The Hydrogen Atom Relativistic Model

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Preface

In this paper the hydrogen atom is solved using Einstein Relativity Theory without any approximation. The solution is very short and simple. What I get is the exact behaviors of the hydrogen atom including Relativity the results are not linear, so linear differential equation like Schrödinger Equation Dirac Equation or Klein Gordon Equation deliver complicate expressions. My solution is just simple and elegant.
Most of the result published in the Wikipedia differ from my results and must be checked to establish the correct model

Introduction

According to Einstein relativity (Einstein assumed electrons to be a particles). With a rest mass m_0

This mass corresponds to a rest energy of 0.511 [MeV].

1] $$E_0 = m_0 c^2$$

Where c is the speed of light.
The relativistic mass change depends on mass velocity V

2] $$m = \frac{m_0}{\sqrt{1-\left(\frac{V}{c}\right)^2}}$$

In 1924 De Borglie proposed that electrons and other particles have wave properties
De Borglie hypothesis was that the relation between particle linear momentum and its wavelength is;

3] $$p = mV = \frac{m_0}{\sqrt{1-\left(\frac{V}{c}\right)^2}} V = \frac{h}{\lambda}$$

Where h is Planck Constant, and λ is the wavelength associated with the electron
Let define

4] $$\beta = \frac{V}{c}$$

This is the normalized velocity
So Eq-3 can be written as

5] $$p = \frac{m_0 c}{\sqrt{1-\beta^2}}\beta = \frac{h}{\lambda}$$

Compton in 1912 found that the shortest wavelength of an electron is when the electron is at rest
The linear momentum of an electron at rest is

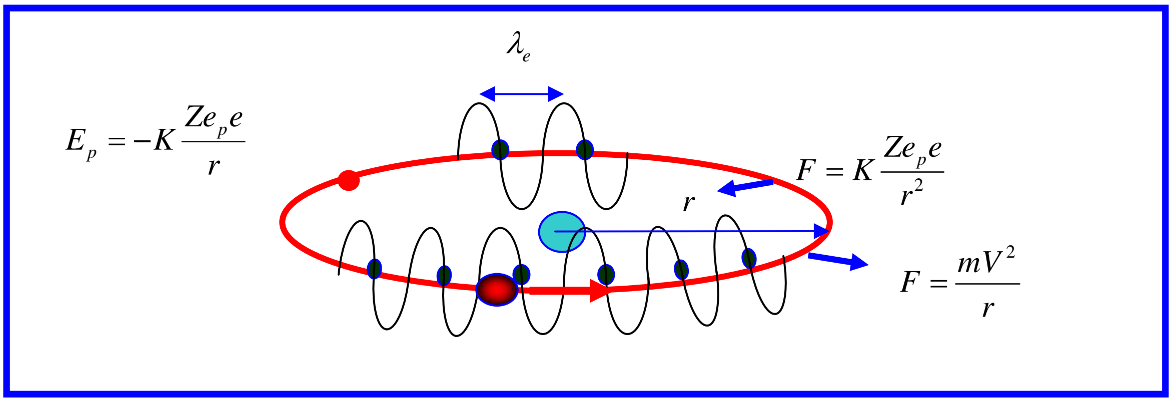
6] $$p_c = m_0 c$$

And its wavelength is named "Compton Wavelength"

7] $$\lambda_c = \frac{h}{p_c} = \frac{h}{m_0 c}$$

$$\lambda_c = 2.4263102175 \cdot 10^{-12} \text{ [m]}$$

The Hydrogen Atom



The electrically neutral hydrogen atom contains a single positively charged proton and a single negatively charged electron bound to the nucleus by the Coulomb force. The electron is circulating around the heavy proton. The Coulomb force is balanced by a centrifugal force so using Eq-2 and Eq-4

8] $$F = \frac{mV^2}{r} = \frac{1}{r}\frac{m_0 c^2}{\sqrt{1-\beta^2}}\beta^2 = \frac{1}{r}\frac{E_0}{\sqrt{1-\beta^2}}\beta^2 = \frac{ZKe_p e}{r^2}$$

The electron relative velocity β can be found from the last equation

9] $$\frac{\beta^2}{\sqrt{1-\beta^2}} = \frac{ZKe_p e}{E_0 r}$$

Since

10] $$V \perp r$$

The radius is not relativistic

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And from Eq –9

11] $$2\pi r = 2\pi \frac{ZKe_p e}{E_0}\frac{\sqrt{1-\beta^2}}{\beta^2}$$

Bohr suggested that the Circumference of a circle must contain a whole number of wavelength to avoid destructive interference of waves

12] $2\pi r = n\lambda \qquad 2\pi r_c = n\lambda_c \qquad n = 1,2,3....$

And from De Borglie hypothesis (Eq-5)

13] $$2\pi r = n\lambda = n\frac{h}{m_0 c}\frac{\sqrt{1-\beta^2}}{\beta} = 2\pi \frac{ZKe_p e}{m_0 c^2}\frac{\sqrt{1-\beta^2}}{\beta^2}$$

