The Clifford Fourier transform in real Clifford algebras

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Abstract

We use the recent comprehensive research [17, 19] on the *manifolds of square roots of* -1 in real Clifford's geometric algebras $Cl(p,q)$ in order to construct the *Clifford Fourier transform*. Basically in the kernel of the complex Fourier transform the imaginary unit $j \in \mathbb{C}$ is replaced by a square root of -1 in $Cl(p,q)$. The Clifford Fourier transform (CFT) thus obtained generalizes previously known and applied CFTs [9, 13, 14], which replaced $j \in \mathbb{C}$ only by blades (usually pseudoscalars) squaring to -1. A major advantage of real Clifford algebra CFTs is their completely real geometric interpretation. We study (left and right) linearity of the CFT for constant multivector coefficients $\in Cl(p,q)$, translation ($\boldsymbol{x}$-shift) and modulation ($\boldsymbol{\omega}$-shift) properties, and signal dilations. We show an inversion theorem. We establish the CFT of vector differentials, partial derivatives, vector derivatives and spatial moments of the signal. We also derive Plancherel and Parseval identities as well as a general convolution theorem.

Keywords: Clifford Fourier transform, Clifford algebra, signal processing, square roots of -1.

1 Introduction

Quaternion, Clifford and geometric algebra Fourier transforms (QFT, CFT, GAFT) [14, 15, 18, 21] have proven *very useful* tools for applications in non-marginal color image processing, image diffusion, electromagnetism, multi-channel processing, vector field processing, shape representation, linear scale invariant filtering, fast vector pattern matching, phase correlation, analysis of non-stationary improper complex signals, flow analysis, partial differential systems, disparity estimation, texture segmentation, as spectral representations for Clifford wavelet analysis, etc.

All these Fourier transforms essentially analyze scalar, vector and multivector signals in terms of sine and cosine waves with multivector coefficients. For this purpose the imaginary unit $j \in \mathbb{C}$ in $e^{j\phi} = \cos\phi + j\sin\phi$ can be replaced by any *square root of* -1 *in a real Clifford algebra* $Cl(p,q)$. The replacement by pure quaternions and blades with negative square [8, 15] has already yielded a wide variety of results with a clear geometric interpretation. It is well-known that there are elements other than blades, squaring to -1. Motivated by their special relevance for new types of CFTs, they have recently been studied thoroughly [17, 19, 25].

We therefore tap into these new results on square roots of -1 in Clifford algebras and fully general construct CFTs, with one general square root of -1 in $Cl(p,q)$. Our new CFTs form therefore a more general class of CFTs, subsuming and generalizing previous results[1]. A further benefit is, that these new CFTs become *fully steerable* within the continuous Clifford algebra submanifolds of square roots of -1. We thus obtain a comprehensive *new mathematical framework* for the investigation and application of Clifford Fourier transforms together with *new properties* (full steerability). Regarding the question of the *most suitable* CFT for a certain application, we are only just beginning to leave the terra cognita of familiar transforms to map out the vast array of possible CFTs in $Cl(p,q)$.

This paper is organized as follows. We first review in Section 2 key notions of Clifford algebra, *multivector signal functions*, and the recent results on *square roots of* -1 in Clifford algebras. Next, in Section 3 we define the central notion of *Clifford Fourier transforms* with respect to any square root of -1 in Clifford algebra. Then we study in Section 4 (left and right) linearity of the CFT for constant multivector coefficients $\in Cl(p,q)$, translation ($\boldsymbol{x}$-shift) and modulation ($\boldsymbol{\omega}$-shift) properties, and signal dilations, followed by an inversion theorem. We establish the CFT of vector differentials, partial derivatives, vector derivatives and spatial moments of the signal. We also show Plancherel and Parseval identities as well as a general convolution theorem.

2 Clifford's geometric algebra

Definition 2.1 (Clifford's geometric algebra [12, 23]) *Let* $\{e_1, e_2, \dots, e_p, e_{p+1}, \dots, e_n\}$*, with* $n = p+q$*,* $e_k^2 = \varepsilon_k$*,* $\varepsilon_k = +1$ *for* $k = 1, \dots, p$*,* $\varepsilon_k = -1$ *for* $k = p+1, \dots, n$*, be an orthonormal base of the inner product vector space*

[1]This is only the first step towards generalization. The non-commutativity of the geometric product of multivectors makes it meaningful to investigate CFTs with several kernel factors to both sides of the signal function. Each kernel factor may use a different square root of -1. Work in this direction has been reported at ICCA9 and will be published in [8].

$\mathbb{R}^{p,q}$ with a geometric product according to the multiplication rules

$$e_k e_l + e_l e_k = 2\varepsilon_k \delta_{k,l}, \qquad k,l = 1,\ldots n, \tag{1}$$

where $\delta_{k,l}$ is the Kronecker symbol with $\delta_{k,l} = 1$ for $k = l$, and $\delta_{k,l} = 0$ for $k \neq l$. This bilinear non-commutative product and the additional axiom of associativity generate the 2^n-dimensional Clifford geometric algebra $Cl(p,q) = Cl(\mathbb{R}^{p,q}) = Cl_{p,q} = \mathscr{G}_{p,q} = \mathbb{R}_{p,q}$ over $\mathbb{R}$. The set $\{e_A : A \subseteq \{1,\ldots,n\}\}$ with $e_A = e_{h_1} e_{h_2} \ldots e_{h_k}$, $1 \leq h_1 < \ldots < h_k \leq n$, $e_\emptyset = 1$ (neutral element of the Clifford geometric product), forms a graded (blade) basis of $Cl(p,q)$. The grades k range from 0 for scalars, 1 for vectors, 2 for bivectors, s for s-vectors, up to n for pseudoscalars. The vector space $\mathbb{R}^{p,q}$ is included in $Cl(p,q)$ as the subset of 1-vectors. The general elements of $Cl(p,q)$ are real linear combinations of basis blades e_A, called Clifford numbers, multivectors or hypercomplex numbers.

In general $\langle A \rangle_k$ denotes the grade k part of $A \in Cl(p,q)$. The parts of grade 0 and $k+s$, respectively, of the geometric product of a k-vector $A_k \in Cl(p,q)$ with an s-vector $B_s \in Cl(p,q)$

$$A_k * B_s := \langle A_k B_s \rangle_0, \qquad A_k \wedge B_s := \langle A_k B_s \rangle_{k+s}, \tag{2}$$

are called *scalar product* and *outer product*, respectively.

For Euclidean vector spaces ($n = p$) we use $\mathbb{R}^n = \mathbb{R}^{n,0}$ and $Cl(n) = Cl(n,0)$. Every k-vector B that can be written as the outer product $B = \boldsymbol{b}_1 \wedge \boldsymbol{b}_2 \wedge \ldots \wedge \boldsymbol{b}_k$ of k vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k \in \mathbb{R}^{p,q}$ is called a *simple* k-vector or *blade*.

Multivectors $M \in Cl(p,q)$ have k-vector parts ($0 \leq k \leq n$): scalar part $Sc(M) = \langle M \rangle = \langle M \rangle_0 = M_0 \in \mathbb{R}$, vector part $\langle M \rangle_1 \in \mathbb{R}^{p,q}$, bi-vector part $\langle M \rangle_2$, ..., and pseudoscalar part $\langle M \rangle_n \in \bigwedge^n \mathbb{R}^{p,q}$

$$M = \sum_A M_A \boldsymbol{e}_A = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \ldots + \langle M \rangle_n \,. \tag{3}$$

Taking the *reverse* is equivalent to reversing the order of products of basis vectors in the basis blades, e.g. $e_1 e_2 \to e_2 e_1 = -e_1 e_2$, etc. The *principal reverse*[2] of $M \in Cl(p,q)$ defined as

$$\widetilde{M} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle \overline{M} \rangle_k, \tag{4}$$

often replaces complex conjugation and quaternion conjugation. The operation $\overline{M}$ means to change in the basis decomposition of M the sign of every

[2]Note that in the current work we use the principal reverse throughout. But depending on the context another involution or anti-involution of Clifford algebra may be more appropriate for specific Clifford algebras, or for the purpose of a specific geometric interpretation.

vector of negative square $\overline{e_A} = \varepsilon_{h_1} e_{h_1} \varepsilon_{h_2} e_{h_2} \dots \varepsilon_{h_k} e_{h_k}$, $1 \le h_1 < \dots < h_k \le n$. Reversion, $\overline{M}$, and principal reversion are all involutions.

