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Some results on Smarandache groupoids

H. J. Siamwalla† and A. S. Muktibodh‡

† Abeda Inamdar Senior College, Azam Campus,
Pune, India
‡ Shri M. Mohota College of Science, Umred Road
Nagpur, India
E-mail: siamwalla.maths@gmail.com amukti2000@yahoo.com

Abstract In this paper we prove some results towards classifying Smarandache groupoids which are in $Z^*(n)$ and not in $Z(n)$ when n is even and n is odd.

Keywords Groupoids, Smarandache groupoids.

§1. Introduction and preliminaries

In [3] and [4], W. B. Kandasamy defined new classes of Smarandache groupoids using Z_n. In this paper we prove some theorems for construction of Smarandache groupoids according as n is even or odd.

Definition 1.1. A non-empty set of elements G is said to form a groupoid if in G is defined a binary operation called the product denoted by $*$ such that $a * b \in G$, $\forall\ a, b \in G$.

Definition 1.2. Let S be a non-empty set. S is said to be a semigroup if on S is defined a binary operation $*$ such that

(i) for all $a, b \in S$ we have $a * b \in S$ (closure).

(ii) for all $a, b, c \in S$ we have $a * (b * c) = (a * b) * c$ (associative law).

$(S, *)$ is a semi-group.

Definition 1.3. A Smarandache groupoid G is a groupoid which has a proper subset $S \subset G$ which is a semi-group under the operation of G.

Example 1.1. Let $(G, *)$ be a groupoid on the set of integer modulo 6, given by the following table.

$*$	0	1	2	3	4	5
0	0	5	0	5	0	5
1	1	3	1	3	1	3
2	2	4	2	4	2	4
3	3	1	3	1	3	1
4	4	2	4	2	4	2
5	5	0	5	0	5	0

Here, $\{0,5\},\{1,3\},\{2,4\}$ are proper subsets of G which are semigroups under $*$.

Definition 1.4. Let $Z_n = \{0,1,2,\cdots,n-1\}$, $n \geq 3$. For $a,b \in Z_n\backslash\{0\}$ define a binary operation $*$ on Z_n as: $a*b = ta+ub \pmod n$ where t,u are 2 distinct elements in $Z_n\backslash\{0\}$ and $(t,u)=1$. Here "+" is the usual addition of two integers and "ta" mean the product of the two integers t and a.

Elements of Z_n form a groupiod with respect to the binary operation. We denote these groupoids by $\{Z_n(t,u),*\}$ or $Z_n(t,u)$ for fixed integer n and varying $t,u \in Z_n\backslash\{0\}$ such that $(t,u)=1$. Thus we define a collection of groupoids $Z(n)$ as follows

$$Z(n) = \{Z_n(t,u),*| \text{ for integers } t,u \in Z_n\backslash\{0\} \text{ such that } (t,u)=1\}.$$

Definition 1.5. Let $Z_n = \{0,1,2,\cdots,n-1\}$, $n \geq 3$. For a, $b \in Z_n\backslash\{0\}$, define a binary operation $*$ on Z_n as: $a*b = ta+ub \pmod n$ where t, u are two distinct elements in $Z_n\backslash\{0\}$ and t and u need not always be relatively prime but $t \neq u$. Here "+" is usual addition of two integers and "ta" means the product of two integers t and a.

For fixed integer n and varying $t,u \in Z_n\backslash\{0\}$ s.t $t \neq u$ we get a collection of groupoids $Z^*(n)$ as: $Z^*(n) = \{Z_n(t,u),*| \text{ for integers } t,u \in Z_n\backslash\{0\} \text{ such that } t \neq u\}$.

Remarks 1.1. (i) Clearly, $Z(n) \subset Z^*(n)$.

(ii) $Z^*(n)\backslash Z(n) = \Phi$ for $n = p+1$ for prime $p = 2,\ 3$.

(iii) $Z^*(n)\backslash Z(n) \neq \Phi$ for $n \neq p+1$ for prime p.

We are interested in Smarandache Groupoids which are in $Z^*(n)$ and not in $Z(n)$ i.e., $Z^*(n)\backslash Z(n)$.

§2. Smarandache groupoids when n is even

Theorem 2.1. Let $Z_n(t,lt) \in Z^*(n)\backslash Z(n)$. If n is even, $n > 4$ and for each $t = 2,3,\cdots,\frac{n}{2}-1$ and $l = 2,3,4,\cdots$ such that $lt < n$, then $Z_n(t,lt)$ is Smarandache groupoid.

