CONIC SECTIONS THROUGH FIVE POINTS CLASSICAL, PROJECTIVE, CONFORMAL

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ABSTRACT

In the so-called conformal model of Euclidean space of geometric algebra, circles receive a very elegant description by the outer product of three general points of that circle, forming what is called a tri-vector. Because circles are a special kind of conic section, the question arises, whether in general some kind of third order outer product of five points on a conic section (or certain linear combinations) may be able to describe other types of conic sections as well. The main idea pursued in this paper is to follow up a formula of Grassmann for conic sections through five points and implement it in the conformal model. Grassmann obviously based his formula on Pascal's theorem. At the end we consider a simple linear combination of circle tri-vectors.

1. INTRODUCTION

Blaise Pascal (1623-1662, Fig. 1) researched the foundations of hydrodynamics, stating that the pressure is the same at all points in a fluid. This is the basis for hydraulic lifts.[1] But Pascal is also famous for his works in mathematics, both in theory and application. He developed and sold e.g. a calculator machine. In his religious writings he famously stated[2]:
If God does not exist, one will lose nothing by believing in him, while if he does exist, one will lose everything by not believing.

Our present point of interest is Pascal's work on conic sections: points, x-shaped pairs of lines, circles, ellipses, parabolas and hyperbolas. These curves have an enormous practical significance. They describe rotation trajectories, the orbits of planets, the trajectories of comets and soccer balls, commercial satellites, the ideal form of well focused antennas (nowadays popular for satellite TV), etc. They are so important that every student in the engineering sciences has to study them as part of his first year curriculum.

Fig. 1 Blaise Pascal (1623-1662)[2]
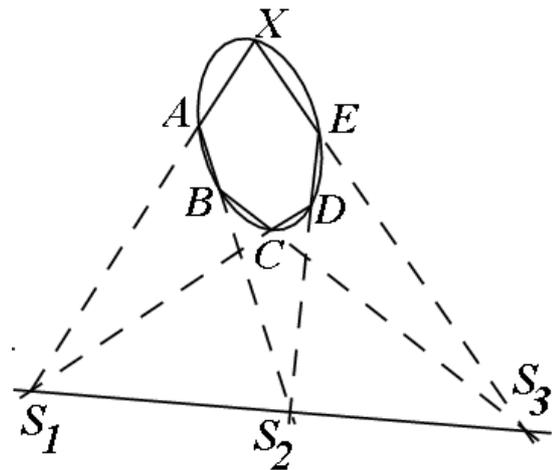
Fig. 2 Pascal's mystic hexagon[3]

At the age of 16 he found what is now called "Pascal's mystic hexagon" or less glamorous

"Pascal's theorem":

If a hexagon (ABCDEX) is inscribed in a conic section, then the three points (S_1, S_2, S_3) where opposite side (lines) meet are collinear.[3]

The theorem is illustrated in Fig. 2. The figure only shows the case of an ellipse, but the theorem is equally true for the other conic sections already mentioned above.

Mathematicians say that Pascal's theorem belongs to the field of "higher geometry," "geometry of position," "descriptive geometry," or in modern terms to "projective geometry."

The six basic axioms of projective geometry are easy to understand[4]:

1. If *A* and *B* are distinct points on a plane, there is at least one line containing both *A* and *B*.
2. If *A* and *B* are distinct points on a plane, there is not more than one line containing both *A* and *B*.
3. Any two lines in a plane have at least one point of the plane (which may be the point at infinity) in common.
4. There is at least one line on a plane.
5. Every line contains at least three points of the plane.
6. All the points of the plane do not belong to the same line.

2. A NEW BRANCH OF MATHEMATICS

In 1844, just about 200 years after Pascal discovered his theorem, the German mathematics school teacher Hermann Grassmann (1809-1877) invented his "Extension Theory"[5], which he republished in 1862[6]. He saw his "new branch of mathematics" indeed to "… form the keystone of the entire structure of mathematics."[6]

Also the popular mathematician Albrecht Beutelspacher considers the extension theory to comprise a number of "theoretical milestones" and "gems." Amongst the latter he counts: "*Without using coordinates, he could represent the equation of a conic section through five points (A,B,C,D,E) in general position in a plane.*"[7]

Later we will state Grassmann's representation of conic sections in precise formal terms. But before doing that we shall follow up the historical development of this new "keystone of mathematics."