The principal reverse of every basis element $e_A \in Cl(p,q)$, $1 \le A \le 2^n$, has the property

$$\widetilde{e_A} * e_B = \delta_{AB}, \qquad 1 \le A, B \le 2^n, \tag{5}$$

where the Kronecker delta $\delta_{AB} = 1$ if $A = B$, and $\delta_{AB} = 0$ if $A \ne B$. For the vector space $\mathbb{R}^{p,q}$ this leads to a reciprocal basis e^l, $1 \le l, k \le n$

$$e^l := \widetilde{e_l} = \varepsilon_l e_l, \quad e^l * e_k = e^l \cdot e_k = \begin{cases} 1, & \text{for } l = k \\ 0, & \text{for } l \ne k \end{cases}. \tag{6}$$

For $M, N \in Cl(p,q)$ we get $M * \widetilde{N} = \sum_A M_A N_A$. Two multivectors $M, N \in Cl(p,q)$ are *orthogonal* if and only if $M * \widetilde{N} = 0$. The modulus $|M|$ of a multivector $M \in Cl(p,q)$ is defined as

$$|M|^2 = M * \widetilde{M} = \sum_A M_A^2. \tag{7}$$

2.1 Multivector signal functions

A multivector valued function $f : \mathbb{R}^{p,q} \to Cl(p,q)$, has 2^n blade components $(f_A : \mathbb{R}^{p,q} \to \mathbb{R})$

$$f(\boldsymbol{x}) = \sum_A f_A(\boldsymbol{x}) \boldsymbol{e}_A, \qquad \boldsymbol{x} = \sum_{l=1}^n x_l e^l = \sum_{l=1}^n x^l e_l. \tag{8}$$

We define the *inner product* of two functions $f, g : \mathbb{R}^{p,q} \to Cl(p,q)$ by

$$(f,g) = \int_{\mathbb{R}^{p,q}} f(\boldsymbol{x}) \widetilde{g(\boldsymbol{x})} \, d^n\boldsymbol{x} = \sum_{A,B} \boldsymbol{e}_A \widetilde{\boldsymbol{e}_B} \int_{\mathbb{R}^{p,q}} f_A(\boldsymbol{x}) g_B(\boldsymbol{x}) \, d^n\boldsymbol{x}, \tag{9}$$

with the *symmetric scalar part*

$$\langle f,g \rangle = \int_{\mathbb{R}^{p,q}} f(\boldsymbol{x}) * \widetilde{g(\boldsymbol{x})} \, d^n\boldsymbol{x} = \sum_A \int_{\mathbb{R}^{p,q}} f_A(\boldsymbol{x}) g_A(\boldsymbol{x}) \, d^n\boldsymbol{x}, \tag{10}$$

and the $L^2(\mathbb{R}^{p,q}; Cl(p,q))$-*norm*[3]

$$\|f\|^2 = \langle (f,f) \rangle = \int_{\mathbb{R}^{p,q}} |f(\boldsymbol{x})|^2 d^n\boldsymbol{x} = \sum_A \int_{\mathbb{R}^{p,q}} f_A^2(\boldsymbol{x}) \, d^n\boldsymbol{x}, \tag{11}$$

$$L^2(\mathbb{R}^{p,q}; Cl(p,q)) = \{ f : \mathbb{R}^{p,q} \to Cl(p,q) \mid \|f\| < \infty \}. \tag{12}$$

[3]Note, that we do prefer in (11) the notation $\langle (f,f) \rangle$ over $\langle f, f \rangle$, because the round brackets are useful to clearly indicate the application of the inner product integral of (9). This helps to avoid confusion, because the angular brackets alone are often used for indicating the scalar part of a multivector.

The *vector derivative* ∇ of a function $f : \mathbb{R}^{p,q} \to Cl(p,q)$ can be expanded in a basis of $\mathbb{R}^{p,q}$ as [27]

$$\nabla = \sum_{l=1}^{n} e^l \partial_l \quad \text{with} \quad \partial_l = \partial_{x_l} = \frac{\partial}{\partial x_l}, \ 1 \le l \le n. \tag{13}$$

2.2 Square roots of -1 in Clifford algebras

We briefly summarize the new results on square roots of -1 in Clifford algebras. For details and explicit proofs, please see [17, 19]. The material in the following Section 2.3 for the conformal geometric algebra $Cl(4,1)$ is newly added.

Every Clifford algebra $Cl(p,q)$, $s_8 = (p-q) \bmod 8$, is isomorphic to one of the following (square) matrix algebras[4] $\mathscr{M}(2d,\mathbb{R})$, $\mathscr{M}(d,\mathbb{H})$, $\mathscr{M}(2d,\mathbb{R}^2)$, $\mathscr{M}(d,\mathbb{H}^2)$ or $\mathscr{M}(2d,\mathbb{C})$. The first argument of $\mathscr{M}$ is the dimension, the second the associated ring[5] $\mathbb{R}$ for $s_8 = 0,2$, $\mathbb{R}^2$ for $s_8 = 1$, $\mathbb{C}$ for $s_8 = 3,7$, $\mathbb{H}$ for $s_8 = 4,6$, and $\mathbb{H}^2$ for $s_8 = 5$. For even n: $d = 2^{(n-2)/2}$, for odd n: $d = 2^{(n-3)/2}$.

It has been shown [17, 19] that[6] $Sc(f) = 0$ for every square root of $-\mathbf{1}$ in every matrix algebra $\mathscr{A}$ isomorphic to $Cl(p,q)$. One can distinguish *ordinary* square roots of $-\mathbf{1}$, and *exceptional* ones. All square roots of $-\mathbf{1}$ in $Cl(p,q)$ can be computed using the package CLIFFORD for Maple [1, 3, 20, 24].

Exceptional square roots of $-\mathbf{1}$ *only* exist if $\mathscr{A} \cong \mathscr{M}(2d,\mathbb{C})$, and have a non-zero pseudoscalar part. In all other cases the *ordinary* square roots f of $-\mathbf{1}$ constitute a *unique conjugacy class* of dimension $\dim(\mathscr{A})/2$, which has *as many connected components as the group* $\mathrm{G}(\mathscr{A})$ of invertible elements in $\mathscr{A}$. Furthermore, we have $\mathrm{Spec}(f) = 0$ (zero pseudoscalar part) if the associated ring is $\mathbb{R}^2$, $\mathbb{H}^2$, or $\mathbb{C}$. The manifolds of square roots of -1 in $Cl(p,q)$, $n = p+q = 2$, compare Table 1 of [17], are visualized[7] in Fig. 1.

For $\mathscr{A} = \mathscr{M}(2d,\mathbb{R})$, the centralizer (set of all elements in $Cl(p,q)$ commuting with f) and the conjugacy class of a square root f of $-\mathbf{1}$ both have $\mathbb{R}$-dimension $2d^2$ with *two connected components*. For the simplest case $d = 1$ we have the algebra $Cl(2,0)$ isomorphic to $\mathscr{M}(2,\mathbb{R})$, pictured in Fig. 1 (left) and alternatively in Fig. 2.

