The orthogonal representation of the Poincare group on the Majorana spinor field

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Abstract

The irreducibility of a representation of a real Lie algebra may depend on whether the representation space is a real or complex Hilbert space.

The unitary projective representations of the Poincare group on complex Hilbert spaces were studied by Wigner and many others. Although the Poincare group has a real Lie algebra, we do not know of any study of the orthogonal projective representations of the Poincare group on real Hilbert spaces.

The Majorana spinor field, a space-time dependent element of a 4 dimensional real vector space, is a solution of the free Dirac equation. Our goal is to study the projective representation of the Poincare group on the real Hilbert space of Majorana spinor fields.

The Majorana-Fourier and Majorana-Hankel orthogonal transforms of Majorana spinor fields are defined and related to the linear and angular momentums of a spin one-half projective representation of the Poincare group.

Then we show that the projective representation of the Poincare group on the Majorana spinor field, whether we include the parity and time reversal or not, is orthogonal and irreducible. This contrasts with the unitary projective representations of the Poincare group on the Dirac and Weyl spinor fields, whose properties change when including or excluding the parity and time reversal transformations.

Keywords: Majorana spinors, unitary operator, hilbert space

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1. Introduction

The Poincare group, also called inhomogeneous Lorentz group, has a real Lie algebra [1]. The irreducibility of a representation of a real Lie algebra may depend on whether the representation space is a real or complex Hilbert space. In a physicists language, the complex Hilbert spaces have twice the number of degrees of freedom of the real ones.

The unitary projective representations of the Poincare group on complex Hilbert spaces were studied by Wigner and many others [2–4]. Since Quantum Mechanics is based on complex Hilbert spaces [5], these studies were very important in the evolution of the role of symmetry in the Quantum theory[6]. We do not know of any study of the orthogonal projective representations of the Poincare group on real Hilbert spaces.

The Majorana spinor field[7], a space-time dependent Majorana spinor, is a solution of the free Dirac equation [8]. The space of Majorana spinors is a 4 dimensional real vector

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space, while the space of Majorana spinor fields is an infinite dimensional real Hilbert space. To study a system of many neutral particles with spin one-half, Majorana spinor fields are extended with second quantization operators and are called Majorana quantum fields. There are important applications of the Majorana quantum field in theories trying to explain phenomena in neutrino physics, dark matter searches, the fractional quantum Hall effect and superconductivity [9]. Note that Majorana quantum fields are related to but are different from the Majorana spinor fields. The Majorana, Dirac and Weyl spinors and quantum fields were extensively studied [10–12]. Yet, we do not know of a detailed study of the real Hilbert space of Majorana spinor fields, without second quantization operators.

In the context of Clifford Algebras, there are studies on the geometric square roots of -1 [13, 14] and on the generalizations of the Fourier transform [15], with applications to image processing.

The Poincare group is the semi-direct product of the translations and Lorentz groups. Whether or not the Lorentz and Poincare groups include the parity and time reversal transformations depends on the context and authors. To be clear, we use the prefixes full/restricted when including/excluding parity and time reversal transformations. The Pin(3,1) group is a double cover of the full Lorentz group[16]. The SL(2,C) subgroup is the double cover of the restricted Lorentz subgroup. It is already known that the Majorana spinor representation of both SL(2,C) and Pin(3,1) is irreducible [17].

Our goal is to study the projective representation of the Poincare group on the real Hilbert space of Majorana spinor fields.

We will show that the Majorana spinor representations of the groups SU(2), SL(2,C) and Pin(3,1) are irreducible and faithful.

The Majorana-Fourier and Majorana-Hankel orthogonal transforms of Majorana spinor fields are defined and related to the linear and angular momentums of a spin one-half projective representation of the Poincare group.

Then we show that the projective representation of the Poincare group on the Majorana spinor field, whether we include the parity and time reversal or not, is orthogonal and irreducible. This contrasts with the unitary projective representations of the Poincare group on the Dirac and Weyl spinor fields, whose properties change when including or excluding the parity and time reversal transformations.

In chapter 2 we define the Majorana matrices and spinors. In chapter 3 we study the Majorana spinor projective representation of the Lorentz group. In chapter 4 we relate the Majorana and Pauli spinor fields. In 5 and 6 we define the Majorana-Fourier and Majorana-Hankel transforms of a Majorana spinor. In 7 we show that the projective Poincare group representation on the Majorana spinor field is orthogonal and irreducible. In 8, we extend the Majorana transforms to include the energy. In 9, by comparison with the Dirac spinor field solutions of the free Dirac equation, we show that the Majorana transforms are related with the linear and angular momentums of a free particle with spin one-half.

2. Majorana, Dirac and Pauli Matrices and Spinors

Definition 2.1. $\mathbb{F}^{m\times n}$ is the vector space of $m \times n$ matrices whose entries are elements of the field $\mathbb{F}$.

In the next remark we state the Pauli's fundamental theorem of gamma matrices. The proof can be found in [18].

Remark 2.2. *Let A^μ, B^μ, $\mu \in \{0,1,2,3\}$, be two sets of 4×4 complex matrices verifying:*

$$A^\mu A^\nu + A^\nu A^\mu = -2\eta^{\mu\nu} \tag{2.1}$$
$$B^\mu B^\nu + B^\nu B^\mu = -2\eta^{\mu\nu} \tag{2.2}$$

Where $\eta^{\mu\nu} \equiv diag(+1,-1,-1-1)$ is the Minkowski metric.

1) There is a complex matrix S, with $|detS| = 1$, such that $B^\mu = SA^\mu S^{-1}$, for all $\mu \in \{0,1,2,3\}$. S is unique up to a complex phase.

2) If A^μ and B^μ are all unitary, then S is unitary.

Proposition 2.3. *Let α^μ, β^μ, $\mu \in \{0,1,2,3\}$, be two sets of 4×4 real matrices verifying:*

$$\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu = -2\eta^{\mu\nu} \tag{2.3}$$
$$\beta^\mu \beta^\nu + \beta^\nu \beta^\mu = -2\eta^{\mu\nu} \tag{2.4}$$

Then there is a real matrix S, with $|detS| = 1$, such that $\beta^\mu = S\alpha^\mu S^{-1}$, for all $\mu \in \{0,1,2,3\}$. S is unique up to a signal.