3. GEOMETRIC ALGEBRA

In 1878, one year after Grassmann's death, William K. Clifford (1845-1879) published his "*Applications of Grassmann's extensive algebra.*"[8], in which he successfully unified Grassmann's extensive algebra with Hamilton's quaternion [9] description of rotations. This was the birth of (Clifford) *Geometric Algebra,* (which needs to be thoroughly distinguished from algebraic geometry.)

During the last 50 years or so geometric algebra has become quite popular as a rather universal tool for mathematics and its applications, including engineering.[10] But the development seems not finished yet. Projective geometry is by now well integrated in geometric algebra.[11] Especially for applications in computer vision and robotics it proves very versatile to adopt a higher dimensional geometric algebra, the socalled of *conformal* geometric algebra.[12]

4. CONFORMAL MODEL OF EUCLIDEAN SPACE

The conformal model of Euclidean space simply interprets the point of origin and spatial infinity as two extra linear dimensions of space, whose vectors have the peculiar property that they square to zero.[12] This can be seen as borrowing from the description of the propagation of light in space and time. Light propagates at the invariant vacuum speed of light and is therefore relativistic. The propagation of light in four-dimensional space-time also happens along vectors which square to zero. For a point light source, all these vectors form together the light cone.

Defining an even higher dimensional "light cone", the socalled *horosphere* in our five dimensional space of origin, 3-space and infinity, we get the socalled conformal model of Euclidean space. In this conformal model, every point on the horosphere is in one-to-one correspondence with every point in Euclidean space. This idea can be implemented with a host of geometric and computational benefits for areas like: computer vision, computer graphics, robotics, etc. The idea of the horosphere is not at all new, it was already defined by F.A. Wachter (1792-1817), an assistant of Gauss.[13]

Based on the conformal model, a number of computer programs have been developed for various applications, using object oriented programming languages, such as C++ and Java.[14,15] The description of points, pairs of points, lines, planes, circles and spheres is of

great elegance, just using one, two, three or four points. (In the case of lines and planes one of these points will be at infinity.) But a yet unsolved question is, whether we can find in the conformal model a similarly elegant description for conic sections, only using the five general points A,B,C,D,E (comp. Fig. 2) which Grassmann used.

The answer will be worked out in this paper. We will find, that in the conformal model, the implementation of Grassmann's formula for the conic sections given by five general points in the plane is indeed possible. But the resulting description will continue to be "quadratic" in each point and not "linear". This is in contrast to the "linear" descriptions of e.g. circles in the conformal model.

5. CONIC SECTIONS FROM 5 POINTS

Pascal's construction of Fig. 2 can be interpreted in two important ways, an *analytic* and a *constructive* way. The analytic interpretation was given in the introduction.

The constructive interpretation means using the theorem for the construction of a conic section from five general points on a plane. Assume five points A,B,C,D,E to be given. Construct the four lines AB, BC, CD, DE and the point of intersection S_2 of the lines AB and DE. Next draw any line g through the point S_2 and construct the intersection points S_1 and S_3 of the line g with CD and BC, respectively. After that draw the lines S_1A and S_3E. The point X of intersection of the lines S_1A and S_3E is also a point on the conic section. By conducting this construction for every angle of the line g through the point S_2, X will sweep out the whole conic section.

Table 1 Construction of S_1, S_2, and S_3

Point	= Intersection of lines
S_1	XA and CD
S_2	AB and DE
S_3	BC and EX

A consequence is, that any point X in the plane will be part of the conic section iff it can be reached by changing the angle of line g through point S_2. Therefore to decide whether a point X is on the conic section or not, we only need to check, whether S_1, S_2, and S_3 are collinear (on g) or not. The positions of S_1, and S_3 in this check will critically depend on the position of X (compare Table 1).

5.1 Grassmann's formula

Grassmann used precisely this method for obtaining his *"equation of a conic section that goes through the five points A,B,C,D,E, no three of which lie on the same straight line"*[6].

For this purpose he stated: "By *planimetric* multiplication I mean relative multiplication in the plane {as a domain of third order}, …"[6] This hint to the plane *as a domain of third order* is very important, because it shows that Grassmann actually expands the plane projectively by adding an extra dimension, commonly interpreted as infinity.

Grassmann further obtains the expression AB of a line from the outer product of two points A and B on the line. Grassmann omits the product symbol. Therefore the expression AB comes to mean both the product of two projective points A and B that results in an algebraic representation of a line and the common symbolic representation AB of a line through two points A and B.