14] $$\beta_n = \frac{Z}{n}\frac{2\pi Ke_p e}{hc} = \frac{Z}{n}\alpha$$

15] $$\alpha = \frac{2\pi Ke_p e}{hc}$$

α is the Fine Structure Constant

16] $\alpha = 1/137.035999074$

And is dimensionless

From the last equation we learn that the electron velocity is quantized

The electron move only with discrete velocities and therefore stay in discrete distances from the proton

According to Einstein Relativity Theory

17] $$E^2 = \left[E_{kinetic} + E_0\right]^2 = p^2c^2 + E_0^2$$

using Eq-5

18] $$p^2c^2 + E_0^2 = \frac{(m_0 c)^2 \beta^2}{1-\beta^2}c^2 + E_0^2 = \frac{E_0^2\beta^2}{1-\beta^2} + E_0^2$$

Now let find the kinetic energy of the electron

Pay attention that according to Relativity

19] $$E_k \neq \frac{mV^2}{2}$$

But the result is very close to this result

To find the kinetic energy let use

20] $$E^2 = \left[E_k + E_0\right]^2 = p^2c^2 + E_0^2$$

Expanding the last Eq and from Eq-18

21] $$E^2 = \left[E_k + E_0\right]^2 = E_k{}^2 + 2E_kE_0 + E_0^2 = \frac{E_0^2\beta^2}{1-\beta^2} + E_0^2$$

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We get a quadratic equation

22] $$E_k^{\,2} + 2E_0 E_k - E_0^2 \frac{\beta^2}{1-\beta^2} = 0$$

And the solution is

23] $$E_k = -E_0 \pm \frac{E_0}{\sqrt{1-\beta^2}} = \frac{-E_0\sqrt{1-\beta^2} \pm E_0}{\sqrt{1-\beta^2}} =$$

Using Taylor expansion

24] $$\sqrt{1-\beta^2} \approx 1 - \frac{\beta^2}{2} - \frac{\beta^4}{8} - \frac{3\beta^6}{48}$$

We find that

25] $$E_k = \frac{-E_0\left[1 - \frac{\beta^2}{2} - \frac{\beta^4}{8} - \frac{3\beta^6}{48}\right] + m_0c^2}{\sqrt{1-\beta^2}} \approx \frac{m_0c^2\left[\frac{\beta^2}{2} + \frac{\beta^4}{8} + \frac{3\beta^6}{48}\right]}{\sqrt{1-\beta^2}}$$

And from the first term we find that the energy is approximately the Newtonian energy

26] $$E_k = \frac{m_0c^2\frac{\beta^2}{2}}{\sqrt{1-\beta^2}} \approx \frac{mc^2\beta^2}{2} = \frac{mV^2}{2}$$

Summery

Since

27] $$\beta_n = \frac{Z}{n}\alpha$$

The quantized kinetic energy is given by

28] $$E_{kn} = \frac{-m_0c^2\sqrt{1-\beta_n^2} + m_0c^2}{\sqrt{1-\beta_n^2}} = \frac{-m_0c^2\sqrt{1-\left(\frac{Z}{n}\alpha\right)^2} + m_0c^2}{\sqrt{1-\left(\frac{Z}{n}\alpha\right)^2}}$$

The radius is computed from

29]

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$$r_n = \frac{Zhc}{2\pi m_0 c^2}\frac{2\pi K e_p e}{hc}\frac{\sqrt{1-\beta_n^2}}{\beta_n^2} = \lambda_c \frac{Z}{2\pi}\alpha\frac{\sqrt{1-\left(\frac{Z}{n}\alpha\right)^2}}{\left(\frac{Z}{n}\alpha\right)^2} = \lambda_c \frac{n^2}{2\pi}\frac{\sqrt{1-\left(\frac{Z}{n}\alpha\right)^2}}{Z\alpha}$$

And the quantized momentum is

30]
$$p_n = mV_n = \frac{m_0 c}{\sqrt{1-\beta_n^2}}\beta_n = \frac{m_0 c}{\sqrt{1-\left(\frac{Z}{n}\alpha\right)^2}}\left(\frac{Z}{n}\alpha\right)$$

The velocity is quantized

31]
$$V_n = \frac{Z}{n}c\alpha$$

And the energy is also quantized

32]
$$E_n = \sqrt{p_n^2 c^2 + E_0^2} = \sqrt{\frac{E_0^2\beta_n^2}{1-\beta_n^2} + E_0^2} = \frac{E_0}{\sqrt{1-\beta_n^2}} = \frac{E_0}{\sqrt{1-\left(\frac{Z}{n}\alpha\right)^2}}$$

The electrostatic potential is derived from Eq -8

32]
$$E_p = -\frac{ZKe_p e}{r} = -\frac{E_0\beta^2}{\sqrt{1-\beta^2}}$$

the electrostatic potential combine with the kinetic energy sum up to the total energy