Proof. From remark 2.2, we know that there is a complex matrix T, unique up to a complex phase, such that $\beta^\mu = T\alpha^\mu T^{-1}$.

Conjugating the previous equation, we get $\beta^\mu = T^*\alpha^\mu T^{*-1}$. Then $T^* = e^{i2\theta}T$ for some real number θ. Therefore $S \equiv e^{i\theta}T$ is a real matrix, unique up to a signal. □

Definition 2.4. The Majorana matrices, $i\gamma^\mu$, $\mu \in \{0,1,2,3\}$, are 4×4 complex unitary matrices verifying:

$$(i\gamma^\mu)(i\gamma^\nu) + (i\gamma^\nu)(i\gamma^\mu) = -2\eta^{\mu\nu} \tag{2.5}$$

The Dirac matrices are $\gamma^\mu \equiv -i(i\gamma^\mu)$.

In the Majorana bases, the Majorana matrices are 4×4 real orthogonal matrices. An example of the Majorana matrices in a particular Majorana basis is:

$$\begin{aligned}
i\gamma^1 &= \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \quad i\gamma^2 = \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix} \quad i\gamma^3 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\
i\gamma^0 &= \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad i\gamma^5 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad = -\gamma^0\gamma^1\gamma^2\gamma^3
\end{aligned} \tag{2.6}$$

Definition 2.5. The Dirac spinor is a 4×1 complex column matrix, $\mathbb{C}^{4\times 1}$.

The space of Dirac spinors is a 4 dimensional complex vector space.

Definition 2.6. Let S be an invertible matrix such that $Si\gamma^\mu S^{-1}$ is real, for $\mu = 0,1,2,3$.

The set of Majorana spinors, *Pinor*, is the set of Dirac spinors verifying the Majorana condition:

$$Pinor \equiv \{u \in \mathbb{C}^{4\times 1} : S^* u^* = Su\} \tag{2.7}$$

Where * denotes complex conjugation.

The set of Majorana spinors is a 4 dimensional real vector space. Note that the linear combinations of Majorana spinors with complex scalars do not verify the Majorana condition. The Majorana spinor, in the Majorana bases, is a 4×1 real column matrix.

There are 16 linear independent products of Majorana matrices. These form a basis of the real vector space of endomorphisms of Majorana spinors, $End(Pinor)$. In the Majorana bases, $End(Pinor)$ is the vector space of 4×4 real matrices.

Definition 2.7. The Pauli matrices $\sigma^k,\ k \in \{1,2,3\}$ are 2×2 hermitian, unitary, anti-commuting, complex matrices. The Pauli spinor is a 2×1 complex column matrix. The space of Pauli spinors is denoted by $Pauli$.

The space of Pauli spinors, $Pauli$, is a 2 dimensional complex vector space and a 4 dimensional real vector space.

3. Majorana spinor representation of the Lorentz group

Remark 3.1. *The Lorentz group, $O(1,3) \equiv \{\lambda \in \mathbb{R}^{4\times 4} : \lambda^T \eta \lambda = \eta\}$, is the set of real matrices that leave the metric, $\eta = diag(1,-1,-1,-1)$, invariant.*

The proper orthochronous Lorentz subgroup is defined by $SO^+(1,3) \equiv \{\lambda \in O(1,3) : det(\lambda) = 1, {\lambda^0}_0 > 0\}$. It is a normal subgroup. The discrete Lorentz subgroup of parity and time-reversal is $\Delta \equiv \{1, \eta, -\eta, -1\}$.

The Lorentz group is the semi-direct product of the previous subgroups, $O(1,3) = \Delta \ltimes SO^+(1,3)$.

Definition 3.2. The set Maj is the 4 dimensional real space of the linear combinations of the Majorana matrices, $i\gamma^\mu$:

$$Maj \equiv \{a_\mu i\gamma^\mu : a_\mu \in \mathbb{R},\ \mu \in \{0,1,2,3\}\} \tag{3.1}$$

Definition 3.3. $Pin(3,1)$ [16] is the group of endomorphisms of Majorana spinors that leave the space Maj invariant, that is:

$$Pin(3,1) \equiv \Big\{S \in End(Pinor) :\ |detS| = 1,\ S^{-1}(i\gamma^\mu)S \in Maj,\ \mu \in \{0,1,2,3\}\Big\} \tag{3.2}$$

Proposition 3.4. *The map $\Lambda : Pin(3,1) \to O(1,3)$ defined by:*

$$(\Lambda(S))^\mu{}_\nu i\gamma^\nu \equiv S^{-1}(i\gamma^\mu)S \tag{3.3}$$

is two-to-one and surjective. It defines a group homomorphism.

Proof. 1) Let $S \in Pin(3,1)$. Since the Majorana matrices are a basis of the real vector space Maj, there is an unique real matrix $\Lambda(S)$ such that:

$$(\Lambda(S))^\mu{}_\nu i\gamma^\nu = S^{-1}(i\gamma^\mu)S \tag{3.4}$$

Therefore, Λ is a map with domain $Pin(3,1)$. Now we can check that $\Lambda(S) \in O(1,3)$:

$$(\Lambda(S))^\mu{}_\alpha \eta^{\alpha\beta}(\Lambda(S))^\nu{}_\beta = -\frac{1}{2}(\Lambda(S))^\mu{}_\alpha\{i\gamma^\alpha, i\gamma^\beta\}(\Lambda(S))^\nu{}_\beta = \tag{3.5}$$

$$= -\frac{1}{2}S\{i\gamma^\mu, i\gamma^\nu\}S^{-1} = S\eta^{\mu\nu}S^{-1} = \eta^{\mu\nu} \tag{3.6}$$

We have proved that Λ is a map from $Pin(3,1)$ to $O(1,3)$.