Proof. Let $x = \frac{n}{2}$.

Case 1. t is even.

$x*x = xt+ltx = (l+1)tx \equiv 0 \bmod n$.

$x*0 = xt \equiv 0 \bmod n$.

$0*x = lxt \equiv 0 \bmod n$.

$0*0 = 0 \bmod n$.

$\therefore$ $\{0,x\}$ is semigroup in $Z_n(t,lt)$.

$\therefore$ $Z_n(t,lt)$ is Smarandache groupoid when t is even.

Case 2. t is odd.

(a) If l is even.

$x*x = xt+ltx = (l+1)tx \equiv x \bmod n$.

$\{x\}$ is semigroup in $Z_n(t,lt)$.

$\therefore$ $Z_n(t,lt)$ is Smarandache groupoid when t is odd and l is even.

(b) If l is odd then $(l+1)$ is even.
$x * x = xt + ltx = (l+1)tx \equiv 0 \bmod n$.
$x * 0 = xt \equiv x \bmod n$.
$0 * x = ltx \equiv x \bmod n$.
$0 * 0 \equiv 0 \bmod n$.
$\Rightarrow \{0, x\}$ is semigroup in $Z_n(t, lt)$.
$\therefore$ $Z_n(t, lt)$ is Smarandache groupoid when t is odd and l is odd.

Theorem 2.2. Let $Z_n(t, u) \in Z^*(n)\backslash Z(n)$, n is even $n > 4$ where $(t, u) = r$ and $r \neq t, u$ then $Z_n(t, u)$ is Smarandache groupoid.

Proof. Let $x = \frac{n}{2}$.

Case 1. Let r be even i.e t and u are even.
$x * x = tx + ux = (t+u)x \equiv 0 \bmod n$.
$0 * x = ux \equiv 0 \bmod n$.
$x * 0 = tx \equiv 0 \bmod n$.
$0 * 0 = 0 \bmod n$.
$\{0, x\}$ is semigroup in $Z_n(t, lt)$.
$\therefore$ $Z_n(t, lt)$ is Smarandache groupoid when t is even and u is even.

Case 2. Let r be odd.

(a) when t is odd and u is odd,
$\Rightarrow t + u$ is even.
$x * x = tx + ux = (t+u)x \equiv 0 \bmod n$.
$x * 0 = tx \equiv x \bmod n$.
$0 * x \equiv ux \equiv x \bmod n$.
$0 * 0 \equiv 0 \bmod n$.
$\{0, x\}$ is a semigroup in $Z_n(t, u)$.
$\therefore$ $Z_n(t, u)$ is Smarandache groupoid when t is odd and u is odd.

(b) when t is odd and u is even,
$\Rightarrow t + u$ is odd.
$x * x = tx + ux = (t+u)x \equiv x \bmod n$.
$\{x\}$ is a semigroup in $Z_n(t, u)$.
$\therefore$ $Z_n(t, u)$ is Smarandache groupoid when t is odd and u is even.

(c) when t is even and u is odd,
$\Rightarrow t + u$ is odd.
$x * x = tx + ux = (t+u)x \equiv x \bmod n$.
$\{x\}$ is a semigroup in $Z_n(t, u)$.
$\therefore$ $Z_n(t, u)$ is Smarandache groupoid when t is even and u is odd.

By the above two theorems we can determine Smarandache groupoids in $Z^*(n)\backslash Z(n)$ when n is even and $n > 4$.

We find Smarandache groupoids in $Z^*(n)\backslash Z(n)$ for $n = 22$ by Theorem 2.1.