33]
$$E = E_k + E_p = \frac{mV^2}{2} - \frac{E_0\beta^2}{\sqrt{1-\beta^2}} = \frac{m_0 c^2\beta^2}{2\sqrt{1-\beta^2}} - \frac{E_0\beta^2}{\sqrt{1-\beta^2}} = -\frac{E_0\beta^2}{2\sqrt{1-\beta^2}}$$

in the quantized energy using Eq-27 is

34]
$$E_n = E_{kn} + E_{pn} = -\frac{E_0\beta_n^2}{2\sqrt{1-\beta_n^2}} = -\left(\frac{Z}{n}\alpha\right)^2 \frac{E_0}{2\sqrt{1-\left(\frac{Z}{n}\alpha\right)^2}}$$

35]

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Relativistic

$$E_1 = -\frac{E_0\alpha^2}{2\sqrt{1-\alpha^2}} = -\frac{0.511MeV\cdot(1/137.035999074)^2}{2\sqrt{1-(1/137.035999074)^2}} = -13.606083 \text{ [Volt]}$$

Non Relativistic

$$E_1 = -\frac{E_0\alpha^2}{2} = -\frac{0.511MeV\cdot(1/137.035999074)^2}{2} = -13.605721 \text{ [Volt]}$$

And the radius for E_1

36]

Relativistic

$$r_1 = \frac{\lambda_c}{2\pi}\frac{\sqrt{1-\alpha^2}}{\alpha} = \frac{2.4263102175\cdot 10^{-12}}{2\pi}\frac{\sqrt{1-(1/137.035999074)^2}}{1/137.035999074} = 5.291631\cdot 10^{-11}[m]$$

Non Relativistic

$$r_1 = \frac{\lambda_c}{2\pi\alpha} = \frac{2.4263102175\cdot 10^{-12}}{2\pi}\cdot\frac{1}{1/137.035999074} = 5.291772\cdot 10^{-11}[m]$$

The Spectrum of the Hydrogen Atom

the spectral lines of hydrogen correspond to particular jumps of the electron between energy levels. The simplest model of the hydrogen atom is given by the Bohr model. When an electron jumps from a higher energy to a lower, a photon of a specific wavelength is emitted.

The energy differences between levels in the Bohr model, and hence the wavelengths of emitted/absorbed photons, is given by the Rydberg formula which is non relativistic

$$n_s, n_t \in 1,2,3.......$$

37] Not Relativistic

$$\frac{1}{\lambda_{ns,nt}} = Z^2\frac{\alpha^2 E_0}{2hc}\left[\left(-\frac{1}{n_s^{\,2}}\right)-\left(-\frac{1}{n_t^{\,2}}\right)\right]$$

To get a relativistic formula, the difference between to energy levels is computed using Eq-34

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38]

$$\Delta E_{ns,nt} = h\upsilon = h\frac{c}{\lambda_{ns,nt}} = E_{ns} - E_{nt} = \left[\frac{-\left(\frac{Z}{n_s}\alpha\right)^2 E_0}{2\sqrt{1-\left(\frac{Z}{n_s}\alpha\right)^2}}\right] - \left[\frac{-\left(\frac{Z}{n_t}\alpha\right)^2 E_0}{2\sqrt{1-\left(\frac{Z}{n_t}\alpha\right)^2}}\right]$$

Eq-38 can be made more elegant

39]

$$\frac{1}{\lambda_{ns,nt}} = Z^2 R_\infty \left[\frac{-\frac{1}{n_s^{\,2}}}{\sqrt{1-\left(\frac{Z}{n_s}\alpha\right)^2}} - \frac{-\frac{1}{n_t^{\,2}}}{\sqrt{1-\left(\frac{Z}{n_t}\alpha\right)^2}}\right]$$

The Rydberg Constant for relativistic and not relativistic formula is the same

40]

$$R_\infty = \frac{\alpha^2 E_0}{2hc} = \frac{\alpha^2 m_0 c^2}{2hc} = \frac{\alpha^2 m_0 c}{2h} = \frac{\alpha^2}{2\lambda_c}$$

41]

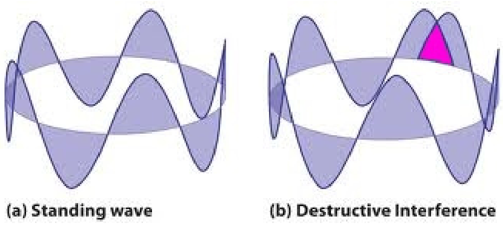
$$R_\infty = \frac{\alpha^2}{2\lambda_c} = 1.097373 \cdot 10^{-3} [1/A^0]$$

$$1\text{m} = 1 \cdot 10^{10} \text{ [Angstrem]}$$

Conclusions

The spectrum of the hydrogen atom and the Rydberg formula are the main tools to find the energy levels of the electron. I found that the non relativistic Rydberg formula differs from the relativistic formula derived in this paper. And since in heavy metals like Mercury the electron velocity compared to light velocity can't be neglected.

I found that Relativity produces non linear effects. Therefore we must be suspicious about Schrödinger and Dirac Equation and check again their results



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