ZETA REGULARIZATION METHOD APPLIED TO THE CALCULATION OF DIVERGENT INTEGRALS $\int_0^\infty x^s dx$

Jose Javier Garcia Moreta
Graduate student of Physics at the UPV/EHU (University of Basque country)
In Solid State Physics
Addres: Practicantes Adan y Grijalba 2 6 G
P.O 644 48920 Portugalete Vizcaya (Spain)
Phone: (00) 34 685 77 16 53
E-mail: josegarc2002@yahoo.es

MSC: 97M50, 40G99 , 40G10 , 81T55

- ABSTRACT: We study a generalization of the zeta regularization method applied to the case of the regularization of divergent integrals $\int_0^\infty x^s dx$ for positive 's' , using the Euler Maclaurin summation formula, we manage to express a divergent integral in term of a linear combination of divergent series , these series can be regularized using the Riemann Zeta function $\zeta(s)$ s >0 , in the case of the pole at s=1 we use a property of the Functional determinant to obtain the regularization $\sum_{n=0}^{\infty}\frac{1}{(n+a)}=-\frac{\Gamma'}{\Gamma}(a)$, with the aid of the Laurent series in one and several variables we can extend zeta regularization to the cases of integrals $\int_0^\infty f(x)dx$, we believe this method can be of interest in the regularization of the divergent UV integrals in Quantum Field theory since our method would not have the problems of the Analytic regularization or dimensional regularization
- *Keywords:* = Riemann Zeta function, Functional determinant, Zeta regularization, divergent series .

ZETA REGULARIZATION FOR DIVERGENT INTEGRALS:

Sometimes in mathematics and physics , we must evaluate divergent series of the form $\sum_{n=1}^{\infty} n^k$, of course this series is divergent unles Re (k) >1 , however cases like k=1 or k=3 appear in several calculations of string theory and Casimir effect , for the case of Casimir effect [3] the result $\sum_{n=1}^{\infty} n^3 = \frac{1}{120}$ appears to give the correct result for the

Casimir force $\frac{F_c}{A} = -\frac{\hbar c\pi^2}{240a^4}$ here A is the area and 'd' the separation between the 2 plates , c and $\hbar$ are the speed of ligth and the Planck's constant. The idea behind the Zeta regularization method is to take for granted that for every 's' the identity $\sum_{n=1}^{\infty} n^{-s} = \zeta(s)$, follows although this formula is valid just for Re (s) > 1 , to extend the definition of the Riemann Zeta function to negative real numbers, one need to use the functional equation for the Riemann function

$$\zeta(1-s) = 2\left(2\pi\right)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s) \qquad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \qquad (1)$$

This gives the expressions $\sum_{n=i}^{\infty} n^0 = -\frac{1}{2}$, $\sum_{n=i}^{\infty} n = -\frac{1}{12}$ and $\sum_{n=i}^{\infty} n^2 = 0$ due to the pole at s=1 , the Harmonic series $\sum_{n=1}^{\infty} n^{-1}$ is NOT zeta regularizable, although it can be given a finite value $\sum_{n=1}^{\infty} n^{-1} = \gamma = 0.577215..$, this value can be justified by using the theory of Zeta-regularized infinite products (determinants) , as we shall see later in the paper

- *Zeta regularization for divergent integrals:*

Let be $f(x) = x^{m-s}$ with Re(m-s) < -1 , then the Euler-Maclaurin summation formula (see [1] and appendix A formula (A.2))for this function reads

$$\sum_{n=a+1}^{\infty} f(n) = \int_a^{\infty} f(x)dx - \frac{f(a)+f(\infty)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(\infty) - f^{(2k-1)}(a)\right) \qquad (2)$$

$$\begin{aligned}\int_a^{\infty} x^{m-s}dx &= \frac{m-s}{2}\int_a^{\infty} x^{m-1-s}dx + \zeta(s-m) - \sum_{i=1}^{a} i^{m-s} + a^{m-s} \\ &-\sum_{r=1}^{\infty} \frac{B_{2r}\Gamma(m-s+1)}{(2r)!\Gamma(m-2r+2-s)}(m-2r+1-s)\int_a^{\infty} x^{m-2r-s}dx\end{aligned} \qquad a \in N \qquad (3)$$