2) Since any $\lambda \in O(1,3)$ conserve the metric η, the matrices $\alpha^\mu \equiv \lambda^\mu{}_\nu i\gamma^\nu$ verify:

$$\{\alpha^\mu, \alpha^\nu\} = -2\lambda^\mu{}_\alpha \eta^{\alpha\beta} \lambda^\nu{}_\beta = -2\eta^{\mu\nu} \tag{3.7}$$

In a basis where the Majorana matrices are real, from Proposition 2.3 there is a real invertible matrix S_λ, with $|detS_\Lambda| = 1$, such that $\lambda^\mu{}_\nu i\gamma^\nu = S_\lambda^{-1}(i\gamma^\mu)S_\lambda$. The matrix S_Λ is unique up to a sign. So, $\pm S_\lambda \in Pin(3,1)$ and we proved that the map $\Lambda : Pin(3,1) \to O(1,3)$ is two-to-one and surjective.

3) The map defines a group homomorphism because:

$$\Lambda^\mu{}_\nu(S_1)\Lambda^\nu{}_\rho(S_2)i\gamma^\rho = \Lambda^\mu{}_\nu S_2^{-1} i\gamma^\nu S_2 \tag{3.8}$$

$$= S_2^{-1}S_1^{-1} i\gamma^\mu S_1 S_2 = \Lambda^\mu{}_\rho(S_1S_2)i\gamma^\rho \tag{3.9}$$

□

Remark 3.5. *The group $SL(2,\mathbb{C}) = \{e^{\theta^j i\sigma^j + b^j\sigma^j} : \theta^j, b^j \in \mathbb{R},\ j \in \{1,2,3\}\}$, where σ^j are the Pauli matrices.*

There is a two-to-one, surjective map $\Upsilon : SL(2,\mathbb{C}) \to SO^+(1,3)$, defined by:

$$\Upsilon^\mu{}_\nu(T)\sigma^\nu \equiv T^\dagger \sigma^\mu T \tag{3.10}$$

Where $T \in SL(2,\mathbb{C})$, $\sigma^0 = 1$ and σ^j, $j \in \{1,2,3\}$ are the Pauli matrices.

Lemma 3.6. *Consider that $\{M_+, M_-, i\gamma^5 M_+, i\gamma^5 M_-\}$ and $\{P_+, P_-, iP_+, iP_-\}$ are orthonormal basis of the 4 dimensional real vector spaces Pinor and Pauli, respectively, verifying:*

$$\gamma^0\gamma^3 M_\pm = \pm M_\pm,\ \sigma^3 P_\pm = \pm P_\pm \tag{3.11}$$

The isomorphism $\Sigma : Pauli \to Pinor$ is defined by:

$$\Sigma(P_+) = M_+,\ \Sigma(iP_+) = i\gamma^5 M_+ \tag{3.12}$$

$$\Sigma(P_-) = M_-,\ \Sigma(iP_-) = i\gamma^5 M_- \tag{3.13}$$

The group $Spin^+(3,1) \equiv \{\Sigma \circ A \circ \Sigma^{-1} : A \in SL(2,\mathbb{C})\}$ is a subgroup of $Pin(1,3)$. For all $S \in Spin^+(1,3)$, $\Lambda(S) = \Upsilon(\Sigma^{-1} \circ S \circ \Sigma)$.

Proof. From remark 3.5, $Spin^+(3,1) = \{e^{\theta^j i\gamma^5\gamma^0\gamma^j + b^j\gamma^0\gamma^j} : \theta^j, b^j \in \mathbb{R},\ j \in \{1,2,3\}\}$. Then, for all $T \in SL(2,C)$:

$$-i\gamma^0 \Sigma \circ T^\dagger \circ \Sigma^{-1} i\gamma^0 = \Sigma \circ T^{-1} \circ \Sigma^{-1} \tag{3.14}$$

Now, the map $\Upsilon : SL(2,\mathbb{C}) \to SO^+(1,3)$ is given by:

$$\Upsilon^\mu{}_\nu(T)i\gamma^\nu = (\Sigma \circ T^{-1} \circ \Sigma^{-1})i\gamma^\mu(\Sigma \circ T \circ \Sigma^{-1}) \tag{3.15}$$

Then, all $S \in Spin^+(3,1)$ leaves the space Maj invariant:

$$S^{-1} i\gamma^\mu S = \Upsilon^\mu{}_\nu(\Sigma^{-1} \circ S \circ \Sigma)i\gamma^\nu \in Maj \tag{3.16}$$

Since all the products of Majorana matrices, except the identity, are traceless, then $det(S) = 1$. So, $Spin^+(3,1)$ is a subgroup of $Pin(1,3)$ and $\Lambda(S) = \Upsilon(\Sigma^{-1} \circ S \circ \Sigma)$. □

Definition 3.7. The discrete Pin subgroup $\Omega \subset Pin(3,1)$ is:

$$\Omega \equiv \{\pm 1, \pm i\gamma^0, \pm\gamma^0\gamma^5, \pm i\gamma^5\} \tag{3.17}$$

The previous lemma implies that $Spin^+(1,3)$ is a double cover of $SO^+(3,1)$. We can check that for all $\omega \in \Omega$, $\Lambda(\pm\omega) \in \Delta$. That is, the discrete Pin subgroup is the double cover of the discrete Lorentz subgroup. Therefore, $Pin(3,1) = \Omega \ltimes Spin^+(1,3)$

Since there is a two-to-one surjective group homomorphism, $Pin(3,1)$ is a double cover of $O(1,3)$, $Spin^+(3,1)$ is a double cover of $SO^+(1,3)$ and $Spin^+(1,3) \cap SU(4)$ is a double cover of $SO(3)$. We can check that $Spin^+(1,3) \cap SU(4)$ is isomorphic to $SU(2)$.