Let us go into further geometric and algebraic details. The three-dimensional basis of the projective space of a plane is given in terms of three orthonormal vectors $\{e_1, e_2, e_i\}$. The first two vectors span the familiar non-projective Euclidean plane. The third vector e_i, is the additional third dimension of infinity. We represent a point in the Euclidean plane by a linear combination of e_1 and e_2:

$$a = a_1e_1 + a_2 e_2, \qquad (1)$$

where a_1 and a_2 are simply the two-dimensional Cartesian coordinates. The projective representation[16] of the point A is obtained by adding e_i

$$A = a + e_i . \qquad (2)$$

I now deliberately introduce the product symbol "^" for the exterior product in order to ease he distinction of the symbolic representation of a line AB and Grassmann's algebraic representation A^B. The exterior product is antisymmetric:

$$A\text{^}B = (a + e_i)\text{^}(b + e_i)$$
$$= a\text{^}b + (a\text{-}b)\text{^}e_i . \qquad (3)$$

a^b is the moment bivector of a line and (a-b) is its direction vector. The two entities suffice to

construct the line.[15]

Grassmann's planimetric product of two lines AB and DE can be realized in the geometric algebra of the three-dimensional space spanned by $\{e_1, e_2, e_i\}$ by the inner product

$$S_2 = (A{\wedge}B) \cdot [I_3 (D{\wedge}E)], \qquad (4)$$

where the dot ($\cdot$) represents the inner product[17,18] and $I_3 = e_1{\wedge}e_2{\wedge}e_i = ie_i$ is the volume 3-vector of the projective space ($i = e_1{\wedge}e_2$). $[I_3 (D{\wedge}E)]$ results in the *dual* complement vector perpendicular to the projective line bivector $D{\wedge}E$. Finally the inner product with the line bivector $A{\wedge}B$ results in the element S_2 in $A{\wedge}B$, which is perpendicular to $[I_3 (D{\wedge}E)]$ in $A{\wedge}B$, and therefore also contained in $D{\wedge}E$.

Inserting (2) and simplifying the expressions algebraically, we get for the intersection

$$S_2 = \lambda_2 s_2 + \lambda_2 e_i = (a\text{-}b)[i(d{\wedge}e)]\text{-}(d\text{-}e)[i(a{\wedge}b)]$$
$$+i[(d\text{-}e){\wedge}(a\text{-}b)]e_i \qquad (5)$$

In projective geometry points are identical up to scalar factors. We therefore divide by

$$\lambda_2 = i[(d\text{-}e){\wedge}(a\text{-}b)] = (d_1\text{-}e_1)(a_2\text{-}b_2)\text{-}(d_2\text{-}e_2)(a_1\text{-}b_1) \quad (6)$$

to get the plane Euclidean vector

$$s_2 = \{(a\text{-}b)[i(d{\wedge}e)]\text{-}(d\text{-}e)[i(a{\wedge}b)]\}/\lambda_2 \quad (7)$$

Inserting coordinates (2) we explicitly get

$$s_2 = \{(d_1e_2\text{-}d_2e_1)(a\text{-}b)\text{-}(a_1b_2\text{-}a_2b_1)(d\text{-}e)\}/\lambda_2 \quad (8)$$

In the very same way Grassmann calculates S_1 and S_3 by planimetric products as

$$S_1 = (X{\wedge}A) \cdot [I_3(C{\wedge}D)],\ S_3 = (B{\wedge}C) \cdot [I_3(E{\wedge}X)], \ (9)$$

So we can finally express the collinearity of S_1, S_2 and S_3 by

$$S_1{\wedge} S_2{\wedge} S_3 = 0, \quad \text{i.e.}$$
$$\{(X{\wedge}A) \cdot [I_3(C{\wedge}D)]\}{\wedge} \{(A{\wedge}B) \cdot [I_3 (D{\wedge}E)]\}{\wedge}$$
$$\{(B{\wedge}C) \cdot [I_3(E{\wedge}X)]\} = 0. \qquad (10)$$