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[4]Compare chapter 16 on *matrix representations and periodicity of 8*, as well as Table 1 on p. 217 of [23].

[5]Associated ring means, that the matrix elements are from the respective ring $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{H}^2$.

[6]In Sections 2.2 and 2.3 we use the symbol f for square roots of -1 in Clifford algebras. In this way we follow the notation of [19]. But in order to avoid confusion with multivector functions, we use the symbol i in the rest of the paper.

[7]The identity (modulo a 90 degree rotation) of the manifolds of square roots of -1 of $Cl(2,0)$ (left) and $Cl(1,1)$ (center) in Fig. 1 is a manifestation of the isomorphism between the two Clifford algebras .

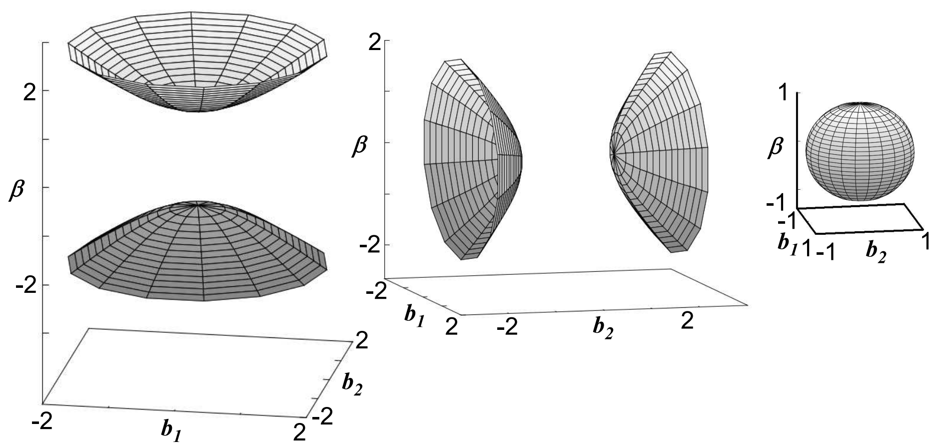

Figure 1: Manifolds [19] of square roots f of -1 in $Cl(2,0)$ (left), $Cl(1,1)$ (center), and $Cl(0,2) \cong \mathbb{H}$ (right). The square roots are $f = \alpha + b_1 e_1 + b_2 e_2 + \beta e_{12}$, with $\alpha, b_1, b_2, \beta \in \mathbb{R}$, $\alpha = 0$, and $\beta^2 = b_1^2 e_2^2 + b_2^2 e_1^2 + e_1^2 e_2^2$.

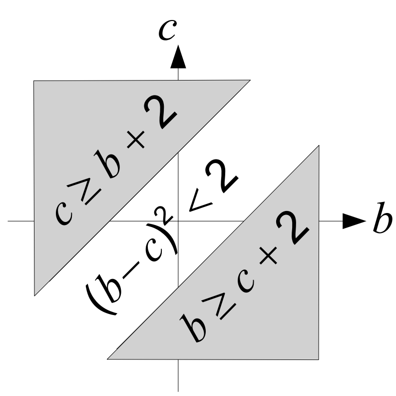

Figure 2: Two components of square roots of $-\mathbf{1}$ in $\mathscr{M}(2,\mathbb{R}) \cong Cl(2,0)$, see [19] for details.

For $\mathscr{A} = \mathscr{M}(2d, \mathbb{R}^2) = \mathscr{M}(2d, \mathbb{R}) \times \mathscr{M}(2d, \mathbb{R})$, the square roots of $(-\mathbf{1}, -\mathbf{1})$ are pairs of two square roots of $-\mathbf{1}$ in $\mathscr{M}(2d, \mathbb{R})$. They constitute a unique conjugacy class with *four connected components*, each of dimension $4d^2$. Regarding the four connected components, the group of inner automorphisms $\mathrm{Inn}(\mathscr{A})$ induces the permutations of the Klein group, whereas the quotient group $\mathrm{Aut}(\mathscr{A})/\mathrm{Inn}(\mathscr{A})$ is isomorphic to the group of isometries of a Euclidean square in 2D. The simplest example with $d = 1$ is $Cl(2,1)$ isomorphic to $M(2, \mathbb{R}^2) = \mathscr{M}(2, \mathbb{R}) \times \mathscr{M}(2, \mathbb{R})$.

For $\mathscr{A} = \mathscr{M}(d, \mathbb{H})$, the submanifold of the square roots f of $-\mathbf{1}$ is a *single connected conjugacy class* of $\mathbb{R}$-dimension $2d^2$ equal to the $\mathbb{R}$-dimension of the centralizer of every f. The easiest example is $\mathbb{H} \cong Cl(0,2)$ itself for $d = 1$, pictured in Fig. 1 (right).

For $\mathscr{A} = \mathscr{M}(d, \mathbb{H}^2) = \mathscr{M}(d, \mathbb{H}) \times \mathscr{M}(d, \mathbb{H})$, the square roots of $(-\mathbf{1}, -\mathbf{1})$ are pairs of two square roots (f, f') of $-\mathbf{1}$ in $\mathscr{M}(d, \mathbb{H})$ and constitute a *unique connected conjugacy class* of $\mathbb{R}$-dimension $4d^2$. The group $\mathrm{Aut}(\mathscr{A})$ has two connected components: the neutral component $\mathrm{Inn}(\mathscr{A})$ connected to the identity and the second component containing the swap automorphism $(f, f') \mapsto (f', f)$. The simplest case for $d = 1$ is $\mathbb{H}^2$ isomorphic to $Cl(0,3)$.

For $\mathscr{A} = \mathscr{M}(2d, \mathbb{C})$, the square roots of $-\mathbf{1}$ are in *bijection to the idempotents* [2]. First, the *ordinary* square roots of $-\mathbf{1}$ (with $k = 0$, i.e. zero pseudoscalar part) constitute a conjugacy class of $\mathbb{R}$-dimension $4d^2$ of a *single connected component* which is invariant under $\mathrm{Aut}(\mathscr{A})$. Second, there are $2d$ *conjugacy classes* of *exceptional* square roots of $-\mathbf{1}$, each composed of a *single connected component*, characterized by the equality $\mathrm{Spec}(f) = k/d$ (the pseudoscalar coefficient) with $\pm k \in \{1, 2, \ldots, d\}$, and their $\mathbb{R}$-dimensions are $4(d^2 - k^2)$. The group $\mathrm{Aut}(\mathscr{A})$ includes conjugation of the pseudoscalar $\omega \mapsto -\omega$ which maps the conjugacy class associated with k to the class associated with $-k$. The simplest case for $d = 1$ is the Pauli matrix algebra isomorphic to the geometric algebra $Cl(3,0)$ of 3D Euclidean space $\mathbb{R}^3$, and to complex biquaternions [25]. The square roots of $-\mathbf{1}$ in conformal geometric algebra $Cl(4,1) \cong \mathscr{M}(4, \mathbb{C})$, $d = 2$ are considered separately in Section 2.3.

With respect to any square root $i \in Cl(p,q)$ of -1, $i^2 = -1$, every multivector $A \in Cl(p,q)$ can be split into *commuting* and *anticommuting* parts [19].

