Uncertainty Principle for Clifford Geometric Algebras $Cl_{n,0}$, $n = 3 \pmod 4$ based on Clifford Fourier Transform

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Abstract. First, the basic concepts of the multivector functions, vector differential and vector derivative in geometric algebra are introduced. Second, we define a generalized real Fourier transform on Clifford multivector-valued functions ($f : \mathbb{R}^n \to Cl_{n,0},\ n = 3 \pmod 4$). Third, we introduce a set of important properties of the Clifford Fourier transform on $Cl_{n,0},\ n = 3 \pmod 4$ such as differentiation properties, and the Plancherel theorem. Finally, we apply the Clifford Fourier transform properties for proving a *directional* uncertainty principle for $Cl_{n,0},\ n = 3 \pmod 4$ multivector functions.

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1. Introduction

In the field of applied mathematics the Fourier transform has developed into an important tool. It is a powerful method for solving partial differential equations. The Fourier transform provides also a technique for signal analysis where the signal from the original domain is transformed to the spectral or frequency domain. In the frequency domain many characteristics of the signal are revealed. With these facts in mind, we extend the Fourier transform in geometric algebra.

Brackx et al. [1] extended the Fourier transform to multivector valued function-distributions in $Cl_{0,n}$ with compact support. They also showed some properties of this generalized Fourier transform. A related applied approach for hypercomplex Clifford Fourier transformations in $Cl_{0,n}$ was followed by Bülow et. al. [2].

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By extending the classical trigonometric exponential function $\exp(j\,\boldsymbol{x}*\boldsymbol{\xi})$ (where $*$ denotes the scalar product of $\boldsymbol{x}\in\mathbb{R}^m$ with $\boldsymbol{\xi}\in\mathbb{R}^m$, j the imaginary unit) in [3, 4], McIntosh et. al. generalized the classical Fourier transform. Applied to a function of m real variables this generalized Fourier transform is holomorphic in m complex variables and its inverse is *monogenic* in $m+1$ real variables, thereby effectively extending the function of m real variables to a monogenic function of $m+1$ real variables (with values in a *complex* Clifford algebra). This generalization has significant applications to harmonic analysis, especially to singular integrals on surfaces in $\mathbb{R}^{m+1}$. Based on this approach Kou and Qian obtained a Clifford Payley-Wigner theorem and derived Shannon interpolation of band-limitted functions using the monogenic sinc function [5, and references therein]. The Clifford Payley-Wigner theorem also allows to derive left-entire (left-monogenic in the whole $\mathbb{R}^{m+1}$) functions from square integrable functions on $\mathbb{R}^m$ with compact support.

In this paper we adopt and expand[1] to $\mathcal{G}_n,\ n=3\,(\mathrm{mod}\,4)$ the generalization of the Fourier transform in Clifford geometric algebra $\mathcal{G}_3$ recently suggested by Ebling and Scheuermann [7]. We introduce detailed properties of the real[2] Clifford geometric algebra Fourier transform (CFT), which we subsequently use to define and prove a general directional uncertainty principle for $\mathcal{G}_n$ multivector functions.

2. Clifford's Geometric Algebra $\mathcal{G}_n$ of $\mathbb{R}^n$

Let us consider now and in the following an orthonormal vector basis $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_n\}$ of the real n-dimensional Euclidean vector space $\mathbb{R}^n$ with $n=3\,(\mathrm{mod}\,4)$. Each basis vector has unit square, i.e. $\boldsymbol{e}_k^2=1,\ 1\le k\le n$. The geometric algebra over $\mathbb{R}^n$ denoted by $\mathcal{G}_n$ then has the graded 2^n-dimensional basis

$$\{1,\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_n,\boldsymbol{e}_{12},\boldsymbol{e}_{31},\boldsymbol{e}_{23},\ldots,i_n=\boldsymbol{e}_1\boldsymbol{e}_2\ldots\boldsymbol{e}_n\}. \tag{2.1}$$