Remark 3.8. *A representation (M_G, V) of a group G is defined by:*
1) the representation space V, which is a vector space;
2) the representation map $M : G \to GL(V)$ from the group elements to the automorphisms of the representation space, verifying for $\Lambda_1, \Lambda_2 \in G$:

$$M(\Lambda_1)M(\Lambda_2) = M(\Lambda_1\Lambda_2) \tag{3.18}$$

Definition 3.9. The Majorana spinor representation of $Pin(3,1)$ is defined by:
1) the representation space $V = Pinor$ is the space of Majorana spinors;
2) The representation map is:

$$M(S) = S, \ S \in Pin(3,1) \tag{3.19}$$

Remark 3.10. *A unitary matrix representation of a group is irreducible iff there is a basis where all the matrices of the representation can be block diagonalized.*

Proposition 3.11. *The Majorana spinor representation of $Spin^+(1,3) \cap SU(4)$ (isomorphic to $SU(2)$), is irreducible.*

Proof. In a Majorana basis, the automorphisms of Majorana spinors are 4×4 non-singular real matrices. We can check that $i\gamma^5\gamma^0\gamma^j \in Spin^+(1,3) \cap SU(4)$, $j \in \{1,2,3\}$. These matrices square to -1 and anti-commute. If there is a basis where they are all block diagonal, then the blocks also square to -1 and anti-commute. But there is only one 2×2 real matrix that squares to -1 and no 1×1 real matrix that squares to -1. Therefore, the representation is irreducible. □

4. Hilbert spaces of Majorana and Pauli spinor fields

Definition 4.1. The complex Hilbert space of Pauli spinors, *Pauli*, has the internal product:

$$< \phi, \psi >= \phi^\dagger\psi; \ \phi, \psi \in Pauli \tag{4.1}$$

Definition 4.2. The real Hilbert space of Majorana spinors, *Pinor*, has the internal product:

$$< \Phi, \Psi >= \Phi^\dagger\Psi; \ \Phi, \Psi \in Pinor \tag{4.2}$$

Definition 4.3. Consider that $\{M_+, M_-, i\gamma^0 M_+, i\gamma^0 M_-\}$ and $\{P_+, P_-, iP_+, iP_-\}$ are orthonormal basis of the 4 dimensional real vector spaces *Pinor* and *Pauli*, respectively, verifying:

$$\gamma^3\gamma^5 M_\pm = \pm M_\pm,\ \sigma^3 P_\pm = \pm P_\pm \tag{4.3}$$

Let H be a real Hilbert space. For all $h \in H$, the bijective linear map $\Theta_H : Pauli \otimes_{\mathbb{R}} H \to Pinor \otimes_{\mathbb{R}} H$ is defined by:

$$\Theta_H(h \otimes_{\mathbb{R}} P_+) = h \otimes_{\mathbb{R}} M_+,\ \Theta_H(h \otimes_{\mathbb{R}} iP_+) = h \otimes_{\mathbb{R}} i\gamma^0 M_+ \tag{4.4}$$
$$\Theta_H(h \otimes_{\mathbb{R}} P_-) = h \otimes_{\mathbb{R}} M_-,\ \Theta_H(h \otimes_{\mathbb{R}} iP_-) = h \otimes_{\mathbb{R}} i\gamma^0 M_- \tag{4.5}$$

Definition 4.4. Let H_n, with $n \in \{1, 2\}$, be two real Hilbert spaces and $U : Pauli \otimes_{\mathbb{R}} H_1 \to Pauli \otimes_{\mathbb{R}} H_2$ be an operator. The operator $U^\Theta : \Theta_{H_2} \circ U \circ \Theta_{H_1}^{-1} : Pinor \otimes_{\mathbb{R}} H_1 \to Pinor \otimes_{\mathbb{R}} H_2$ is defined as $U^\Theta \equiv \Theta_{H_2} \circ U \circ \Theta_{H_1}^{-1}$.

Remark 4.5. *Let H_n, with $n \in \{1, 2\}$, be two Hilbert spaces with internal products $<,>: H_n \times H_n \to \mathbb{F}$,($\mathbb{F} = \mathbb{R}, \mathbb{C}$). A linear operator $U : H_1 \to H_2$ is unitary iff:*
1) it is surjective;
2) for all $x \in H_1$, $< U(x), U(x) >=< x, x >$.

Remark 4.6. *Given two real Hilbert spaces H_1, H_2 and an unitary operator $U : H_1 \to H_2$, the inverse operator $U^{-1} : H_2 \to H_1$ is defined by:*

$$< x, U^{-1}y >=< Ux, y >,\ x \in H_1, y \in H_2 \tag{4.6}$$

Proposition 4.7. *Let H_n, with $n \in \{1, 2\}$, be two real Hilbert spaces. The following two statements are equivalent:*
1) The operator $U : Pauli \otimes_{\mathbb{R}} H_1 \to Pauli \otimes_{\mathbb{R}} H_2$ is unitary;
2) The operator $U^\Theta : Pinor \otimes_{\mathbb{R}} H_1 \to Pinor \otimes_{\mathbb{R}} H_2$ is unitary.

Proof. Because Θ_{H_n} is bijective, U is surjective iff $\Theta_{H_2} \circ U \circ \Theta_{H_1}^{-1}$ is surjective.
For all $g \in Pauli \otimes_{\mathbb{R}} H_1$, we have:

$$< g, g >=< \Theta_{H_1}(g), \Theta_{H_1}(g) > \tag{4.7}$$

$$< U(g), U(g) >=< \Theta_{H_2}(U(g)), \Theta_{H_2}(U(g)) > \tag{4.8}$$

Since Θ_{H_n} is bijective, we get that the following two statements are equivalent:
1) for all $g \in Pauli \otimes_{\mathbb{R}} H_1$, $< g, g >=< U(g), U(g) >$;
2) for all $g' \in Pinor \otimes_{\mathbb{R}} H_1$, $< g', g' >=< \Theta_{H_2}(U(\Theta_{H_1}^{-1}(g'))), \Theta_{H_2}(U(\Theta_{H_1}^{-1}(g'))) >$. □

Definition 4.8. The space of Majorana spinor fields over a set S, $Pinor(S) \equiv Pinor \otimes_{\mathbb{R}} L^2(S)$, is the real Hilbert space of Majorana spinors whose entries, in a Majorana basis, are real Lebesgue square integrable functions of S.

Definition 4.9. The space of Pauli spinor fields over a set S, $Pauli(S) \equiv Pauli \otimes_{\mathbb{R}} L^2(S)$ is the complex Hilbert space of Pauli spinors whose components are complex Lebesgue square integrable functions of S.