Lemma 2.2 *Every multivector $A \in Cl(p,q)$ has, with respect to a square root $i \in Cl(p,q)$ of -1, i.e., $i^{-1} = -i$, the unique decomposition*

$$A_{+i} = \frac{1}{2}(A + i^{-1}Ai), \qquad A_{-i} = \frac{1}{2}(A - i^{-1}Ai)$$
$$A = A_{+i} + A_{-i}, \qquad A_{+i}\, i = iA_{+i}, \qquad A_{-i}\, i = -iA_{-i}. \tag{14}$$

2.3 Square roots of -1 in conformal geometric algebra $Cl(4,1)$

We pay special attention to the square roots of -1 in conformal geometric algebra $Cl(4,1)$, because of the enormous practical importance of this algebra in applications to robotics, computer graphics, robot and computer vision, virtual reality, visualization, and the like [22]. See Table 1 for representative exceptional ($k \neq 0$) square roots of $-\mathbf{1}$ in conformal geometric algebra $Cl(4,1)$ of three-dimensional Euclidean space [19].

k	f_k	$\Delta_k(t)$
2	$\omega = e_{12345}$	$(t-i)^4$
1	$\frac{1}{2}(e_{23}+e_{123}-e_{2345}+e_{12345})$	$(t-i)^3(t+i)$
0	e_{123}	$(t-i)^2(t+i)^2$
-1	$\frac{1}{2}(e_{23}+e_{123}+e_{2345}-e_{12345})$	$(t-i)(t+i)^3$
-2	$-\omega = -e_{12345}$	$(t+i)^4$

Table 1: Square roots of $-\mathbf{1}$ in conformal geometric algebra $Cl(4,1) \cong \mathscr{M}(4,\mathbb{C})$, $d=2$, with characteristic polynomials $\Delta_k(t)$. See [19] for details.

2.3.1 Ordinary square roots of -1 in $Cl(4,1)$ with $k=0$

In the algebra basis of $Cl(4,1)$ there are nine blades which represent ordinary square roots of -1:

$$\begin{aligned} &e_5, \\ &e_{234}, e_{134}, e_{124}, e_{123}, \\ &e_{2345}, e_{1345}, e_{1245}, e_{1235}. \end{aligned} \tag{15}$$

But remembering the work in [17], we know that even if we only look at the subalgebras $Cl(4,0)$ or $Cl(3,1)$, which do not contain the pseudoscalar e_{12345}, and contain therefore only ordinary square roots of -1 for $Cl(4,1)$, we have long parametrized expressions for ordinary square roots of -1. But because of the high dimensionality it may not be easy to compute a complete expression for the whole 16D submanifold of ordinary square roots of -1 in $Cl(4,1)$ by hand.

2.3.2 Exceptional square roots of -1 in $Cl(4,1)$ with $k=1$

In this case we can generalize Table 1 to patches of the twelve dimensional submanifold of exceptional square roots of -1 in $Cl(4,1)$. In the future a complete parametrized expression obtained, e.g., with Clifford for Maple would be very desirable.

We begin with the general expression

$$f_1 = (\frac{1+u}{2}E + \frac{1-u}{2})\omega, \qquad \omega = e_{12345}, \tag{16}$$

where we assume that $E, u \in Cl(4,1)$, $E^2 = u^2 = +1$. This makes the expressions $\frac{1\pm u}{2}$ become idempotents $\left(\frac{1\pm u}{2}\right)^2 = \frac{1\pm u}{2}$. In the following we put forward certain values for E and u which will yield linearly independent patches of the twelve-dimensional submanifold of $\sqrt{-1}$.

- $E = ve_5$, $v \in \mathbb{R}^4$, $v^2 = 1$, $u \in \mathbb{R}^3_{\perp v}$, $u^2 = 1$ gives a 3D × 2D = 6D submanifold. As a concrete example in this submanifold we can e.g. set $v = e_4$, $u = e_1$ and get

$$f_1 = \frac{1}{2}[(1+e_1)e_{45} + 1 - e_1]\omega = \frac{1}{2}[e_{45} + e_{145} + 1 - e_1]\omega. \tag{17}$$

- $E = e_{1234}$, $u \in \mathbb{R}^4$, $u^2 = 1$ gives a 3D submanifold of $\sqrt{-1}$. A concrete example is e.g. $u = e_1$, then

$$f_1 = \frac{1}{2}[(1+e_1)e_{1234} + 1 - e_1]\omega = \frac{1}{2}[e_{1234} + e_{234} + 1 - e_1]\omega. \tag{18}$$

- $E = v$, $v \in \mathbb{R}^4$, $v^2 = 1$, $u = e_{1234}$ gives another 3D submanifold. A concrete example is e.g. $v = e_1$ and gives

$$f_1 = \frac{1}{2}[(1+e_{1234})e_1 + 1 - e_{1234}]\omega = \frac{1}{2}[e_1 - e_{234} + 1 - e_{1234}]\omega. \tag{19}$$

2.3.3 Exceptional square roots of -1 in $Cl(4,1)$ with $k = -1$

This is completely analogous to $k = +1$ by starting with

$$f_{-1} = (\frac{1+u}{2}E - \frac{1-u}{2})\omega, \qquad \omega = e_{12345}. \tag{20}$$

2.3.4 Exceptional square roots of -1 in $Cl(4,1)$ with $k = \pm 2$

The exceptional square roots of -1 are zero-dimensional in this case and therefore uniquely given by

$$f_{\pm 2} = \pm e_{12345}. \tag{21}$$

3 The Clifford Fourier transform

The *general Clifford Fourier transform* (CFT), to be introduced now, can be understood as a generalization of known CFTs [14] to a general real Clifford algebra setting. Most previously known CFTs use in their kernels specific square roots of -1, like bivectors, pseudoscalars, unit pure quaternions, or blades [8]. For an introduction to known CFTs see [4], and for their various applications see [21]. We will *remove all these restrictions* on the square root of -1 used in a CFT[8].

Definition 3.1 (CFT with respect to one square root of -1) *Let $i \in Cl(p,q)$, $i^2 = -1$, be any square root of -1. The general Clifford Fourier transform (CFT) of $f \in L^1(\mathbb{R}^{p,q}; Cl(p,q))$, with respect to i is*

$$\mathscr{F}^i\{f\}(\omega) = \int_{\mathbb{R}^{p,q}} f(\boldsymbol{x})\, e^{-iu(\boldsymbol{x},\omega)} d^n\boldsymbol{x}, \tag{22}$$

where $d^n\boldsymbol{x} = dx_1 \dots dx_n$, $\boldsymbol{x}, \omega \in \mathbb{R}^{p,q}$, and $u : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \mathbb{R}$.

Since square roots of -1 in $Cl(p,q)$ populate *continuous submanifolds* in $Cl(p,q)$, the CFT of Definition 3.1 is generically *steerable* within these manifolds, see (38). In Definition 3.1, the square roots $i \in Cl(p,q)$ of -1 may be from any component of any conjugacy class. The choice of the geometric product in the integrand of (22) is very important. Because only this choice allowed, e.g. in [9], to define and apply a holistic vector field convolution, without loss of information.

4 Properties of the CFT

We now study important properties of the general CFT of Definition 3.1. The proofs in this section may seem deceptively similar to standard proofs of properties of the classical complex Fourier transform. But the inherent non-commutativity of the geometric product of multivectors, makes it necessary to carefully respect the order of factors. Already for the first property of left and right linearity in (23) and (24), respectively, the order of factors leads to crucial differences. We therefore give detailed proofs of all properties.