For the simplest case of $n=3$ the basis reduces to

$$\begin{aligned} & \{1,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_{23},\boldsymbol{e}_{31},\boldsymbol{e}_{12},i_3=\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\} \\ = \; & \{1,i_3,\boldsymbol{e}_1,i_3\boldsymbol{e}_1=\boldsymbol{e}_{23},\boldsymbol{e}_2,i_3\boldsymbol{e}_2=\boldsymbol{e}_{31},\boldsymbol{e}_3=\boldsymbol{e}_2\boldsymbol{e}_{23},\boldsymbol{e}_{12}=\boldsymbol{e}_{13}\boldsymbol{e}_{23}\} \\ \doteq \; & \{1,i_3,\boldsymbol{e}_{23},i_3\boldsymbol{e}_{23}=-\boldsymbol{e}_1,\boldsymbol{e}_{31},i_3\boldsymbol{e}_{31}=-\boldsymbol{e}_2,\boldsymbol{e}_{12}=\boldsymbol{e}_{13}\boldsymbol{e}_{23},i_3\boldsymbol{e}_{12}=-\boldsymbol{e}_3\}. \end{aligned} \tag{2.2}$$

Equation (2.2) exemplifies for $n=3$ the general isomorphisms

$$\mathcal{G}_n\approx\mathcal{G}_{n-1}\times\mathbb{C}\approx\mathcal{G}_{0,n-1}\times\mathbb{C}, \tag{2.3}$$

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[1]For further details and proofs in the case of $n=3$ compare [6]. In the geometric algebra literature [8] instead of the mathematical notation $Cl_{p,q}$ the notation $\mathcal{G}_{p,q}$ is widely in use. It is convention to abbreviate $\mathcal{G}_{n,0}$ to $\mathcal{G}_n$.

[2]The meaning of *real* in this context is, that we use the n-dimensional volume element $i_n=\boldsymbol{e}_1\boldsymbol{e}_2\ldots\boldsymbol{e}_n$ of the geometric algebra $\mathcal{G}_n$ over the field of the reals $\mathbb{R}$ to construct the kernel of the Clifford Fourier transformation of definition 4.1. This i_n has a clear geometric interpretation. Note that $i_n^2=-1$ for $n=2,3\,(\mathrm{mod}\,4)$.

which can be exploited to transfer results from a complexified Clifford algebra $\mathcal{G}_{0,n-1} \times \mathbb{C}$ to the real geometric algebra $\mathcal{G}_n$.

The *grade selector* is defined as $\langle M \rangle_k$ for the k-vector part of M, especially $\langle M \rangle = \langle M \rangle_0$. Then M can be expressed as

$$M = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \ldots + \langle M \rangle_n. \tag{2.4}$$

The *reverse* of M is defined by the anti-automorphism

$$\widetilde{M} = \sum_{k=0}^{k=n} (-1)^{k(k-1)/2} \langle M \rangle_k. \tag{2.5}$$

The *square norm* of M is defined by

$$\|M\|^2 = \langle M\widetilde{M} \rangle, \tag{2.6}$$

where

$$\langle M\widetilde{N} \rangle = M * \widetilde{N} = \sum_A \alpha_A \beta_A \tag{2.7}$$

is a real valued (inner) *scalar product* for any M, N in $\mathcal{G}_n$ with $M = \sum_A \alpha_A \boldsymbol{e}_A$ and $N = \sum_A \beta_A \boldsymbol{e}_A$, $A \in \{0, 1, 2, \ldots, n, 12, 31, 23, \ldots, 12\ldots n\}$, $\alpha_A, \beta_A \in \mathbb{R}$, and $\boldsymbol{e}_A$ the basis elements of (2.1). Especially for vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ we get (using the costumary dot)

$$\boldsymbol{a} \cdot \boldsymbol{b} = \langle \boldsymbol{a}\boldsymbol{b} \rangle = \boldsymbol{a} * \boldsymbol{b} = \sum_{A=1}^{n} \alpha_A \beta_A \tag{2.8}$$

As a consequence we obtain the *multivector Cauchy-Schwarz inequality*

$$|\langle M\widetilde{N} \rangle|^2 \le \|M\|^2 \, \|N\|^2 \quad \forall\, M, N \in \mathcal{G}_n. \tag{2.9}$$