t	l	$lt < 22$	$Z_n(t, lt)$	Proper subset which is semigroup	Smarandache groupoid in $Z^*(n)\backslash Z(n)$
2	2	4	$Z_{22}(2,4)$	$\{0,11\}$	$Z_{22}(2,4)$
	3	6	$Z_{22}(2,6)$	$\{0,11\}$	$Z_{22}(2,6)$
	4	8	$Z_{22}(2,8)$	$\{0,11\}$	$Z_{22}(2,8)$
	5	10	$Z_{22}(2,10)$	$\{0,11\}$	$Z_{22}(2,10)$
	6	12	$Z_{22}(2,12)$	$\{0,11\}$	$Z_{22}(2,12)$
	7	14	$Z_{22}(2,14)$	$\{0,11\}$	$Z_{22}(2,14)$
	8	16	$Z_{22}(2,16)$	$\{0,11\}$	$Z_{22}(2,16)$
	9	18	$Z_{22}(2,18)$	$\{0,11\}$	$Z_{22}(2,18)$
	10	20	$Z_{22}(2,20)$	$\{0,11\}$	$Z_{22}(2,20)$
3	2	6	$Z_{22}(3,6)$	$\{11\}$	$Z_{22}(3,6)$
	3	9	$Z_{22}(3,9)$	$\{0,11\}$	$Z_{22}(3,9)$
	4	12	$Z_{22}(3,12)$	$\{11\}$	$Z_{22}(3,12)$
	5	15	$Z_{22}(3,15)$	$\{0,11\}$	$Z_{22}(3,15)$
	6	18	$Z_{22}(3,18)$	$\{11\}$	$Z_{22}(3,18)$
	7	21	$Z_{22}(3,21)$	$\{0,11\}$	$Z_{22}(3,21)$
4	2	8	$Z_{22}(4,8)$	$\{0,11\}$	$Z_{22}(4,8)$
	3	12	$Z_{22}(4,12)$	$\{0,11\}$	$Z_{22}(4,12)$
	4	16	$Z_{22}(4,16)$	$\{0,11\}$	$Z_{22}(4,16)$
	5	20	$Z_{22}(4,20)$	$\{0,11\}$	$Z_{22}(4,20)$
5	2	10	$Z_{22}(5,10)$	$\{11\}$	$Z_{22}(5,10)$
	3	15	$Z_{22}(5,15)$	$\{0,11\}$	$Z_{22}(5,15)$
	4	20	$Z_{22}(5,20)$	$\{11\}$	$Z_{22}(5,20)$
6	2	12	$Z_{22}(6,12)$	$\{0,11\}$	$Z_{22}(6,12)$
	3	18	$Z_{22}(6,18)$	$\{0,11\}$	$Z_{22}(6,18)$
7	2	14	$Z_{22}(7,14)$	$\{11\}$	$Z_{22}(7,14)$
	3	21	$Z_{22}(7,21)$	$\{0,11\}$	$Z_{22}(7,21)$
8	2	16	$Z_{22}(8,16)$	$\{0,11\}$	$Z_{22}(8,16)$
9	2	18	$Z_{22}(9,18)$	$\{11\}$	$Z_{22}(9,18)$
10	2	20	$Z_{22}(10,20)$	$\{0,11\}$	$Z_{22}(10,20)$

Next, we find Smarandache groupoids in $Z^*(n)\backslash Z(n)$ for $n = 22$ by Theorem 2.2.