4.1 Linearity, shift, modulation, dilation, and powers of f, g, steerability

Regarding *left and right linearity* of the general CFT of Definition 3.1 we can establish with the help of Lemma 2.2 that for $h_1, h_2 \in L^1(\mathbb{R}^{p,q}; Cl(p,q))$,

[8] For example, the use of the square root $i = f_1$ of Table 1 would lead to a new type of CFT, which has so far not been studies anywhere in the literature.

and constants $\alpha, \beta \in Cl(p,q)$

$$\mathscr{F}^i\{\alpha h_1 + \beta h_2\}(\omega) = \alpha \mathscr{F}^i\{h_1\}(\omega) + \beta \mathscr{F}^i\{h_2\}(\omega), \tag{23}$$

$$\begin{aligned}\mathscr{F}^i\{h_1\alpha + h_2\beta\}(\omega) = \mathscr{F}^i\{h_1\}(\omega)\alpha_{+i} + \mathscr{F}^{-i}\{h_1\}(\omega)\alpha_{-i} \\ + \mathscr{F}^i\{h_2\}(\omega)\beta_{+i} + \mathscr{F}^{-i}\{h_2\}(\omega)\beta_{-i}\,.\end{aligned} \tag{24}$$

Proof. Based on Lemma 2.2 we have

$$\begin{aligned}&\alpha = \alpha_{+i} + \alpha_{-i}, \qquad \alpha_{+i}i = i\alpha_{+i}, \qquad \alpha_{-i}i = -i\alpha_{-i} \\ \Rightarrow\ & \alpha e^{-iu} = (\alpha_{+i} + \alpha_{-i})e^{-iu} = \alpha_{+i}e^{-iu} + \alpha_{-i}e^{-iu} \\ & \qquad = e^{-iu}\alpha_{+i} + e^{-(-i)u}\alpha_{-i},\end{aligned} \tag{25}$$

and similarly

$$\beta = \beta_{+i} + \beta_{-i}, \qquad \beta e^{-iu} = e^{-iu}\beta_{+i} + e^{-(-i)u}\beta_{-i}. \tag{26}$$

We apply Definition 3.1 and get

$$\begin{aligned}\mathscr{F}^i\{\alpha h_1 + \beta h_2\}(\omega) &= \int_{\mathbb{R}^{p,q}} \{\alpha h_1 + \beta h_2\}\, e^{-iu} d^n\boldsymbol{x} \\ &= \alpha \mathscr{F}^i\{h_1\}(\omega) + \beta \mathscr{F}^i\{h_2\}(\omega).\end{aligned} \tag{27}$$

By inserting (25) and (26) into Definition 3.1 we can further derive

$$\begin{aligned}\mathscr{F}^i\{h_1\alpha + h_2\beta\}(\omega) = \mathscr{F}^i\{h_1\}(\omega)\alpha_{+i} + \mathscr{F}^{-i}\{h_1\}(\omega)\alpha_{-i} \\ + \mathscr{F}^i\{h_2\}(\omega)\beta_{+i} + \mathscr{F}^{-i}\{h_2\}(\omega)\beta_{-i}\,.\end{aligned} \tag{28}$$

□

For *i power factors* in $h_{a,b}(\boldsymbol{x}) = i^a h(\boldsymbol{x}) i^b$, $a, b \in \mathbb{Z}$, we obtain as an application of linearity

$$\mathscr{F}^i\{h_{a,b}\}(\omega) = i^a \mathscr{F}^i\{h\}(\omega) i^b. \tag{29}$$

Regarding the $\boldsymbol{x}$-*shifted* function $h_0(\boldsymbol{x}) = h(\boldsymbol{x} - \boldsymbol{x}_0)$ we obtain with constant $\boldsymbol{x}_0 \in \mathbb{R}^{p,q}$, assuming linearity of $u(\boldsymbol{x}, \omega)$ in its vector space argument $\boldsymbol{x}$,

$$\mathscr{F}^i\{h_0\}(\omega) = \mathscr{F}^i\{h\}(\omega)\, e^{-iu(\boldsymbol{x}_0, \omega)}. \tag{30}$$

Proof. We assume linearity of $u(\boldsymbol{x}, \omega)$ in its vector space argument $\boldsymbol{x}$. Inserting $h_0(\boldsymbol{x}) = h(\boldsymbol{x} - \boldsymbol{x}_0)$ in Definition 3.1 we obtain

$$\begin{aligned}\mathscr{F}^i\{h_0\}(\omega) &= \int_{\mathbb{R}^{p,q}} h(\boldsymbol{x} - \boldsymbol{x}_0)\, e^{-iu(\boldsymbol{x}, \omega)} d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^{p,q}} h(\boldsymbol{y})\, e^{-iu(\boldsymbol{y} + \boldsymbol{x}_0, \omega)} d^n\boldsymbol{y} \\ &= \int_{\mathbb{R}^{p,q}} h(\boldsymbol{y})\, e^{-iu(\boldsymbol{y}, \omega)} e^{-iu(\boldsymbol{x}_0, \omega)} d^n\boldsymbol{y} \\ &= \int_{\mathbb{R}^{p,q}} h(\boldsymbol{y})\, e^{-iu(\boldsymbol{y}, \omega)} d^n\boldsymbol{y}\, e^{-iu(\boldsymbol{x}_0, \omega)} \\ &= \mathscr{F}^i\{h\}(\omega)\, e^{-iu(\boldsymbol{x}_0, \omega)},\end{aligned} \tag{31}$$

where we have substituted $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}_0$ for the second equality, we used the linearity of $u(\boldsymbol{x}, \omega)$ in its vector space argument $\boldsymbol{x}$ for the third equality, and that $e^{-iu(\boldsymbol{x}_0,\omega)}$ is independent of $\boldsymbol{y}$ for the fourth equality.
□

For the purpose of *modulation* we make the special assumption, that the function $u(\boldsymbol{x}, \omega)$ is linear in its frequency argument ω. Then we obtain for $h_m(\boldsymbol{x}) = h(x)\, e^{-iu(\boldsymbol{x},\omega_0)}$, and constant $\omega_0 \in \mathbb{R}^{p,q}$ the modulation formula

$$\mathscr{F}^i\{h_m\}(\omega) = \mathscr{F}^i\{h\}(\omega + \omega_0). \tag{32}$$

Proof. We assume, that the function $u(\boldsymbol{x}, \omega)$ is linear in its frequency argument ω. Inserting $h_m(\boldsymbol{x}) = h(x)\, e^{-iu(\boldsymbol{x},\omega_0)}$ in Definition 3.1 we obtain

$$\begin{aligned}
\mathscr{F}^i\{h_m\}(\omega) &= \int_{\mathbb{R}^{p,q}} h_m(\boldsymbol{x})\, e^{-iu(\boldsymbol{x},\omega)} d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^{p,q}} h(x)\, e^{-iu(\boldsymbol{x},\omega_0)}\, e^{-iu(\boldsymbol{x},\omega)} d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^{p,q}} h(x)\, e^{-iu(\boldsymbol{x},\omega+\omega_0)} d^n\boldsymbol{x} \\
&= \mathscr{F}^i\{h\}(\omega + \omega_0),
\end{aligned} \tag{33}$$

where we used the linearity of $u(\boldsymbol{x}, \omega)$ in its frequency argument ω for the third equality.
□

Regarding *dilations*, we make the special assumption, that for constants $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$, and $\boldsymbol{x}' = \sum_{k=1}^n a_k x^k \boldsymbol{e}_k$, we have $u(\boldsymbol{x}', \omega) = u(\boldsymbol{x}, \omega')$, with $\omega' = \sum_{k=1}^n a_k \omega^k \boldsymbol{e}_k$. We then obtain for $h_d(\boldsymbol{x}) = h(\boldsymbol{x}')$ that

$$\mathscr{F}^i\{h_d\}(\omega) = \frac{1}{|a_1 \ldots a_n|} \mathscr{F}^i\{h\}(\omega_d), \qquad \omega_d = \sum_{k=1}^n \frac{1}{a_k} \omega^k \boldsymbol{e}_k. \tag{34}$$