5. Linear Momentum of Majorana spinor fields

Definition 5.1. $L^2(\mathbb{R}^n)$ is the real Hilbert space of real functions of n real variables whose square is Lebesgue integrable in $\mathbb{R}^n$. The internal product is:

$$< f, g >\equiv \int d^n x f(x) g(x),\ \ f, g \in L^2(\mathbb{R}^n) \tag{5.1}$$

Remark 5.2. *The Pauli-Fourier Transform $\mathcal{F}_P : Pauli(\mathbb{R}^n) \to Pauli(\mathbb{R}^n)$ is an unitary operator defined by:*

$$\mathcal{F}_P\{\psi\}(\vec{p}) \equiv \int d^n\vec{x} \frac{e^{-i\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^n}}\psi(\vec{x}),\ \ \psi \in Pauli(\mathbb{R}^n) \tag{5.2}$$

Where the domain of the integral is $\mathbb{R}^n$.

Definition 5.3. The Majorana-Fourier Transform $\mathcal{F}_M : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{R}^3)$ is an operator defined by:

$$\mathcal{F}_M\{\Psi\}(\vec{p}) \equiv \int d^3\vec{x}\ \frac{e^{-i\gamma^0\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3}} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}}\Psi(\vec{x}),\ \ \Psi \in Pinor(\mathbb{R}^3) \tag{5.3}$$

Where the domain of the integral is $\mathbb{R}^3$, $m \geq 0$, $E_p \equiv \sqrt{\vec{p}^2 + m^2}$ and $\not{p} = E_p\gamma^0 - \vec{p}\cdot\vec{\gamma}$.

Proposition 5.4. *The Majorana-Fourier Transform is an unitary operator.*

Proof. The Majorana-Fourier Transform can be written as:

$$\mathcal{F}_M\{\Psi\}(\vec{p}) \equiv \sqrt{\frac{E_p + m}{2E_p}}\Big(\int d^3\vec{x}\ \frac{e^{-i\gamma^0\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3}}\Psi(\vec{x})\Big) \tag{5.4}$$

$$-\sqrt{\frac{E_p - m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|}\Big(\int d^3\vec{x}\ \frac{e^{+i\gamma^0\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3}}\Psi(\vec{x})\Big) \tag{5.5}$$

So, one gets:

$$\mathcal{F}_M\{\Psi\} = S \circ \mathcal{F}_P^{\Theta}\{\Psi\} \tag{5.6}$$

Where $S : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{R}^3)$ is a bijective linear map defined by:

$$\left[\begin{array}{c} S\{\Psi\}(+\vec{p}) \\ S\{\Psi\}(-\vec{p}) \end{array}\right] \equiv \left[\begin{array}{cc} \sqrt{\frac{E_p+m}{2E_p}} & -\sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} \\ \sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} & \sqrt{\frac{E_p+m}{2E_p}} \end{array}\right] \left[\begin{array}{c} \Psi(+\vec{p}) \\ \Psi(-\vec{p}) \end{array}\right] \tag{5.7}$$

We can check that the 2×2 matrix appearing in the equation above is orthogonal. Therefore S is an unitary operator. Since $\mathcal{F}_P^{\Theta}$ is also unitary, $\mathcal{F}_M$ is unitary. □

Proposition 5.5. *The inverse Majorana-Fourier Transform verifies:*

$$(\gamma^0\vec{\gamma}\cdot\vec{\partial} + i\gamma^0 m)\mathcal{F}_M^{-1}\{\Psi\}(\vec{x}) = (\mathcal{F}_M^{-1} \circ R)\{\Psi\}(\vec{x}) \tag{5.8}$$

Where $\Psi \in Pinor(\mathbb{R}^3)$ and $R : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{R}^3)$ is a bijective linear map defined by $R\{\Psi\}(\vec{p}) = i\gamma^0 E_p\Psi(\vec{p})$.

Proof. We have $\mathcal{F}_M^{-1} = (\mathcal{F}_P^{\Theta})^{-1} \circ S^{-1}$. Then:

$$(\gamma^0 \vec{\gamma} \cdot \vec{\partial} + i\gamma^0 m)(\mathcal{F}_P^{\Theta})^{-1}\{\Psi\}(\vec{x}) = ((\mathcal{F}_P^{\Theta})^{-1} \circ Q)\{\Psi\}(\vec{x}) \tag{5.9}$$

Where $Q : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{R}^3)$ is a bijective linear map defined by:

$$\begin{bmatrix} Q\{\Psi\}(+\vec{p}) \\ Q\{\Psi\}(-\vec{p}) \end{bmatrix} \equiv \begin{bmatrix} i\gamma^0 m & i\vec{p}\cdot\vec{\gamma} \\ -i\vec{p}\cdot\vec{\gamma} & i\gamma^0 m \end{bmatrix} \begin{bmatrix} \Psi(+\vec{p}) \\ \Psi(-\vec{p}) \end{bmatrix} \tag{5.10}$$

Now we show that $Q \circ S^{-1} = S^{-1} \circ R$:

$$\begin{bmatrix} i\gamma^0 m & i\vec{p}\cdot\vec{\gamma} \\ -i\vec{p}\cdot\vec{\gamma} & i\gamma^0 m \end{bmatrix} \begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} \\ -\sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix} = \tag{5.11}$$

$$= \begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} \\ -\sqrt{\frac{E_p-m}{2E_p}}\frac{\vec{p}\cdot\vec{\gamma}\gamma^0}{|\vec{p}|} & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix} \begin{bmatrix} i\gamma^0 E_p & 0 \\ 0 & i\gamma^0 E_p \end{bmatrix} \tag{5.12}$$

□

6. Angular momentum of Majorana spinor fields

Definition 6.1. Let $\vec{x} \in \mathbb{R}^3$. The spherical coordinates parametrization is:

$$\vec{x} = r(\sin(\theta)\sin(\varphi)\vec{e_1} + \sin(\theta)\sin(\varphi)\vec{e_2} + \cos(\theta)\vec{e_3}) \tag{6.1}$$

where $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a fixed orthonormal basis of $\mathbb{R}^3$ and $r \in [0, +\infty[$ $\theta \in [0, \pi]$, $\varphi \in [-\pi, \pi]$.