3. Multivector Functions, Vector Differential and Vector Derivative

Let $f = f(\boldsymbol{x})$ be a multivector-valued function of a vector variable $\boldsymbol{x}$ in $\mathcal{G}_n$. For an arbitrary vector $\boldsymbol{a}$ we define[3] the *vector differential* in the $\boldsymbol{a}$ direction as

$$\boldsymbol{a} \cdot \nabla f(\boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{f(\boldsymbol{x} + \epsilon \boldsymbol{a}) - f(\boldsymbol{x})}{\epsilon} \tag{3.1}$$

provided this limit exists and is well defined. The basis independent linear *vector derivative* ∇ defined in [8, 9] obeys equation (3.1) for all vectors $\boldsymbol{a}$ and can be expanded as

$$\nabla = \boldsymbol{e}_k \partial_k = \boldsymbol{e}_1 \partial_1 + \boldsymbol{e}_2 \partial_2 + \ldots + \boldsymbol{e}_n \partial_n, \tag{3.2}$$

For use in later sections we state a number of elementary properties of the vector differential and the vector derivative (compare [8, 9])

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[3]Bracket convention: $A \cdot BC = (A \cdot B)C \neq A \cdot (BC)$ and $A \wedge BC = (A \wedge B)C \neq A \wedge (BC)$ for multivectors $A, B, C \in \mathcal{G}_{p,q}$. The vector variable index $\boldsymbol{x}$ of the vector derivative is dropped: $\boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{\nabla}$ and $\boldsymbol{a} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{a} \cdot \boldsymbol{\nabla}$, but not when differentiating with respect to a different vector variable (compare e.g. proposition 3.2).

Proposition 3.1 (Chain rule for $g \circ \lambda, \lambda \in \mathbb{R}$). *For $f(\boldsymbol{x}) = g(\lambda(\boldsymbol{x}))$, $\lambda(\boldsymbol{x}) \in \mathbb{R}$,*

$$\boldsymbol{a} \cdot \nabla f = \{\boldsymbol{a} \cdot \nabla \lambda(\boldsymbol{x})\} \frac{\partial g}{\partial \lambda}\,. \tag{3.3}$$

Proposition 3.2 (Derivative from differential).

$$\nabla f = \nabla_{\boldsymbol{a}} (\boldsymbol{a} \cdot \nabla f)\,. \tag{3.4}$$

Differentiating twice with the vector derivative, we get the differential Laplacian operator ∇^2. We can write $\nabla^2 = \nabla \cdot \nabla + \nabla \wedge \nabla$. But for integrable functions $\nabla \wedge \nabla = 0$. In this case we have $\nabla^2 = \nabla \cdot \nabla$.

The following form of the product rule deviates from [8] insofar as we do not use the perhaps unfamiliar overdot notation of Hestenes and Sobczyk.

Proposition 3.3 (Product rule).

$$\nabla(fg) = (\nabla f)g + \nabla_{\boldsymbol{a}} f(\boldsymbol{a} \cdot \nabla g) = (\nabla f)g + \sum_{k=1}^{n} \boldsymbol{e}_k f(\partial_k g)\,. \tag{3.5}$$

Note that the multivector functions f and g in (3.5) do not necessarily commute.

Proposition 3.4 (Integration by parts).

$$\int_{\mathbb{R}^n} g(\boldsymbol{x})[\boldsymbol{a} \cdot \nabla h(\boldsymbol{x})] d^n\boldsymbol{x} = \left[\int_{\mathbb{R}^{n-1}} g(\boldsymbol{x}) h(\boldsymbol{x}) d^{n-1}\boldsymbol{x}\right]_{a\cdot x=-\infty}^{a\cdot x=\infty} - \int_{\mathbb{R}^n} [\boldsymbol{a} \cdot \nabla g(\boldsymbol{x})] h(\boldsymbol{x}) d^n\boldsymbol{x}. \tag{3.6}$$

Remark 3.5. Proposition 3.4 reduces to the familiar coordinate form, if we insert for $\boldsymbol{a}$ the grade 1 basis vectors $\boldsymbol{e}_k, 1 \leq k \leq n$ of (2.1), because

$$\boldsymbol{e}_k \cdot \nabla = \partial_k \quad \text{and} \quad \boldsymbol{e}_k \cdot \boldsymbol{x} = x_k. \tag{3.7}$$

But since the introduction of a coordinate system is arbitrary, we can conversly always rotate every chosen coordinate vector into the direction of the vector $\boldsymbol{a}$ of proposition 3.4, which shows that the generalized form 3.4 for the integration by parts formula is valid. Proposition 3.4 is used in the proof of the directional uncertainty principle 5.1.