t	u	$(t,u)=r$ $r\neq t,u$	$Z_n(t,u)$	**Proper subset which is semigroup**	**Smarandache groupoid in**$Z^*(n)\backslash Z(n)$
	6	(4,6)=2	$Z_{22}(4,6)$	$\{0,11\}$	$Z_{22}(4,6)$
4	10	(4,10=2	$Z_{22}(4,10)$	$\{0,11\}$	$Z_{22}(4,10)$
	14	(4,14)=2	$Z_{22}(4,14)$	$\{0,11\}$	$Z_{22}(4,14)$
	18	(4,18)=2	$Z_{22}(4,18)$	$\{0,11\}$	$Z_{22}(4,18)$
	8	(6,8)=2	$Z_{22}(6,8)$	$\{0,11\}$	$Z_{22}(6,8)$
	9	(6,9)=3	$Z_{22}(6,9)$	$\{11\}$	$Z_{22}(6,9)$
	10	(6,10)=2	$Z_{22}(6,10)$	$\{0,11\}$	$Z_{22}(6,10)$
6	14	(6,14)=2	$Z_{22}(6,14)$	$\{0,11\}$	$Z_{22}(6,14)$
	16	(6,16)=2	$Z_{22}(6,16)$	$\{0,11\}$	$Z_{22}(6,16)$
	20	(6,20)=2	$Z_{22}(6,20)$	$\{0,11\}$	$Z_{22}(6,20)$
	21	(6,21)=3	$Z_{22}(6,21)$	$\{11\}$	$Z_{22}(6,21)$
	10	(8,10)=2	$Z_{22}(8,10)$	$\{0,11\}$	$Z_{22}(8,10)$
	12	(8,12)=4	$Z_{22}(8,12)$	$\{0,11\}$	$Z_{22}(8,12)$
8	14	(8,14)=2	$Z_{22}(8,14)$	$\{0,11\}$	$Z_{22}(8,14)$
	18	(8,18)=2	$Z_{22}(8,18)$	$\{0,11\}$	$Z_{22}(8,18)$
	20	(8,20)=4	$Z_{22}(8,20)$	$\{0,11\}$	$Z_{22}(8,20)$
9	21	(9,21)=3	$Z_{22}(9,21)$	$\{0,11\}$	$Z_{22}(9,21)$
	12	(10,12)=2	$Z_{22}(10,12)$	$\{0,11\}$	$Z_{22}(10,12)$
	14	(10,14)=2	$Z_{22}(10,14)$	$\{0,11\}$	$Z_{22}(10,14)$
10	16	(10,16)=2	$Z_{22}(10,16)$	$\{0,11\}$	$Z_{22}(10,16)$
	18	(10,18)=2	$Z_{22}(10,18)$	$\{0,11\}$	$Z_{22}(10,19)$
	14	(12,14)=2	$Z_{22}(12,14)$	$\{0,11\}$	$Z_{22}(12,14)$
	15	(12,15)=3	$Z_{22}(12,15)$	$\{11\}$	$Z_{22}(12,15)$
12	16	(12,16)=4	$Z_{22}(12,16)$	$\{0,11\}$	$Z_{22}(12,16)$
	18	(12,18)=6	$Z_{22}(12,18)$	$\{0,11\}$	$Z_{22}(12,18)$
	20	(12,20)=4	$Z_{22}(12,20)$	$\{0,11\}$	$Z_{22}(12,20)$
	21	(12,21)=3	$Z_{22}(12,21)$	$\{11\}$	$Z_{22}(12,21)$
	16	(14,16)=2	$Z_{22}(14,16)$	$\{0,11\}$	$Z_{22}(14,16)$
	18	(14,18)=2	$Z_{22}(14,18)$	$\{0,11\}$	$Z_{22}(14,18)$
14	20	(14,20)=2	$Z_{22}(14,20)$	$\{0,11\}$	$Z_{22}(14,20)$
	21	(14,21)=7	$Z_{22}(14,21)$	$\{11\}$	$Z_{22}(14,21)$
15	20	(15,20)=5	$Z_{22}(15,20)$	$\{11\}$	$Z_{22}(15,20)$
	18	(16,18)=2	$Z_{22}(16,18)$	$\{0,11\}$	$Z_{22}(16,18)$
16	20	(16,20)=4	$Z_{22}(16,20)$	$\{0,11\}$	$Z_{22}(16,20)$
18	20	(18,20)=2	$Z_{22}(18,20)$	$\{0,11\}$	$Z_{22}(18,20)$

§3. Smarandache groupoids when n is odd

Theorem 3.1. Let $Z_n(t,u) \in Z^*(n)\backslash Z(n)$. If n is odd, $n > 4$ and for each $t = 2, \cdots, \frac{n-1}{2}$, and $u = n-(t-1)$ such that $(t,u) = r$ then $Z_n(t,u)$ is Smarandache groupoid.

Proof. Let $x \in \{0, \cdots, n-1\}$.

$$x * x = xt + xu = (n+1)x \equiv x \bmod n.$$

$\therefore$ $\{x\}$ is semigroup in Z_n.
$\therefore$ $Z_n(t,u)$ is Smarandanche groupoid.

By the above theorem we can determine the Smarandache groupoids in $Z^*(n)\backslash Z(n)$ when n is odd and $n > 4$.

Also we note that all $\{x\}$ where $x \in \{0, \cdots, n-1\}$ are proper subsets which are semigroups in $Z_n(t,u)$.