For $a_1 = \ldots = a_n = a \in \mathbb{R} \setminus \{0\}$ this simplifies under the same special assumption to

$$\mathscr{F}^i\{h_d\}(\omega) = \frac{1}{|a|^n} \mathscr{F}^i\{h\}(\frac{1}{a}\omega). \tag{35}$$

Note, that the above assumption would, e.g., be fulfilled for $u(\boldsymbol{x}, \omega) = \boldsymbol{x} * \widetilde{\omega} = \sum_{k=1}^n x^k \omega^k = \sum_{k=1}^n x_k \omega_k$.
Proof. We assume for constants $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$, and $\boldsymbol{x}' = \sum_{k=1}^n a_k x^k \boldsymbol{e}_k$, that we have $u(\boldsymbol{x}', \omega) = u(\boldsymbol{x}, \omega')$, with $\omega' = \sum_{k=1}^n a_k \omega^k \boldsymbol{e}_k$. Inserting $h_d(\boldsymbol{x}) =$

$h(\boldsymbol{x}')$ in Definition 3.1 we obtain

$$\begin{aligned}
\mathscr{F}^i\{h_d\}(\omega) &= \int_{\mathbb{R}^{p,q}} h_d(\boldsymbol{x})\, e^{-iu(\boldsymbol{x},\omega)} d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^{p,q}} h(\boldsymbol{x}')\, e^{-iu(\boldsymbol{x},\omega)} d^n\boldsymbol{x} \\
&= \frac{1}{|a_1 \ldots a_n|} \int_{\mathbb{R}^{p,q}} h(\boldsymbol{y})\, e^{-iu(\boldsymbol{y}',\omega)} d^n\boldsymbol{y} \\
&= \frac{1}{|a_1 \ldots a_n|} \int_{\mathbb{R}^{p,q}} h(\boldsymbol{y})\, e^{-iu(\boldsymbol{y},\omega_d)} d^n\boldsymbol{y} \\
&= \frac{1}{|a_1 \ldots a_n|} \mathscr{F}^i\{h\}(\omega_d),
\end{aligned} \tag{36}$$

where we substituted $\boldsymbol{y} = \boldsymbol{x}' = \sum_{k=1}^n a_k x^k \boldsymbol{e}_k$ and $\boldsymbol{x} = \sum_{k=1}^n \frac{1}{a_k} y^k \boldsymbol{e}_k = \boldsymbol{y}'$ for the third equality. Note that in this step each negative $a_k < 0, 1 \leq k \leq n$, leads to a factor $\frac{1}{|a_k|}$, because the negative sign is absorbed by interchanging the resulting integration boundaries $\{+\infty, -\infty\}$ to $\{-\infty, +\infty\}$. For the fourth equality we applied the assumption $u(\boldsymbol{y}', \omega) = u(\boldsymbol{y}, \omega')$, and defined $\omega_d = \omega' = \sum_{k=1}^n a_k \omega^k \boldsymbol{e}_k$.
□

Within the same conjugacy class of square roots of -1 the CFTs of Definition 3.1 are related by the following equation, and therefore steerable. Let $i, i' \in Cl(p,q)$ be any two square roots of -1 in the same conjugacy class, i.e. $i' = a^{-1}ia$, $a \in Cl(p,q)$, a being invertible. As a consequence of this relationship we also have

$$e^{-i'u} = a^{-1}e^{-iu}a, \ \forall u \in \mathbb{R}. \tag{37}$$

This in turn leads to the following *steerability relationship* of all CFTs with square roots of -1 from the same conjugacy class:

$$\mathscr{F}^{i'}\{h\}(\omega) = \mathscr{F}^i\{ha^{-1}\}(\omega)a, \tag{38}$$

where ha^{-1} means to multiply the signal function h by the constant multivector $a^{-1} \in Cl(p,q)$.

4.2 CFT inversion, moments, derivatives, Plancherel, Parseval

For establishing an inversion formula, moment and derivative properties, Plancherel and Parseval identities, certain *assumptions* about the phase function $u(\boldsymbol{x}, \omega)$ need to be made. In principle these assumptions could be made based on the desired properties of the resulting CFT. One possibility is, e.g., to assume

$$u(\boldsymbol{x}, \omega) = \boldsymbol{x} * \widetilde{\omega} = \sum_{l=1}^n x^l \omega^l = \sum_{l=1}^n x_l \omega_l, \tag{39}$$

which will be assumed for the current subsection.

We then get the following *inversion* formula[9]

$$h(\boldsymbol{x}) = \mathscr{F}^i_{-1}\{\mathscr{F}^i\{h\}\}(\boldsymbol{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{p,q}} \mathscr{F}^i\{h\}(\boldsymbol{\omega})\, e^{iu(\boldsymbol{x},\omega)} d^n\boldsymbol{\omega}, \tag{40}$$

where $d^n\omega = d\omega_1 \dots d\omega_n$, $\boldsymbol{x}, \omega \in \mathbb{R}^{p,q}$. For the existence of (40) we need $\mathscr{F}^i\{h\} \in L^1(\mathbb{R}^{p,q};\; Cl(p,q))$.

Proof. By direct computation we find

$$\begin{aligned}
&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^{p,q}} \mathscr{F}^i\{h\}(\boldsymbol{\omega})\, e^{iu(\boldsymbol{x},\omega)} d^n\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{p,q}}\int_{\mathbb{R}^{p,q}} h(\boldsymbol{y})\, e^{-iu(\boldsymbol{y},\omega)} e^{iu(\boldsymbol{x},\omega)} d^n\boldsymbol{y} d^n\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{p,q}}\int_{\mathbb{R}^{p,q}} h(\boldsymbol{y})\, e^{iu(\boldsymbol{x}-\boldsymbol{y},\omega)} d^n\boldsymbol{\omega} d^n\boldsymbol{y} \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{p,q}}\int_{\mathbb{R}^{p,q}} h(\boldsymbol{y})\, e^{i\sum_{m=1}^n (x_m-y_m)\omega_m} d^n\boldsymbol{\omega} d^n\boldsymbol{y} \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{p,q}}\int_{\mathbb{R}^{p,q}} h(\boldsymbol{y}) \prod_{m=1}^n e^{i(x_m-y_m)\omega_m} d^n\boldsymbol{\omega} d^n\boldsymbol{y} \\
&= \int_{\mathbb{R}^{p,q}} h(\boldsymbol{y}) \prod_{m=1}^n \delta(x_m-y_m) d^n\boldsymbol{y} \\
&= h(\boldsymbol{x}),
\end{aligned} \tag{41}$$

where we have inserted Definition 3.1 for the first equality, used the linearity of u according to (39) for the second equality, as well as inserted (39) for the third equality, and that $\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x_m-y_m)\omega_m} d\omega_m = \delta(x_m-y_m)$, $1 \leq m \leq n$, for the fifth equality.

□

Additionally, we get the transformation law for *partial derivatives* $h'_l(\boldsymbol{x}) = \partial_{x_l} h(\boldsymbol{x})$, $1 \leq l \leq n$, for h piecewise smooth and integrable, and $h, h'_l \in L^1(\mathbb{R}^{p,q};\; Cl(p,q))$ as

$$\mathscr{F}^i\{h'_l\}(\omega) = \omega_l \mathscr{F}^i\{h\}(\omega) i, \quad \text{for } 1 \leq l \leq n. \tag{42}$$

[9]Note, that we show the inversion symbol -1 as lower index in $\mathscr{F}^i_{-1}$, in order to avoid a possible confusion by using two upper indice. The inversion could also be written with the help of the CFT itself as $\mathscr{F}^i_{-1} = \frac{1}{(2\pi)^n}\mathscr{F}^{-i}$.