This is Grassmann's formula for the conic sections through five general points (A,B,C,D,E) in a plane. Every point X, that fulfills equation (10) will be on the conic section. The equation is quadratic in X and in all of the five points A,B,C,D,E. With the help of the *anticommutator*

$$B_1 \times B_2 = (B_1B_2\text{-}B_2B_1)/2 \qquad (11)$$

we can rewrite (10) as

$$\{[(X{\wedge}A) \times (C{\wedge}D)] \times [(A{\wedge}B) \times (D{\wedge}E)]\}{\wedge}$$
$$\{I_3[(B{\wedge}C) \times (E{\wedge}X)]\} = 0. \qquad (12)$$

5.2 Grassmann's formula for the conformal model

The five-dimensional conformal model adds to three-dimensional space two dimensions: one for representing the origin and one for representing infinity. This is done by adding two null-vectors, which square to zero and are perpendicular to the vectors of Euclidean space:

$$\{\underline{n}, e_1, e_2, e_3, n\}, \qquad (13)$$

where $\underline{n}$ and n represent the origin and infinity, respectively.

The conformal representation of a point A is obtained by adding two contributions

$$A = a + (a^2/2)n + \underline{n}. \qquad (14)$$

A straight Euclidean line AB in the conformal model is given by the outer product of two points on the line with infinity

$$A{\wedge}B{\wedge}n = a{\wedge}b{\wedge}n + (b\text{-}a)N, \qquad (15)$$

where $N = n{\wedge}\underline{n}$ represents the additional two dimensional space. A point X is on the line AB iff

$$X{\wedge}A{\wedge}B{\wedge}n = 0. \qquad (16)$$

The intersection S^c_2 of two lines AB and DE is obtained in a fashion very similar to (4)

$$S^c_2 = (A{\wedge}B{\wedge}n) \cdot [I_4(D{\wedge}E{\wedge}n)], \qquad (17)$$

where $I_4 = iN$, with i being the bivector that represents the plane shared by A,B,D and E.

The explicit calculation of (17) yields

$$S^c_2 = \lambda_2 \{s_2 + (s_2{}^2/2)n + \underline{n}\}{\wedge}n. \qquad (18)$$

Note that $n{\wedge}n = 0$, but the term $(s_2{}^2/2)n$ is inserted for conveniently recognizing the structure of S^c_2. S^c_2 actually represents a pair of conformal points, the point $s_2 + (s_2{}^2/2)n + \underline{n}$ and infinity n.

Similar to (17) we obtain

$$S^c_1 = (X \wedge A \wedge n) \cdot [I_4(C \wedge D \wedge n)], \qquad (19)$$
$$S^c_3 = (B \wedge C \wedge n) \cdot [I_4(E \wedge X \wedge n)]. \qquad (20)$$

Using the three conformal points of intersection $s_1+(s_1^2/2)n+\underline{n}$, $s_2+(s_2^2/2)n+\underline{n}$, and $s_3+(s_3^2/2)n+\underline{n}$ we can finally give the equation for the conic sections through five general points on a plane in the conformal model as

$$(s_1+(s_1^2/2)n+\underline{n}) \wedge (s_2+(s_2^2/2)n+\underline{n}) \wedge S^c_3 = 0. \quad (21)$$

This is the conformal equivalent of Grassmann's formula for conic sections, which in turn has been seen to be based on Pascal's theorem. Every conformal point $X = x+(x^2/2)n+\underline{n}$ that fulfills (21) is on the plane conic section through A,B,C,D,E. By construction, equation (21) is again quadratic in X and in all of the five points A,B,C,D,E.

6. CIRCLES

This quadratic property of (21) is in contrast to the simpler representation of circles[15] by three conformal points A_1, A_2 and A_3

$$A_1 \wedge A_2 \wedge A_3. \qquad (22)$$

All points X on the circle through A_1, A_2 and A_3 are simply obtained from

$$X \wedge A_1 \wedge A_2 \wedge A_3 = 0. \qquad (23)$$

This equation is linear in X and in the three general defining points A_1, A_2 and A_3.
The explicit form of (22) becomes

$$0.5(a_1^2 a_2 \wedge a_3 + a_2^2 a_3 \wedge a_1 + a_3^2 a_1 \wedge a_2)n$$
$$+(a_3 \wedge a_1 + a_2 \wedge a_3 + a_1 \wedge a_2)\underline{n} \qquad (24)$$
$$0.5([a_2^2-a_3^2]a_1+[a_3^2-a_1^2]a_2+[a_1^2-a_2^2]a_3)N.$$