Let us consider the examples when n is odd. We will find the Smarandache groupoids in $Z^*(n)\backslash Z(n)$ by Theorem 3.1.

n	**t**	$\boldsymbol{u = n-(t-1)}$	$\boldsymbol{(t,u) = r}$	$\boldsymbol{Z_n(t,u)}$ **Smarandache groupoid (S.G.) in** $\boldsymbol{Z^*(n)\backslash Z(n)}$
5	2	4	$(2,4)=2$	$Z_5(2,4)$ is S.G. in $Z^*(5)\backslash Z(5)$
7	2	6	$(2,6)=3$	$Z_7(2,6)$ is S.G. in $Z^*(7)\backslash Z(7)$
9	2	8	$(2,8)=2$	$Z_9(2,8)$ is S.G. in $Z^*(9)\backslash Z(9)$
	4	6	$(4,6)=2$	$Z_9(4,6)$ is S.G. in $Z^*(9)\backslash Z(9)$
11	2	10	$(2,10)=2$	$Z_{11}(2,10)$ is S.G. in $Z^*(11)\backslash Z(11)$
	3	9	$(3,9)=3$	$Z_{11}(3,9)$ is S.G. in $Z^*(11)\backslash Z(11)$
	4	8	$(4,8)=4$	$Z_{11}(4,8)$ is S.G. in $Z^*(11)\backslash Z(11)$
13	2	12	$(2,12)=2$	$Z_{13}(2,12)$ is S.G. in $Z^*(13)\backslash Z(13)$
	4	10	$(4,10)=2$	$Z_{13}(4,10)$ is S.G. in $Z^*(13)\backslash Z(13)$
	6	8	$(6,8)=2$	$Z_{13}(6,8)$ is S.G. in $Z^*(13)\backslash Z(13)$
15	2	14	$(2,14)=2$	$Z_{15}(2,14)$ is S.G. in $Z^*(15)\backslash Z(15)$
	4	12	$(4,12)=4$	$Z_{15}(4,12)$ is S.G. in $Z^*(15)\backslash Z(15)$
	6	10	$(6,10)=2$	$Z_{15}(6,10)$ is S.G. in $Z^*(15)\backslash Z(15)$
17	2	16	$(2,16)=2$	$Z_{17}(2,16)$ is S.G. in $Z^*(17)\backslash Z(17)$
	3	15	$(3,15)=3$	$Z_{17}(3,15)$ is S.G. in $Z^*(17)\backslash Z(17)$
	4	14	$(4,14)=2$	$Z_{17}(4,14)$ is S.G. in $Z^*(17)\backslash Z(17)$
	6	12	$(6,12)=6$	$Z_{17}(6,12)$ is S.G. in $Z^*(17)\backslash Z(17)$
	8	10	$(8,10)=2$	$Z_{17}(8,10)$ is S.G. in $Z^*(17)\backslash Z(17)$

n	t	$u = n - (t-1)$	$(t, u) = r$	$Z_n(t, u)$ Smarandache groupoid (S.G.) in $Z^*(n)\backslash Z(n)$
19	2	18	$(2, 18) = 2$	$Z_{19}(2, 18)$ is S.G. in $Z^*(19)\backslash Z(19)$
	4	16	$(4, 16) = 4$	$Z_{19}(4, 16)$ is S.G. in $Z^*(19)\backslash Z(19)$
	5	15	$(5, 15) = 5$	$Z_{19}(5, 15)$ is S.G. in $Z^*(19)\backslash Z(19)$
	6	14	$(6, 14) = 2$	$Z_{19}(6, 14)$ is S.G. in $Z^*(19)\backslash Z(19)$
	8	12	$(8, 12) = 4$	$Z_{19}(8, 12)$ is S.G. in $Z^*(19)\backslash Z(19)$
21	2	20	$(2, 20) = 2$	$Z_{21}(2, 20)$ is S.G. in $Z^*(21)\backslash Z(21)$
	4	18	$(4, 18) = 2$	$Z_{21}(4, 18)$ is S.G. in $Z^*(21)\backslash Z(21)$
	6	16	$(6, 16) = 2$	$Z_{21}(6, 16)$ is S.G. in $Z^*(21)\backslash Z(21)$
	8	14	$(8, 14) = 2$	$Z_{21}(8, 14)$ is S.G. in $Z^*(21)\backslash Z(21)$
	10	12	$(10, 12) = 2$	$Z_{21}(10, 12)$ is S.G. in $Z^*(21)\backslash Z(21)$

Open Problems:

1. Let n be a composite number. Are all groupoids in $Z^*(n)\backslash Z(n)$ Smarandache groupoids?
2. Which class will have more number of Smarandache groupoids in $Z^*(n)\backslash Z(n)$?
   (a) When $n + 1$ is prime.
   (b) When n is prime.

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