Proof. We have

$$\begin{aligned}
\mathscr{F}^i\{h_l'\}(\omega) &= \int_{\mathbb{R}^{p,q}} h_l'(\boldsymbol{x})\, e^{-iu(\boldsymbol{y},\omega)} d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^{p,q}} \partial_{x_l} h(\boldsymbol{x})\, e^{-iu(\boldsymbol{y},\omega)} d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^{p,q}} \partial_{x_l} h(\boldsymbol{x})\, e^{-i\sum_{l=1}^n x_l\omega_l} d^n\boldsymbol{x} \\
&= -\int_{\mathbb{R}^{p,q}} h(\boldsymbol{x})\, \partial_{x_l}\left(e^{-i\sum_{l=1}^n x_l\omega_l}\right) d^n\boldsymbol{x} \\
&= -\int_{\mathbb{R}^{p,q}} h(\boldsymbol{x})\, e^{-i\sum_{l=1}^n x_l\omega_l} d^n\boldsymbol{x}(-i\omega_l) \\
&= \omega_l \mathscr{F}^i\{h\}(\omega) i, \qquad (43)
\end{aligned}$$

where we inserted u of (39) for the third equality and performed integration by parts for the fourth equality.

□

The *vector derivative* of $h \in L^1(\mathbb{R}^{p,q};\ Cl(p,q))$ with $h_l' \in L^1(\mathbb{R}^{p,q};\ Cl(p,q))$ gives therefore due to the linearity (23) of the CFT integral

$$\mathscr{F}^i\{\nabla h\}(\omega) = \mathscr{F}^i\{\sum_{l=1}^n e^l h_l'\}(\omega) = \omega \mathscr{F}^i\{h\}(\omega) i. \qquad (44)$$

For the transformation of the *spatial moments* with $h_l(\boldsymbol{x}) = x_l h(\boldsymbol{x})$, $1 \leq l \leq n$, $h, h_l \in L^1(\mathbb{R}^{p,q};\ Cl(p,q))$, we obtain

$$\mathscr{F}^i\{h_l\}(\omega) = \partial_{\omega_l} \mathscr{F}^i\{h\}(\omega) i. \qquad (45)$$

Proof. We compute

$$\begin{aligned}
-h_l(\boldsymbol{x}) i = h(\boldsymbol{x})(-ix_l) &= \mathscr{F}^i_{-1}\{\mathscr{F}^i\{h\}\}(\boldsymbol{x})(-ix_l) \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \mathscr{F}^i\{h\}(\omega)\, e^{iu(\boldsymbol{x},\omega)} d^n\omega(-ix_l) \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \mathscr{F}^i\{h\}(\omega)\, e^{i\sum_{l=1}^n x_l\omega_l}(-ix_l) d^n\omega \\
&= -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \mathscr{F}^i\{h\}(\omega)\, \partial_{\omega_l}\left(e^{i\sum_{l=1}^n x_l\omega_l}\right) d^n\omega \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \left[\partial_{\omega_l}\mathscr{F}^i\{h\}(\omega)\right] e^{i\sum_{l=1}^n x_l\omega_l} d^n\omega \\
&= \mathscr{F}^i_{-1}\left[\partial_{\omega_l}\mathscr{F}^i\{h\}\right](\boldsymbol{x}), \qquad (46)
\end{aligned}$$

where we used the inversion formula (40) for the second equality, integration by parts for the sixth equality, and (40) again for the seventh equality. Moreover, by applying the CFT $\mathscr{F}^i$ to both sides of (46) we finally obtain

$$\mathscr{F}^i\{h_l(-i)\}(\omega) = \partial_{\omega_l}\mathscr{F}^i\{h\}(\omega) \;\Leftrightarrow\; \mathscr{F}^i\{h_l\}(\omega) = \partial_{\omega_l}\mathscr{F}^i\{h\}(\omega) i, \quad (47)$$

because $\mathscr{F}^i\{h_l(-i)\} = \mathscr{F}^i\{h_l\}(-i)$. Note that in (47) the notation $(-i)$ indicates a constant right side multivector factor and not an argument of the function h_l.
□

For the *spatial vector moment* we obtain due to the linearity (23) of the CFT integral

$$\mathscr{F}^i\{\boldsymbol{x}h\}(\boldsymbol{\omega}) = \mathscr{F}^i\{\sum_{l=1}^{n} e^l x_l h\}(\boldsymbol{\omega}) = \nabla_{\boldsymbol{\omega}} \mathscr{F}^i\{h\}(\boldsymbol{\omega})i, \tag{48}$$

Note that for $Cl(p,q) \cong \mathscr{M}(2d,\mathbb{C})$ or $\mathscr{M}(d,\mathbb{H})$ or $\mathscr{M}(d,\mathbb{H}^2)$, or for i being a blade in $Cl(p,q) \cong \mathscr{M}(2d,\mathbb{R})$ or $\mathscr{M}(2d,\mathbb{R}^2)$, we have $\widetilde{i} = -i$. We assume this for the CFT $\mathscr{F}^i$ in the following Plancherel and Parseval identities.

For the functions $h_1, h_2, h \in L^2(\mathbb{R}^{p,q};\ Cl(p,q))$ we obtain the *Plancherel* identity

$$\langle h_1, h_2 \rangle = \frac{1}{(2\pi)^n} \langle \mathscr{F}^i\{h_1\}, \mathscr{F}^i\{h_2\} \rangle, \tag{49}$$

as well as the *Parseval* identity

$$\|h\| = \frac{1}{(2\pi)^{n/2}} \left\| \mathscr{F}^i\{h\} \right\|. \tag{50}$$

Proof. We only need to proof the Plancherel identity, because the Parseval identity follows from it by setting $h_1 = h_2 = h$ and by taking the square root on both sides. Assume that $\widetilde{i} = -i$. We abbreviate $\int = \int_{\mathbb{R}^{p,q}}$, and compute

$$\begin{aligned}
&\langle \mathscr{F}^i\{h_1\}, \mathscr{F}^i\{h_2\} \rangle \\
&= \int \langle \mathscr{F}^i\{h_1\}(\boldsymbol{\omega}) [\mathscr{F}^i\{h_2\}(\boldsymbol{\omega})]^{\sim} \rangle d^n\boldsymbol{\omega} \\
&= \int\int\int \langle h_1(\boldsymbol{x})\, e^{-iu(\boldsymbol{x},\boldsymbol{\omega})} d^n\boldsymbol{x} [h_2(\boldsymbol{y})\, e^{-iu(\boldsymbol{y},\boldsymbol{\omega})} d^n\boldsymbol{y}]^{\sim} \rangle d^n\boldsymbol{\omega} \\
&= \int\int\int \langle h_1(\boldsymbol{x})\, e^{-iu(\boldsymbol{x},\boldsymbol{\omega})} e^{-\widetilde{i}u(\boldsymbol{y},\boldsymbol{\omega})} \widetilde{h_2(\boldsymbol{y})}\, d^n\boldsymbol{y} \rangle d^n\boldsymbol{x} d^n\boldsymbol{\omega} \\
&= \int\int\int \langle h_1(\boldsymbol{x})\, e^{-iu(\boldsymbol{x},\boldsymbol{\omega})} e^{iu(\boldsymbol{y},\boldsymbol{\omega})} \widetilde{h_2(\boldsymbol{y})}\, d^n\boldsymbol{\omega}\, d^n\boldsymbol{y} \rangle d^n\boldsymbol{x} \\
&= \int\int\int \langle h_1(\boldsymbol{x})\, e^{-iu(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{\omega})} \widetilde{h_2(\boldsymbol{y})}\, d^n\boldsymbol{\omega}\, d^n\boldsymbol{y} \rangle d^n\boldsymbol{x} \\
&= (2\pi)^n \int\int\int \langle h_1(\boldsymbol{x}) \frac{e^{-i\sum_{m=1}^{n}(x_m - y_m)\omega_m}}{(2\pi)^n} \widetilde{h_2(\boldsymbol{y})}\, d^n\boldsymbol{\omega}\, d^n\boldsymbol{y} \rangle d^n\boldsymbol{x} \\
&= (2\pi)^n \int\int \langle h_1(\boldsymbol{x}) \prod_{m=1}^{n} \delta(x_m - y_m) \widetilde{h_2(\boldsymbol{y})}\, d^n\boldsymbol{y} \rangle d^n\boldsymbol{x} \\
&= (2\pi)^n \int \langle h_1(\boldsymbol{x}) \widetilde{h_2(\boldsymbol{x})} \rangle d^n\boldsymbol{x} \\
&= (2\pi)^n \langle h_1, h_2 \rangle,
\end{aligned} \tag{51}$$

