What is an imaginary number?

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The previous Japanese emperor is said to have asked this question. Today many students and scientists still ask it, but the traditional canon of mathematics at school and university needs to be widened for the answer. We find it in the works of Hamilton, Grassmann and Clifford. Hamilton introduced quaternions i,j,k, with

$$i^2 = j^2 = k^2 = ijk = -1$$

and

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

for 3D rotations. Grassmann invented the outer product of oriented line segments (vectors) $\mathbf{a},\mathbf{b}$ to give the directed oriented area of the enclosed parallelogram:

$$\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}.$$

Clifford unified their work with the geometric product

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b},$$

leading to geometric algebras.

In two dimensions we have orthogonal, unit vectors $\mathbf{e}_1,\mathbf{e}_2$ as vector space basis with

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1, \quad \mathbf{e}_1\cdot\mathbf{e}_2 = 0.$$

The associative geometric multiplication of the oriented directed unit square

$$i = \mathbf{e}_1\mathbf{e}_2$$

gives:

$$ii = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1(\mathbf{e}_2\mathbf{e}_1)\mathbf{e}_2 = \mathbf{e}_1(\mathbf{e}_2\cdot\mathbf{e}_1+\mathbf{e}_2\wedge\mathbf{e}_1)\mathbf{e}_2 =$$
$$= \mathbf{e}_1(0-\mathbf{e}_1\wedge\mathbf{e}_2)\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -1.$$

NB: We only used

$$\mathbf{e}_2 \cdot \mathbf{e}_1 = \mathbf{e}_1 \cdot \mathbf{e}_2 = 0 \quad \text{and} \quad \mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_2.$$

So the square of the oriented unit area i is –1. Enough to satisfy the emperor's curiosity!

But today's politicians ask for an application. As an answer we calculate:

$$i\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2$$

and

$$i\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = \mathbf{e}_1,$$

which is a clockwise 90° rotation. We can also calculate (NB: the order!)

$$\mathbf{e}_1 i = \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1$$

and

$$\mathbf{e}_2 i = \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -\mathbf{e}_1,$$

which is an anticlockwise (mathematically positiv) 90° rotation. For a general rotation in two dimensions, we simply add trigonometric coefficients:

$$\mathbf{a}\,(\,\cos(\alpha) + i\,\sin(\alpha)\,)$$

rotates the real vector **a** by α degrees. Now even a politician can rotate vectors without using (or even knowing) matrices.

Of what use may the geometric product be for some new advanced technology venture business? As an application to *laser beam optics* let us imagine a laser beam with direction vector **a** hitting a mirror surface element approximated with unit normal vector **n** ($\mathbf{n}^2=1$). We can write **a** in components parallel and perpendicular to **n**:

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_{\perp}.$$

Now

$$\mathbf{a}_{||} \wedge \mathbf{n} = 0,$$

because parallel vectors span no parallelogram, and

$$\mathbf{a}_{\perp} \cdot \mathbf{n} = 0,$$

because of perpendicularity. So we must have

$$\mathbf{a}_{||}\mathbf{n} = \mathbf{a}_{||}\cdot\mathrm{n} + 0 = \mathrm{n}\cdot\mathbf{a}_{||} + 0 = \mathrm{n}\mathbf{a}_{||}$$

and

$$\mathbf{a}_{\perp}\mathbf{n} = 0 + \mathbf{a}_{\perp}\wedge\mathbf{n} = 0 - \mathbf{n}\wedge\mathbf{a}_{\perp} = -\mathbf{n}\mathbf{a}_{\perp}.$$

Reflection only changes the sign of $\mathbf{a}_{||}$. Therefore

$$\begin{aligned}\mathbf{a}' &= -\mathbf{a}_{||}+\mathbf{a}_{\perp} = -\mathbf{nn}(\mathbf{a}_{||}-\mathbf{a}_{\perp}) = -\mathbf{n}(\mathbf{na}_{||}-\mathbf{na}_{\perp}) = \\ &= -\mathbf{n}(\mathbf{a}_{||}\mathbf{n}+\mathbf{a}_{\perp}\mathbf{n}) = -\mathbf{n}(\mathbf{a}_{||}+\mathbf{a}_{\perp})\mathbf{n} = -\mathbf{nan}\end{aligned}$$

is the reflected vector. In a cavity we may want to trace many reflections at a sequence of surface elements with normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$, ... $\mathbf{n}_s$ which simply results in

$$\mathbf{a}' = (-1)^s\ \mathbf{n}_s \ldots\ \mathbf{n}_2\mathbf{n}_1\mathbf{a}\mathbf{n}_1\mathbf{n}_2 \ldots\ \mathbf{n}_s.$$

Nanoscience is a modern buzz word. On this scale mechanics meets quantum mechanics. Geometric algebra provides complete tools for both. From elementary geometry we know that two reflections at planes with normal vectors $\mathbf{n},\mathbf{m}$ enclosing the angle $\theta/2$ result in a rotation by angle θ:

$$\mathbf{a}' = \mathbf{mn}\ \mathbf{a}\ \mathbf{nm}.$$

The general *rot*ation operat*or* (*rotor*) is

$$\mathrm{R} = \mathbf{nm} = \mathbf{n}\cdot\mathbf{m} + \mathbf{n}\wedge\mathbf{m} = \cos(\theta/2) + i\sin(\theta/2) = \exp(i\,\theta/2)$$

with unit area element i in the $\mathbf{n},\mathbf{m}$ rotation plane. Two rotations are given by the geometric product of two rotors RR'. A second $\theta' = 360°$ rotation poduces

$$\mathrm{RR}' = \mathrm{R}\exp(2\pi i/2) = \mathrm{R}\exp(\pi i) = \mathrm{R}\,(\cos(\pi)+i\sin(\pi)) = \mathrm{R}(-1+i0) = -\mathrm{R}.$$

The rotor R itself behaves therefore like the first known *quantum particle*, i.e. the electron described by a Pauli spinor

$$\psi = \rho^{1/2}\,\mathrm{R}.$$

Geometric algebra answers fundamental questions, which the traditional canon of mathematics taught at schools and universities can't. It further provides great methodological simplifications and geometric insight in applications to physics, molecular geometry, image processing, computer graphics, robotics, quantum computing, etc. Geometric algebra is an excellent candidate to restructure mathematical syllabi on all (from school to post graduate) levels. I propose therefore to establish a research institute, dedicated to further develop geometric calculus (with geometric algebra as mathematical grammar) as a general tool for teaching, research and application.

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