Definition 6.2. Let

$$\mathbb{S}^3 \equiv \{(p, l, \mu) : p \in \mathbb{R}_{\geq 0}; l, \mu \in \mathbb{Z}; l \geq 1; -l \leq \mu \leq l-1\} \tag{6.2}$$

The Hilbert space $L^2(\mathbb{S}^3)$ is the real Hilbert space of real Lebesgue square integrable functions of $\mathbb{S}^3$. The internal product is:

$$< f, g >= \sum_{l=1}^{+\infty} \sum_{\mu=-l}^{l-1} \int_0^{+\infty} dp f(p, l, \mu) g(p, l, \mu), \;\; f, g \in L^2(\mathbb{S}^3) \tag{6.3}$$

Definition 6.3. The Pauli-Hankel transform $\mathcal{H}_P : Pauli(\mathbb{R}^3) \to Pauli(\mathbb{S}^3)$ is an operator defined by:

$$\mathcal{H}_P\{\psi\}(p, l, \mu) \equiv \int r^2 dr d(\cos\theta) d\varphi \frac{2p}{\sqrt{2\pi}} \lambda_{l\mu}^{\dagger}(pr, \theta, \varphi)\psi(r, \theta, \varphi), \;\; \psi \in Pauli(\mathbb{R}^3) \tag{6.4}$$

The domain of the integral is $\mathbb{R}^3$. The matrices $\lambda_{l\mu}$, the spherical Bessel function of the first kind j_n [19], the Pauli spherical matrices $\omega_{l\mu}$[20], the spherical harmonics $Y_{l\mu}$ and the associated Legendre functions of the first kind $P_{l\mu}$ are:

$$\lambda_{l\mu}(r,\theta,\varphi) \equiv \omega_{l\mu}(\theta,\varphi)\Big(j_l(r)\frac{1+\sigma^3}{2} + j_{l-1}(r)\frac{1-\sigma^3}{2}\Big) \tag{6.5}$$

$$j_l(r) \equiv r^l\Big(-\frac{1}{r}\frac{d}{dr}\Big)^l\frac{\sin r}{r} \tag{6.6}$$

$$\omega_{l\mu}(\theta,\varphi) \equiv \Big(-\sqrt{\frac{l-\mu}{2l+1}}Y_{l,\mu}(\theta,\varphi) + \sqrt{\frac{l+\mu+1}{2l+1}}Y_{l,\mu+1}(\theta,\varphi)\sigma^1\Big)\frac{1+\sigma^3}{2} \tag{6.7}$$

$$+\Big(\sqrt{\frac{l+\mu}{2l-1}}Y_{l-1,\mu}(\theta,\varphi)\sigma^1 + \sqrt{\frac{l-\mu-1}{2l-1}}Y_{l-1,\mu+1}(\theta,\varphi)\Big)\frac{1-\sigma^3}{2} \tag{6.8}$$

$$Y_{l\mu}(\theta,\varphi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^\mu(\cos\theta)e^{i\mu\varphi} \tag{6.9}$$

$$P_l^\mu(\xi) \equiv \frac{(-1)^\mu}{2^l l!}(1-\xi^2)^{\mu/2}\frac{\mathrm{d}^{l+\mu}}{\mathrm{d}\xi^{l+\mu}}(\xi^2-1)^l \tag{6.10}$$

Remark 6.4. *Due to the properties of spherical harmonics and Bessel functions, the Pauli-Hankel transform is an unitary operator. The inverse Pauli-Hankel Transform verifies:*

$$\vec{\sigma}\cdot\vec{\partial}\ \mathcal{H}_P^{-1}\{\psi\}(\vec{x}) = (\mathcal{H}_P^{-1}\circ R')\{\psi\}(\vec{x}) \tag{6.11}$$

Where $\psi \in Pauli(\mathbb{S}^3)$ and $R' : Pauli(\mathbb{S}^3) \to Pauli(\mathbb{S}^3)$ is a bijective linear map defined by:

$$R'\{\psi\}(p,l,\mu) \equiv p\sigma^1\sigma^3\psi(p,l,\mu) \tag{6.12}$$

Definition 6.5. The Majorana-Hankel transform $\mathcal{H}_M : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{S}^3)$ is an operator defined by:

$$\mathcal{H}_M\{\Psi\}(p,l,\mu) \equiv \int r^2 dr d(\cos\theta)d\varphi\frac{2p}{\sqrt{2\pi}}\Delta^\dagger(p,l,\mu,r,\theta,\varphi)\Psi(r,\theta,\varphi),\ \ \Psi \in Pinor(\mathbb{R}^3) \tag{6.13}$$

$$\Delta(p,l,\mu,r,\theta,\varphi) \equiv \sqrt{\frac{E_p+m}{2E_p}}\Lambda_{l\mu}(pr,\theta,\varphi) + \sqrt{\frac{E_p-m}{2E_p}}(-1)^\mu\Lambda_{l,-\mu-1}(pr,\theta,\varphi)i\gamma^3 \tag{6.14}$$

Where the matrices $\Lambda_{l\mu}(r,\theta,\varphi) \equiv \Theta\circ\lambda_{l\mu}(r,\theta,\varphi)\circ\Theta^{-1}$ are obtained from the Pauli matrices $\lambda_{l\mu}$ replacing (i,σ^1,σ^3) by $(i\gamma^0,\gamma^1\gamma^5,\gamma^3\gamma^5)$.