4. Clifford Fourier Transform (CFT)

Definition 4.1. The Clifford Fourier transform (CFT) of $f(\boldsymbol{x})$ is the function $\mathcal{F}\{f\}$: $\mathbb{R}^n \to \mathcal{G}_n$ given by

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x}, \tag{4.1}$$

where we can write $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2 + \ldots + \omega_n\boldsymbol{e}_n$, $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \ldots + x_n\boldsymbol{e}_n$ with $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ the basis vectors of $\mathbb{R}^n$.

Note that

$$d^n\boldsymbol{x} = \frac{d\boldsymbol{x_1} \wedge d\boldsymbol{x_2} \wedge \ldots \wedge d\boldsymbol{x_n}}{i_n} \tag{4.2}$$

TABLE 1. Properties of the Clifford Fourier transform (CFT)

Property	**Multivector Function**	**CFT**
Linearity	$\alpha f(\boldsymbol{x})+\beta\, g(\boldsymbol{x})$	$\alpha\mathcal{F}\{f\}(\boldsymbol{\omega})+\,\beta\mathcal{F}\{g\}(\boldsymbol{\omega})$
Delay	$f(\boldsymbol{x}-\boldsymbol{a})$	$e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{a}}\mathcal{F}\{f\}(\boldsymbol{\omega})$
Shift	$e^{i_n\boldsymbol{\omega}_0\cdot\boldsymbol{x}}f(\boldsymbol{x})$	$\mathcal{F}\{f\}(\boldsymbol{\omega}-\boldsymbol{\omega}_0)$
Scaling	$f(a\boldsymbol{x}),\ a\in\mathbb{R}\setminus\{0\}$	$\frac{1}{\lvert a\rvert^n}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a})$
Convolution	$(f\star g)(\boldsymbol{x})$	$\mathcal{F}\{f\}(\boldsymbol{\omega})\,\mathcal{F}\{g\}(\boldsymbol{\omega})$
Vec. diff.	$\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})$	$i_n\,\boldsymbol{a}\cdot\boldsymbol{\omega}\mathcal{F}\{f\}(\boldsymbol{\omega})$
	$\boldsymbol{a}\cdot\boldsymbol{x}\ f(\boldsymbol{x})$	$i_n\,\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}\ \mathcal{F}\{f\}(\boldsymbol{\omega})$
	$\boldsymbol{x}f(\boldsymbol{x})$	$i_n\,\nabla_{\boldsymbol{\omega}}\ \mathcal{F}\{f\}(\boldsymbol{\omega})$
Vec. deriv.	$\nabla^m f(\boldsymbol{x})$	$(i_n\,\boldsymbol{\omega})^m\mathcal{F}\{f\}(\boldsymbol{\omega})$
Plancherel T.	$\int_{\mathbb{R}^n} f_1(\boldsymbol{x})\widetilde{f_2(\boldsymbol{x})}\,d^n\boldsymbol{x}$	$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathcal{F}\{f_1\}(\boldsymbol{\omega})\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\,d^n\boldsymbol{\omega}$
sc. Parseval T.	$\int_{\mathbb{R}^n}\ \lVert f(\boldsymbol{x})\rVert^2\,d^n\boldsymbol{x}$	$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\ \lVert\mathcal{F}\{f\}(\boldsymbol{\omega})\rVert^2\,d^n\boldsymbol{\omega}$

is scalar valued ($d\boldsymbol{x_k}=dx_k\boldsymbol{e_k}$, $k=1,2,\ldots,n$, no summation). For the dimension $n=3(\mathrm{mod}\,4)$ the pseudoscalar i_n acts like a commutative[4] imaginary unit ($i_n^2=-1$), i.e. i_n commutes with every element of $\mathcal{G}_n$ (it is *central*), and hence the Clifford Fourier kernel $e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}$ will also commute with every element of $\mathcal{G}_n$. We therefore have the isomorphism (2.3) [exemplified for $n=3$ in (2.2)]. And in consequence, we also have an isomorphism between the presented Fourier transform and the classical Fourier transform, which also provides a straighforward strategy for the proofs of the properties of the CFT listed in table 1. An alternative way would be to generalize the proofs for $n=3$ in [6] to $n=3(\mathrm{mod}\,4)$. Due to the isomorphism, the CFT of equation (4.2) can be broken down to a tupel of 2^{n-1} scalar complex Fourier transforms, which also permits for numerical applications to make use of well-established fast Fourier transform algorithms. This has already been exploited for $n=3$ in [7].