Separating off the circle center vector c and the radius r $(x_1^2= x_2^2= x_3^2=1)$

$$a_1=c+rx_1, \ a_2=c+rx_2, \ a_3=c+rx_3, \qquad (24)$$

we finally get for (24)

$$r^2\{c(c \wedge I_c)n+0.5(r^2-c^2)I_c n+I_c \underline{n}-cI_c N\}, \qquad (25)$$

where we set the bivector of the circle plain to

$$I_c = (x_3-x_2) \wedge (x_1-x_2). \qquad (26)$$

To simplify matters even more, let us assume, that we are just dealing with a plane case, i.e. the circle center c is also in the I_c-plane (i.e. $c \wedge I_c=0$)

$$r^2\{0.5(r^2-c^2)n + \underline{n} - cN\}I_c. \qquad (27)$$

We therefore see how (22) includes component by component the circle plane I_c, the center c and the radius r. Equation (23) is a condition for all points on the circle (22). By inserting $X = x+(x^2/2)n+\underline{n}$ into (23) we get after some algebra

$$(x-c)^2= r^2. \qquad (28)$$

Since a circle is a special conic section, we may ask ourselves, if it might be possible to generate other conic sections by simply "adding" two circles. In order to give it a try, let us assume, the second circle to be in the same plane I_c (including its center vector d)

$$t^2\{0.5(t^2-d^2)n + \underline{n} - dN\}I_c, \qquad (29)$$

where t denotes the radius of the second circle. Naïve addition gives

$$r^2\{0.5(r^2-c^2)n+\underline{n}-cN\}I_c+k't^2\{0.5(t^2-d^2)n+\underline{n}-dN\}I_c$$
$$=r^2\{0.5(r^2-c^2+k[t^2-d^2])n+(1+k)\underline{n}-(c+kd)N\}I_c \ (30)$$

with real constants k' and $k=k't^2/r^2$. In close analogy to (23) we now set zero the outer product of (30) with $X = x+(x^2/2)n+\underline{n}$ in order to see what kind of curve is defined by (30). The zero condition is found to be equivalent to the scalar identity

$$(x-c)^2 + k(x-d)^2= r^2 + kt^2. \qquad (31)$$

Isolating the unknown x into one quadratic term we get from (31)

$$(x-[c+kd]/[1+k])^2=r^2+kt^2 -k(c-d)^2/[1+k]. \qquad (32)$$

The equation (32) for the plane curve obtained from the linear combination of two circles in the conformal model has three cases:
1. If the right hand side term of equation (32) is less than zero, then there will be no solution for equation (32).
2. If the right hand side term of (32) is equal to zero, the solution will be exactly the point

$$x = (c+kd)/(1+k), \qquad (33)$$

This point is the weighted average of the

two circle centers c and d.

3. If the right hand side of (32) is greater than zero, the solution will be a circle with

$$\text{center} = (c+kd)/(1+k),$$
$$\text{and radius}^2 = r^2+kt^2-k(c-d)^2/(1+k). \quad (34)$$

For the special case of k=1 we get

2. $\qquad x = (c+d)/2, \qquad\qquad\qquad (35)$
3. $\qquad \text{center} = (c+d)/2$
$\qquad \text{and radius}^2 = r^2+t^2-(c-d)^2/2. \qquad (36)$

We therefore conclude, that at best the naïve addition (30) of two circles in the conformal model only leads to new circles, but not to equations for the other types of plane conic sections.

7. CONCLUSIONS

- We have introduced Pascal's theorem for hexagons inscribed by plane conic sections. (Fig. 2) This theorem is commonly attributed to the field of projective geometry.
- Subsequently Hamilton, Grassmann and Clifford and later researchers developed Clifford geometric algebra as a popular and fairly comprehensive theory for mathematics and its applications.
- The geometric algebra of the higher dimensional space of origin, Euclidean 3-space and infinity allows to implement an advantageous conformal model of Euclidean space.
- We described how Grassmann used Pascal's theorem for his equation of conic sections through five general points in a plane, by stating the formula in full detail in the (projective) geometric algebra of the Euclidean plane enhanced by an extra infinity dimension.
- We further showed how Grassmann's formula can be implemented in the conformal model of Euclidean space. The result is quadratic in each of the five general points and in the unknown point X.
- Finally the linear description of circles in the conformal model by three general points was recounted.
- The addition of two such circles does only give a new circle (or point), but no other type of conic sections.

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