Proposition 6.6. *The Majorana-Hankel transform is an unitary operator.*

Proof. The Majorana-Hankel transform can be written as:

$$\mathcal{H}_M = S\circ\mathcal{H}_P^\Theta \tag{6.15}$$

Where $S : Pinor(\mathbb{S}^3) \to Pinor(\mathbb{S}^3)$ is a bijective linear map defined by:

$$\begin{bmatrix} S\{\Psi\}(p,l,\mu) \\ S\{\Psi\}(p,l,-\mu-1) \end{bmatrix} \equiv \begin{bmatrix} \sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}(-1)^\mu i\gamma^3 \\ -\sqrt{\frac{E_p-m}{2E_p}}(-1)^\mu i\gamma^3 & \sqrt{\frac{E_p+m}{2E_p}} \end{bmatrix} \begin{bmatrix} \Psi(p,l,\mu) \\ \Psi(p,l,-\mu-1) \end{bmatrix} \tag{6.16}$$

We can check that the 2×2 matrix appearing in the equation above is orthogonal. Therefore S is an unitary operator. Since $\mathcal{H}_P^\Theta$ is also unitary, $\mathcal{H}_M$ is unitary. □

Proposition 6.7. *The inverse Majorana-Hankel Transform verifies:*

$$(\gamma^0\vec{\gamma}\cdot\vec{\partial}+i\gamma^0 m)\mathcal{H}_M^{-1}\{\Psi\}(\vec{x})=(\mathcal{H}_M^{-1}\circ R)\{\Psi\}(\vec{x}) \tag{6.17}$$

Where $\Psi\in Pinor(\mathbb{S}^3)$ *and* $R: Pinor(\mathbb{S}^3)\to Pinor(\mathbb{S}^3)$ *is a bijective linear map defined by:*

$$R\{\Psi\}(p,l,\mu)\equiv i\gamma^0 E_p\Psi(p,l,\mu) \tag{6.18}$$

Proof. We have $\mathcal{H}_M^{-1}=(\mathcal{H}_P^{\Theta})^{-1}\circ S^{-1}$. Then we can check that $i\gamma^5\Lambda_{l\mu}(pr,\theta,\varphi)=-(-1)^{\mu}\Lambda_{l,-\mu-1}(pr,\theta,\varphi)i\gamma^1$.

Therefore, the inverse Pauli-Hankel Transform verifies:

$$(\gamma^0\vec{\gamma}\cdot\vec{\partial}+i\gamma^0 m)\ (\mathcal{H}_P^{\Theta})^{-1}\{\Psi\}(\vec{x})=((\mathcal{H}_P^{\Theta})^{-1}\circ Q)\{\psi\}(\vec{x}) \tag{6.19}$$

Where $\Psi\in Pinor(\mathbb{S}^3)$ and $Q: Pinor(\mathbb{S}^3)\to Pinor(\mathbb{S}^3)$ is a bijective linear map defined by:

$$\begin{bmatrix} Q\{\Psi\}(p,l,\mu) \\ Q\{\Psi\}(p,l,-\mu-1)\end{bmatrix}\equiv\begin{bmatrix} i\gamma^0 m & (-1)^{\mu}\gamma^0\gamma^3 p \\ -(-1)^{\mu}\gamma^0\gamma^3 p & i\gamma^0 m\end{bmatrix}\begin{bmatrix}\Psi(p,l,\mu)\\ \Psi(p,l,-\mu-1)\end{bmatrix} \tag{6.20}$$

Now we show that $Q\circ S^{-1}=S^{-1}\circ R$:

$$\begin{bmatrix} i\gamma^0 m & (-1)^{\mu}\gamma^0\gamma^3 p \\ -(-1)^{\mu}\gamma^0\gamma^3 p & i\gamma^0 m\end{bmatrix}\begin{bmatrix}\sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}(-1)^{\mu}i\gamma^3 \\ -\sqrt{\frac{E_p-m}{2E_p}}(-1)^{\mu}i\gamma^3 & \sqrt{\frac{E_p+m}{2E_p}}\end{bmatrix}= \tag{6.21}$$

$$=\begin{bmatrix}\sqrt{\frac{E_p+m}{2E_p}} & \sqrt{\frac{E_p-m}{2E_p}}(-1)^{\mu}i\gamma^3 \\ -\sqrt{\frac{E_p-m}{2E_p}}(-1)^{\mu}i\gamma^3 & \sqrt{\frac{E_p+m}{2E_p}}\end{bmatrix}\begin{bmatrix} i\gamma^0 E_p & 0 \\ 0 & i\gamma^0 E_p\end{bmatrix} \tag{6.22}$$

□

7. Majorana spinor field representation of the Poincare group

Proposition 7.1. *The Majorana spinor field representation of the Poincare group is unitary.*

Proof. 1) The representation of the Poincare group is surjective. That is, for all Ψ, there is a $\Phi(x)=S^{-1}\Psi(\Lambda_S^{-1}(x-a))$ such that:

$$\Psi(x)=S\Phi(\Lambda_S x+a) \tag{7.1}$$

2) The only part of the Poincare representation that is not easy to see that is unitary are the boosts. Let S be a boost transformation:

$$<\mathcal{F}_M\circ S\{\Psi\},\mathcal{F}_M\circ S\{\Psi\}>=\int d^3\vec{x}d^3\vec{y}\Psi^{\dagger}(\vec{y})F(\vec{y},\vec{x})\Psi(\vec{x}) \tag{7.2}$$

$$F(\vec{y},\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} S^\dagger \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{\Lambda}(p)\cdot(\vec{y}-\vec{x})} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} S \tag{7.3}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} S^\dagger e^{i\frac{\not{p}}{m}\vec{\Lambda}(p)\cdot(\vec{y}-\vec{x})} \frac{\not{p}\gamma^0}{E_p} S \tag{7.4}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\frac{\not{\Lambda}(p)}{m}\vec{\Lambda}(p)\cdot(\vec{y}-\vec{x})} \frac{\not{\Lambda}(p)\gamma^0}{E_p} \tag{7.5}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\frac{\not{p}}{m}\vec{p}\cdot(\vec{y}-\vec{x})} \frac{\not{p}\gamma^0}{E_p} \tag{7.6}$$

Therefore:

$$< \mathcal{F}_M \circ S\{\Psi\}, \mathcal{F}_M \circ S\{\Psi\} >=< \mathcal{F}_M\{\Psi\}, \mathcal{F}_M\{\Psi\} > \tag{7.7}$$

Using the fact that the Majorana-Fourier transform is unitary, we conclude that:

$$< S\{\Psi\}, S\{\Psi\} >=< \Psi, \Psi > \tag{7.8}$$

So, the representation of the Poincare group is unitary. □

Proposition 7.2. *The Majorana spinor field representation of the inhomogeneous rotation group is irreducible.*