Theorem 4.2. *The Clifford Fourier transform $\mathcal{F}\{f\}$ of $f\in L^2(\mathbb{R}^n,\mathcal{G}_n)$, $\int_{\mathbb{R}^n}\lVert f\rVert^2 d^n\boldsymbol{x}<\infty$ is invertible and its inverse is calculated by*

$$\mathcal{F}^{-1}[\mathcal{F}\{f\}](\boldsymbol{x})=f(\boldsymbol{x})=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathcal{F}\{f\}(\boldsymbol{\omega})\,e^{i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\,d^n\boldsymbol{\omega}. \tag{4.3}$$

A number of properties of the CFT are listed in table 1. A related formula for polynomials of the vector derivative (compare line 9) can be found in [4]. The reverse of line 10 and the square norm of line 11 are defined in (2.5) and (2.6), respectively.

[4]It is possible to define the CFT for $n=2(\mathrm{mod}\,4)$ as well, but then care has to be taken of the general non-commutativity of i_n with the elements of $\mathcal{G}_n$.

5. Uncertainty Principle

The uncertainty principle plays an important role in the development and understanding of quantum physics. It is also central for information processing [10].

In quantum physics it states e.g. that particle momentum and position cannot be simultaneously known. The multivector function $f(\boldsymbol{x})$ would represent the spatial part of a separable wave function and its CFT $\mathcal{F}\{f\}(\boldsymbol{\omega})$ the same wave function in momentum space (compare [11, 12, 13]). The variance in space would then be calculated as $(k = 1, 2, 3)$

$$(\Delta x_k)^2 = \int_{\mathbb{R}^3} \langle f(\boldsymbol{x})(\boldsymbol{e}_k \cdot \boldsymbol{x})^2 \tilde{f}(\boldsymbol{x})\rangle \, d^3\boldsymbol{x} = \int_{\mathbb{R}^3} (\boldsymbol{e}_k \cdot \boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x},$$

where it is customary to set without loss of generality the mean value of $\boldsymbol{e}_k \cdot \boldsymbol{x}$ to zero [13]. The variance in momentum space would be calculated as $(l = 1, 2, 3)$

$$\begin{aligned}(\Delta\omega_l)^2 &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \langle \mathcal{F}\{f\}(\boldsymbol{\omega})(\boldsymbol{e}_l \cdot \boldsymbol{\omega})^2 \widetilde{\mathcal{F}}\{f\}(\boldsymbol{\omega})\rangle \, d^3\boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (\boldsymbol{e}_l \cdot \boldsymbol{\omega})^2 \; \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega}.\end{aligned}$$

Again the mean value of $\boldsymbol{e}_l \cdot \boldsymbol{\omega}$ is costumarily set to zero, it merely corresponds to a phase shift [13]. Using our mathematical units, the space-momentum uncertainty relation of quantum mechanics is then expressed by (compare e.g. with (4.9) of [12, page 86])

$$\Delta x_k \Delta\omega_l = \frac{1}{2}\,\delta_{k,l} F, \tag{5.1}$$

where $\delta_{k,l}$ is the usual Kronecker symbol. Note that we have not normalized the squares of the variances by division with $F = \int_{\mathbb{R}^3} \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x}$, therefore the extra factor F on the right side of (5.1). Further explicit examples from image processing can be found in [17].

