)\underline{f}^{-1}(A) = \det(f)\underline{f}^{-1}(A), \\
&\iff \qquad \underline{f}^{-1}(A) = \frac{1}{\det(f)} I^{-1}\overline{f}(IA)
\end{aligned} \tag{15}$$

[3] An explicit definition for the adjoint linear map can be given as $\overline{f}(a) = \mathbf{e}^k(f(\mathbf{e}_k) * \mathbf{a})$, with $\mathbf{e}^k * \mathbf{e}_l = \delta^k_l$ (the *Kronecker delta*symbol), where $1 \leq k, l \leq n$. Here the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ form (a not necessarily orthonormal nor orthogonal) basis of $\mathbb{R}^n$.

[4] The symbols $(*)$ and $(\wedge)$ denote the (symmetric) scalar and the antisymmetric outer product parts of the geometric product of multivectors.

where we used the fact that right contraction with a pseudoscalar is nothing but the geometric product and that $\underline{f}$ is grade preserving.

In the derivation of $\underline{f}^{-1}$ we tacitly used the following property of the determinant obtained by applying $B * \overline{f}(A) = \underline{f}(B) * A$

$$\det(f) = \underline{f}(I) * I^{-1} = I * \overline{f}(I^{-1}) = \overline{f}(I) * I^{-1} = \det(f), \tag{16}$$

because of the symmetry of the scalar product and because $I^{-1} = (-1)^q (-1)^{n(n-1)/2} I$.

An analogous explicit expression can be derived for $\overline{f}^{-1}$

$$\begin{aligned} \underline{f}^{-1}(A) &= \det(f)^{-1}\overline{f}(AI)I^{-1} = \det(f)^{-1}I^{-1}\overline{f}(IA), \\ \overline{f}^{-1}(A) &= \det(f)^{-1}\underline{f}(AI)I^{-1} = \det(f)^{-1}I^{-1}\underline{f}(IA). \end{aligned} \tag{17}$$

These formulas are very compact and computationally efficient. They show that for invertible maps ($\det(f) \neq 0$) the inverse mappings can be easily constructed as double-dualities. Duality here means multiplication with the pseudoscalar I or I^{-1}.

References

[1] C. J. L. Doran. *Geometric Algebra and its Application to Mathematical Physics*, Ph. D. thesis, University of Cambridge, 181 pages (1994). `http://www.mrao.cam.ac.uk/~clifford/publications/abstracts/chris_thesis.html`

[2] D. Hestenes, G. Sobczyk, Clifford Algebra to Geometric Calculus, Kluwer, Dordrecht, reprinted with corrections 1992.

[3] L. Dorst, *The Inner Products of Geometric Algebra*, in L. Dorst et .al. (eds.), Applications of Geometric Algebra in Computer Science and Engineering, Birkhäuser, Basel, 2002. Preprint: `https://staff.fnwi.uva.nl/l.dorst/clifford/inner.ps`