Proof. Suppose that the representation is reducible. Since it is unitary, there are 2 states Ψ, Φ verifying for all $g \in SU(2)$ and $\vec{a} \in \mathbb{R}^3$:

$$< \Phi, T(\vec{a})R_g\{\Psi\} >= 0 \tag{7.9}$$

Doing a Fourier transform, the above equation can be written as:

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \Phi^\dagger(\vec{p}) e^{i\gamma^0\vec{p}\cdot\vec{a}} R\Psi(\Lambda_R\vec{p}) = 0 \tag{7.10}$$

Integrating the above equation over all possible values of $\vec{a}$, we get:

$$\Phi^\dagger(\vec{0}) R\Psi(\vec{0}) = 0 \tag{7.11}$$

Since the $SU(2)$ representation is irreducible, the above equation cannot be satisfied for all R. Therefore the Majorana spinor field representation of the inhomogeneous rotation group is irreducible. □

8. Energy of Majorana spinor fields

Definition 8.1. The Energy Transform $\mathcal{E} : Pinor(\mathbb{R}) \to Pinor(\mathbb{R})$ is an operator defined by:

$$\mathcal{E}\{\Psi\}(p^0) \equiv \int dx^0 \, \frac{e^{i\gamma^0 p^0 x^0}}{\sqrt{2\pi}} \Psi(x^0), \;\; \Psi \in Pinor(\mathbb{R}) \tag{8.1}$$

Where the domain of the integral is $\mathbb{R}$, $m \geq 0$.

Proposition 8.2. *The Energy transform is an unitary operator.*

Proof. The Energy transform can be written as:

$$\mathcal{E}\{\Psi\}(p^0) = \Theta_{L^2} \circ \mathcal{F}_P(-p^0) \circ \Theta_{L^2}^{-1}\{\Psi\} \tag{8.2}$$

Where $\mathcal{F}_P(-p^0)$ is a Pauli-Fourier transform over $\mathbb{R}$. Since the Pauli-Fourier transform is unitary, so is the Energy transform. □

The energy transform can be applied in the time coordinate of a Majorana spinor field, x^0, after a (linear or spherical) momentum transform on the space coordinates, $\vec{x}$, to define an unitary energy-momentum transform:
- for the linear case $\mathcal{E} \circ \mathcal{F}_M : Pinor(\mathbb{R}^4) \to Pinor(\mathbb{R}^4)$;
- for the spherical case $\mathcal{E} \circ \mathcal{H}_M : Pinor(\mathbb{R}^4) \to Pinor(\mathbb{R} \times \mathbb{S}^3)$.

9. The free Dirac equation

The free Dirac equation can be rewritten as:

$$(\partial_0 + iH)\{\Psi\}(x) = 0 \tag{9.1}$$

$$iH\{\Psi\}(x) \equiv (\gamma^0\vec{\gamma} \cdot \vec{\partial} + i\gamma^0 m)\Psi(x),\ m \geq 0 \tag{9.2}$$

Where $\Psi(x)$ is a space-time dependent Dirac spinor field. Note that iH is time independent and real in a Majorana basis. The solution is:

$$\Psi(x) = e^{-iHx^0}\{\psi\}(\vec{x}) \tag{9.3}$$

Where $\psi(\vec{x})$ is a space dependent Dirac spinor field. So, the study of the free Dirac equation can be done by studying the operator $e^{-iHx^0} : Pinor(\mathbb{R}^3) \to Pinor(\mathbb{R}^3)$.

It can be shown that the operator e^{-iHx^0} is related with the Majorana transforms as:

$$\mathcal{F}_M \circ e^{-iHx^0} \circ \mathcal{F}_M^{-1}\{\psi\}(\vec{p}) = e^{-i\gamma^0 E_p x^0}\psi(\vec{p}) \tag{9.4}$$

$$\mathcal{H}_M \circ e^{-iHx^0} \circ \mathcal{H}_M^{-1}\{\psi\}(p, l, \mu) = e^{-i\gamma^0 E_p x^0}\psi(p, l, \mu) \tag{9.5}$$

Where $E_p = \sqrt{p^2 + m^2}$ (in the linear case $p^2 = \vec{p} \cdot \vec{p}$). Therefore, e^{-iHx^0} is unitary. The solutions of the free Dirac equation can be written as:

$$\Psi(x) = \int d^3\vec{p} \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} \frac{e^{-i\gamma^0(E_p x^0 - \vec{p}\cdot\vec{x})}}{\sqrt{(2\pi)^3}} \psi(\vec{p}) \tag{9.6}$$

$$\Psi(x) = \sum_{l \geq 1,\ -l \leq \mu \leq l-1} \int_0^{+\infty} dp \frac{2p}{\sqrt{2\pi}} \Delta(p, l, \mu, r, \theta, \varphi) e^{-i\gamma^0 E_p x^0} \psi(p, l, \mu) \tag{9.7}$$

If $\psi(\vec{p})$ is a Majorana spinor, then the solution $\Psi(x)$ is also a Majorana spinor. The set of quantum numbers $(\vec{p})$ and (p, l, μ) are related with the linear and spherical momentums of Dirac spinors. For instance, to obtain the Dirac spinor solution for the free electron, we just set $\psi_e(\vec{p}) = \frac{1+\gamma^0}{2}\psi_e(\vec{p})$ and we get:

$$\Psi_e(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\not{p} + m}{\sqrt{E_p + m}\sqrt{2E_p}} e^{-ip\cdot x} \frac{1 + \gamma^0}{2} \psi_e(\vec{p}) \tag{9.8}$$

The matrix γ^0 was replaced by the identity matrix 1, due to the presence of the projector. The same thing happens with the spherical solution and with the spin.

To obtain the Dirac spinor solution for the free positron, we just set $\psi_p(\vec{p}) = \frac{1-\gamma^0}{2}\psi_p(\vec{p})$ and the matrix γ^0 gets replaced by -1.

10. Conclusion

We fulfilled our goal to show that (without second quantization operators) all the kinematic properties of a free spin 1/2 particle with mass are present in the real solutions of the real free Dirac equation.

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