In general in Fourier analysis such conjugate entities correspond to the variances of a function and its Fourier transform which cannot both be simultaneously sharply localized (e.g. [10, 14]). Material on the classical uncertainty principle for the general case of $L_2(\mathbb{R}^n)$ without the additional condition $\lim_{|x|\to\infty} |x|^2|f(x)| = 0$ can be found in [15] and [16]. Felsberg [17] even notes for two dimensions: *In 2D however, the uncertainty relation is still an open problem. In* [18] *it is stated that there is no straightforward formulation for the 2D uncertainty relation.*

From the view point of geometric algebra an uncertainty principle gives us information about how the variance of a multivector valued function and the variance of its Clifford Fourier transform are related. We can shed the restriction to the parallel $(k = l)$ and orthogonal $(k \neq l)$ cases of (5.1) by looking at the $\boldsymbol{x} \in \mathbb{R}^n$ variance in an arbitrary but fixed direction $\boldsymbol{a} \in \mathbb{R}^n$ and at the $\boldsymbol{\omega} \in \mathbb{R}^n$ variance in an arbitrary but fixed direction $\boldsymbol{b} \in \mathbb{R}^n$. This leads to the following theorem.

Theorem 5.1 (Directional uncertainty principle). *Let f be a multivector valued function in $\mathcal{G}_n$, $n = 3 \,(\mathrm{mod}\, 4)$, which has the Clifford Fourier transform $\mathcal{F}\{f\}(\boldsymbol{\omega})$.*

Assume $\int_{\mathbb{R}^n} \|f(\boldsymbol{x})\|^2 \, d^n\boldsymbol{x} = F < \infty$*, then the following inequality holds for arbitrary constant vectors* $\boldsymbol{a}$*,* $\boldsymbol{b}$*:*

$$\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2 \, d^n\boldsymbol{x} \, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \, \geq \, (\boldsymbol{a}\cdot\boldsymbol{b})^2 \, \frac{1}{4} F^2 \quad (5.2)$$

Proof Applying the results stated so far we have[5]

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \, \|f(\boldsymbol{x})\|^2 \, d^n\boldsymbol{x} \, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \\
&\overset{\text{table 1, line 6}}{=} \int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \, \|f(\boldsymbol{x})\|^2 \, d^n\boldsymbol{x} \, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \|\mathcal{F}\{\boldsymbol{b}\cdot\nabla f\}(\boldsymbol{\omega})\|^2 \, d^n\boldsymbol{\omega} \\
&\overset{\text{sc. Parseval}}{=} \int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \, \|f(\boldsymbol{x})\|^2 \, d^n\boldsymbol{x} \int_{\mathbb{R}^n} \|\boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\|^2 \, d^n\boldsymbol{x} \\
&\overset{\text{footnote 5}}{\geq} \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x} \, \|f(\boldsymbol{x})\| \, \|\boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\| \, d^n\boldsymbol{x}\right)^2 \\
&\overset{(2.9)}{\geq} \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x} |\langle \widetilde{f(\boldsymbol{x})} \, \boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\rangle| \, d^n\boldsymbol{x}\right)^2 \\
&\geq \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x} \langle \widetilde{f(\boldsymbol{x})} \, \boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\rangle \, d^n\boldsymbol{x}\right)^2.
\end{aligned}$$

Because of (2.6) and (2.7)

$$(\boldsymbol{b}\cdot\nabla)\|f\|^2 = 2\langle \widetilde{f} \, (\boldsymbol{b}\cdot\nabla) f\rangle, \quad (5.3)$$

we furthermore obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \, \|f(\boldsymbol{x})\|^2 \, d^n\boldsymbol{x} \, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \\
&\geq \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x} \, \frac{1}{2} \, (\boldsymbol{b}\cdot\nabla\|f\|^2) \, d^n\boldsymbol{x}\right)^2 \\
&\overset{\text{Prop. 3.4}}{=} \frac{1}{4}\left(\left[\int_{\mathbb{R}^{n-1}} \boldsymbol{a}\cdot\boldsymbol{x} \|f(\boldsymbol{x})\|^2 d^{n-1}\boldsymbol{x}\right]_{b\cdot x=-\infty}^{b\cdot x=\infty} \right. \\
&\qquad \left. - \int_{\mathbb{R}^n} [(\boldsymbol{b}\cdot\nabla)(\boldsymbol{a}\cdot\boldsymbol{x})] \, \|f(\boldsymbol{x})\|^2 \, d^n\boldsymbol{x}\right)^2 \\
&= \frac{1}{4}\left(0 - \boldsymbol{a}\cdot\boldsymbol{b} \int_{\mathbb{R}^n} \|f(\boldsymbol{x})\|^2 \, d^n\boldsymbol{x})\right)^2 \\
&= (\boldsymbol{a}\cdot\boldsymbol{b})^2 \, \frac{1}{4} \, F^2.
\end{aligned}$$