where we inserted (10) for the first equality, the Definition 3.1 of the CFT $\mathscr{F}^i$ for the second equality, applied the principal reverse for the third equality, and the symmetry of the scalar product and that $\widetilde{i} = -i$ for the fourth equality, the linearity of u according to (39) for the fifth equality, inserted the explicit forms of u of (39) for the sixth equality, and that $\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x_m-y_m)\omega_m} d\omega_m = \delta(x_m - y_m)$, $1 \leq m \leq n$, for the seventh equality, and again (10) for the last equality. Division of both sides with $(2\pi)^n$ finally gives the Plancherel identity (49).
□

4.3 Convolution

We define the *convolution* of two multivector signals $a, b \in L^1(R^{p,q}; Cl(p,q))$ as

$$(a \star b)(\boldsymbol{x}) = \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) b(\boldsymbol{x} - \boldsymbol{y}) d^n\boldsymbol{y}. \tag{52}$$

We assume that the function u is linear with respect to its first argument. The *CFT of the convolution* (52) can then be expressed as

$$\mathscr{F}^i\{a \star b\}(\omega) = \mathscr{F}^{-i}\{a\}(\omega)\mathscr{F}^i\{b_{-i}\}(\omega) + \mathscr{F}^i\{a\}(\omega)\mathscr{F}^i\{b_{+i}\}(\omega). \tag{53}$$

Proof. We now proof (53).

$$\begin{aligned}
&\mathscr{F}^i\{a \star b\}(\omega) \\
&= \int_{\mathbb{R}^{p,q}} (a \star b)(\boldsymbol{x})\, e^{-iu(\boldsymbol{x},\omega)} d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) b(\boldsymbol{x}-\boldsymbol{y}) d^n\boldsymbol{y}\, e^{-iu(\boldsymbol{x},\omega)} d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) b(\boldsymbol{z}) d^n\boldsymbol{y}\, e^{-iu(\boldsymbol{y}+\boldsymbol{z},\omega)} d^n\boldsymbol{z} \\
&= \int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) b(\boldsymbol{z}) d^n\boldsymbol{y}\, e^{-iu(\boldsymbol{y},\omega)} e^{-iu(\boldsymbol{z},\omega)} d^n\boldsymbol{z} \\
&= \int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) [b_{+i}(\boldsymbol{z}) + b_{-i}(\boldsymbol{z})] d^n\boldsymbol{y}\, e^{-iu(\boldsymbol{y},\omega)} e^{-iu(\boldsymbol{z},\omega)} d^n\boldsymbol{z},
\end{aligned} \tag{54}$$

where we used the substitution $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$, $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$. To simplify (54) we expand the inner expression of the integrand to obtain

$$\begin{aligned}
&a(\boldsymbol{y})[b_{+i}(\boldsymbol{z}) + b_{-i}(\boldsymbol{z})]\, e^{-iu(\boldsymbol{y},\omega)} \\
&= a(\boldsymbol{y})[e^{-iu(\boldsymbol{y},\omega)} b_{+i}(\boldsymbol{z}) + e^{+iu(\boldsymbol{y},\omega)} b_{-i}(\boldsymbol{z})] \\
&= a(\boldsymbol{y}) e^{-iu(\boldsymbol{y},\omega)} b_{+i}(\boldsymbol{z}) + a(\boldsymbol{y}) e^{+iu(\boldsymbol{y},\omega)} b_{-i}(\boldsymbol{z}).
\end{aligned} \tag{55}$$

Reinserting (55) into (54) we get

$$\begin{aligned}
&\mathscr{F}^{i}\{a \star b\}(\omega) \\
&= \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) e^{-iu(\boldsymbol{y},\omega)} d^n\boldsymbol{y} \int_{\mathbb{R}^{p,q}} b_{+i}(\boldsymbol{z}) e^{-iu(\boldsymbol{z},\omega)} d^n\boldsymbol{z} \\
&\qquad + \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) e^{+iu(\boldsymbol{y},\omega)} d^n\boldsymbol{y} \int_{\mathbb{R}^{p,q}} b_{-i}(\boldsymbol{z}) e^{-iu(\boldsymbol{z},\omega)} d^n\boldsymbol{z} \\
&= \mathscr{F}^{i}\{a\}(\omega)\mathscr{F}^{i}\{b_{+i}\}(\omega) + \mathscr{F}^{-i}\{a\}(\omega)\mathscr{F}^{i}\{b_{-i}\}(\omega).
\end{aligned} \tag{56}$$

□

We point out that the above convolution theorem of equation (53) is a special case of a more general convolution theorem recently derived in [7].

5 Conclusions

We have established a comprehensive *new mathematical framework* for the investigation and application of Clifford Fourier transforms (CFTs) together with *new properties*. Our new CFTs form a more general class of CFTs, subsuming and generalizing previous results. We have applied new results on square roots of -1 in Clifford algebras to fully general construct CFTs, with a general square root of -1 in real Clifford algebras $Cl(p,q)$. The new CFTs are *fully steerable* within the continuous Clifford algebra submanifolds of square roots of -1. We have thus left the terra cognita of familiar transforms to outline the vast array of possible CFTs in $Cl(p,q)$.

We first reviewed the recent results on *square roots of* -1 in Clifford algebras. Next, we defined the central notion of the *Clifford Fourier transform* with respect to any square root of -1 in real Clifford algebras. Finally, we investigated important *properties* of these new CFTs: linearity, shift, modulation, dilation, moments, inversion, partial and vector derivatives, Plancherel and Parseval formulas, as well as a convolution theorem.

Regarding numerical implementations, usually 2^n complex Fourier transformations (FTs) are sufficient. In some cases this can be reduced to $2^{(n-1)}$ complex FTs, e.g., when the square root of -1 is a pseudoscalar. Further algebraic studies may widen the class of CFTs, where $2^{(n-1)}$ complex FTs are sufficient. Numerical implementation is then possible with 2^n (or $2^{(n-1)}$) discrete complex FTs, which can also be fast Fourier transforms (FFTs), leading to fast CFT implementations.

A well-known example of a CFT is the quaternion FT (QFT) [5, 6, 10, 11, 15, 18, 26], which is particularly used in applications to partial differential systems, color image processing, filtering, disparity estimation (two images differ by local translations), and texture segmentation. Another example is the spacetime FT, which leads to a multivector wave packet analysis of spacetime signals (e.g. electro-magnetic signals), applicable even to relativistic signals [15, 16].

Depending on the choice of the phase functions $u(\boldsymbol{x}, \boldsymbol{\omega})$ the multivector basis coefficient functions of the CFT result carry information on the symmetry of the signal, similar to the special case of the QFT [5].

The convolution theorem allows to design and apply multivector valued filters to multivector valued signals.

Research on the application of CFTs with general square roots of -1 is ongoing. Further results, including special choices of square roots of -1 for certain applications will be published elsewhere.

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