Here in formula (2) all the series and integrals are convergent, formula (2) is usually worthless , since it is trivial to prove that $\int_a^{\infty} x^{-k}dx = \frac{a^{1-k}}{k-1}$ for Re(k) >1 ,and the Riemann zeta function $\zeta\left(m-s\right) = \sum_{i=1}^{\infty} i^{m-s}$, so nothing new can be obtained from (2)

The idea is to use the Functional equation (1) for the Riemann and Zeta function to extend the definition of equation (2) to the whole complex plane except s=1 , in case (m-s) is positive there will be no pole at x=0 , so we can put a=0 and take the limit $s \to 0^+$

$$\int_0^\infty x^m dx = \frac{m}{2}\int_0^\infty x^{m-1}dx + \zeta(-m) - \sum_{r=1}^{\infty}\frac{B_{2r}m!(m-2r+1)}{(2r)!(m-2r+1)!}\int_0^\infty x^{m-2r}\,dx \qquad (4)$$

Formula (4) is the Analytic continuation of formula (3) with a=0 and can be used to obtain a finite definition for otherwise divergent integrals ,apparently this recurrence equation has an infinite number of terms but the Gamma function has a pole at x= 0 and at x being some negative integer , also if the function is of the form $f(x)=x^{\alpha}$ with $0 \ge \alpha > -1$ we can us the Euler-Maclaurin formula directly, since the sum $f(\infty)+f(a)$ and its derivatives are finite.

some examples of formula (4) $I_n = \int_0^\infty x^n dx$

$$I_0 = \zeta(0)+1 = \int_0^\infty dx \qquad I_1 = \frac{I_0}{2} + \zeta(-1) = \int_0^\infty x dx$$

$$I_2 = \left(\frac{I_0}{2} + \zeta(-1)\right) - \frac{B_2}{2}a_{21}I_0 = \int_0^\infty x^2 dx \qquad (5)$$

$$I_3 = \frac{3}{2}\left(\frac{1}{2}\left(I_0 + \zeta(-1)\right) - \frac{B_2}{2}a_{21}I_0\right) + \zeta(-3) - B_2 a_{31} I_0 = \int_0^\infty x^3 dx$$

So our method can provide finite 'regularization' to divergent integrals , with the Aid of the zeta regularization algorithm.

Also our formulae (3) (4) and (5) are consistent with the usual summation properties , in fact if $\int_0^\Lambda x^m dx$ is finite for finite Λ and we use the property of the Riemann and Hurwitz Zeta function [ ] to get the sum of the k-th powers of n on the interval $[0, \Lambda]$ $\sum_{i=0}^{\Lambda-1} i^m = \zeta(-m) - \zeta(-m,\Lambda)$, $\zeta(s,\Lambda) = \sum_{n=0}^{\infty}(n+\Lambda)^{-s}$ defined for Re(s) >1 (of course for positive 's' as $\Lambda \to \infty$ the second term goes to 0)

$$\int_0^\Lambda x^m dx = \frac{m}{2}\int_0^\Lambda x^{m-1}dx + \zeta(-m) - \zeta(-m,\Lambda) - \sum_{r=1}^{\infty}\frac{B_{2r}m!(m-2r+1)}{(2r)!(m-2r+1)!}\int_0^\Lambda x^{m-2r}\,dx \qquad (6)$$