Choosing $\boldsymbol{b} = \pm\boldsymbol{a}$, i.e. $\boldsymbol{b} \parallel \boldsymbol{a}$, with $\boldsymbol{a}^2 = 1$ we get from theorem 5.1 the **uncertainty principle** for parallel variance directions [compare with case $k = l$ of

[5] $\phi, \psi : \mathbb{R}^n \to \mathbb{C}, \quad \int_{\mathbb{R}^n} |\phi(x)|^2 d^n x \int_{\mathbb{R}^n} |\psi(x)|^2 d^n x \geq (\int_{\mathbb{R}^n} \phi(x)\bar{\psi}(x) \, d^n x)^2$

(5.1)]:

$$\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\, \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \;\geq\; \frac{1}{4}\, F^2. \tag{5.4}$$

Remark 5.2. In (5.4) equality holds for *Gaussian* multivector valued functions

$$f(\boldsymbol{x}) = C_0\, e^{-k\,\boldsymbol{x}^2} \tag{5.5}$$

where $C_0 \in \mathcal{G}_n$ is an arbitrary but constant multivector, $0 < k \in \mathbb{R}$. The proof for this follows from the observation that we have for the f of (5.5)

$$-2k\; \boldsymbol{a}\cdot\boldsymbol{x}\; f = \boldsymbol{a}\cdot\nabla f. \tag{5.6}$$

Choosing orthogonal directions $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ we get from theorem 5.1 the **uncertainty principle** for orthogonal variance directions [compare with case $k \neq l$ of (5.1)]:

Theorem 5.3. *For $\boldsymbol{a}\cdot\boldsymbol{b}$ =0, i.e. $\boldsymbol{b}\perp\boldsymbol{a}$, we get*

$$\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\, \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \;\geq\; 0. \tag{5.7}$$

Theorem 5.4. *Under the same assumptions as in theorem* 5.1*, we obtain*

$$\int_{\mathbb{R}^n} \boldsymbol{x^2}\; \|f(\boldsymbol{x})\|^2\; d^n\boldsymbol{x}\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \boldsymbol{\omega^2}\; \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \;\geq\; \frac{n}{4}\, F^2. \tag{5.8}$$

Remark 5.5. For the proof of theorem 5.4 we first insert $\boldsymbol{x}^2 = \sum_{k=1}^n (\boldsymbol{e}_k\cdot\boldsymbol{x})^2, \boldsymbol{\omega}^2 = \sum_{l=1}^n (\boldsymbol{e}_l\cdot\boldsymbol{\omega})^2$. After that we apply (5.4) and (5.7) depending on the relative directions of the vectors $\boldsymbol{e}_k$ and $\boldsymbol{e}_l$.

6. Conclusions

The (real) Clifford Fourier transform extends the traditional Fourier transform on scalar functions to $\mathcal{G}_n$ multivector functions with $n = 3\,(\mathrm{mod}\,4)$. Basic properties and rules for differentiation, convolution, the Plancherel and Parseval theorems were introduced.[6] We then presented a general directional uncertainty principle in the geometric algebra $\mathcal{G}_n$, $n = 3\,(\mathrm{mod}\,4)$ which describes how the variances (in arbitrary but fixed directions) of a multivector-valued function and its Clifford Fourier transform relate. The formula of the uncertainty principle in $\mathcal{G}_n$, $n = 3\,(\mathrm{mod}\,4)$ can be extended to $\mathcal{G}_n$, $n = 2\,(\mathrm{mod}\,4)$ taking due care of the resulting general non-commutativity of i_n with the elements of $\mathcal{G}_n$.

It is known that the Fourier transform is successfully applied to solving equations in all of classical and quantum physics such as the heat equation, wave equations, etc. The same is true for applications of the Fourier transform to problems in image processing and signal theory. Therefore in the future, we can apply geometric algebra and the Clifford Fourier transform to solve such problems involving the whole range of k-vector fields ($k = 0, 1, 2, \ldots, n$) in geometric algebras $\mathcal{G}_n$ with $n = 3\,(\mathrm{mod}\,4)$ and study the inevitably remaining uncertainties of the solutions.

[6]Similar formulas for $n = 2$ are also given and applied in [7].

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