For integer 'm' $\zeta_H(-m,x) = -\frac{B_{m+1}(x)}{m+1}$ we find the Bernoulli Polynomials , the powers of Λ would cancel the integral $\int_0^\Lambda x^m dx = \frac{\Lambda^{m+1}}{m+1}$, so in the end in formula (6) we would get the usual definition of Zeta regularization $\zeta_H(-m) = -\frac{B_{m+1}(0)}{m+1}$ for integer 'm'. Of course one could argue that a 'simpler' regularization of the divergent integrals should be $I(s)=\int_0^\infty dx(x+a)^s=-\frac{a^{s+1}}{s+1}$ and $I(-1)=\int_0^\infty dx(x+a)^{-1}=-\log a$, this is just dropping out the term proportional to $\log\infty$ or ∞^{s+1} inside the integral to make it finite, however if wi plugged this result into the Euler-Maclaurin summation formulae (3) (4) or (6) the terms involving 'a' would cancel and we would finally find that $\zeta_H(-m)=0$ for every 'm' which clearly is against the definition of zeta regularization of a series, for the case of the logarihmic divergence , obtained from differentiation with respect to the external parameter 'a' this is a result of taking the finite part of the integral ,which apparently works. For the case of the integrals $\int_a^\infty x^{m-s}\log^k(x)dx$, we can simply differentiate k-times with respect to regulator 's' inside (3) in order to obtain finite values in terms of $\zeta(-s)$ and $\zeta'(-s)$ for negative values of 's' unless m=-1 (for other negative values of m we can make a change of variable $xq=1$) , this is treated in the next section

- *Zeta-regularized determinants and the Harmonic series:*

Given an operator A with an infinite set of nonzero Eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ we can define a Zeta function and a Zeta-regularized determinant , Voros [10]

$$Tr\left\{A^{-s}\right\}=\zeta_A(s)=\sum_{n=0}^{\infty}\lambda_n^{-s}\qquad \det(A)=\prod_{n=0}^{\infty}\lambda_n=\exp\left(-\frac{d\zeta_A(0)}{ds}\right)\qquad (7)$$

The proof of the second formula inside (7) is pretty easy, the derivative of the Generalized zeta function will be $\zeta_A'(s)=-\sum_{n=0}^{\infty}\frac{\log\lambda_n}{\lambda_n^{s}}$ now let s=0 , use the property of the logarithm $\log(a.b)=\log a+\log b$ and take the exponential on both sides.

For the case of the Eigenvalues of a simple Quantum Harmonic oscillator in one dimension [10] $\lambda_n=n+a$, the Zeta function is just the Hurwitz Zeta function, so we can define a zeta-regularized infinite product in the form

$$\prod_{n=0}^{\infty}(n+a)=\exp\left(-\frac{d\zeta_H(0,a)}{ds}\right)\qquad \frac{d\zeta_H(0,a)}{ds}=\log\Gamma(a)-\log\left(\sqrt{2\pi}\right)\qquad (8)$$

In case we put a=1 we find the zeta-regularized product of all the natural numbers $\prod_{n=0}^{\infty}(n+1)=\sqrt{2\pi}$,see [5] if we take the derivative with respect to 'a' , we would find the same regularized Value Ramanujan did [2] precisely $\sum_{n=0}^{\infty}\frac{1}{(n+a)}=-\frac{\Gamma'}{\Gamma}(a)\quad a>0$

Harmonic series appear due to a logarithmic divergence of the integral $\int_0^\infty\frac{dx}{(n+a)}$, if we put m= -1 inside formula (3) , using a regulator 's' , $s\to 0^+$ we have the Euler Maclaurin summation formula

$$\int_0^\infty \frac{dx}{(n+a)^{s+1}} = -\frac{1}{2a} + \sum_{n=0}^\infty \frac{1}{(n+a)^{1+s}} + \sum_{r=1}^\infty \frac{B_{2r}}{(2r)!} \frac{\partial^{2r-1}}{\partial u^{2r-1}} \left(\frac{1}{(x+a)^{s+1}} \right)_{x=0} \qquad (9)$$

Since s >0 the integral and the series inside (9) will be convergent, now we can integrate over 'a' inside (9) and use the definition of the logarithm $\lim_{s\to 0^+} \frac{x^s-1}{s} = \log x$, to regularize the integral $\int_0^\infty \frac{dx}{(n+a)^{s+1}}$ as $s \to 0^+$ in terms of the function $-\frac{\Gamma'}{\Gamma}(a)$ plus some finite corrections due to the Euler-Maclaurin summation formula.

A faster method is just simple differentiate with respect to 'a' inside the integral $\int_0^\infty \frac{dx}{(n+a)^2} = -\frac{dI}{da}$, now this integral is convergent for every 'a' and equal to $\frac{1}{a}$, integration over 'a' again gives the value $-\log a + c$ plus a constant 'c' that will not depend on the value of a inside the integral in question , the proof that 'c' is unique no matter what a is comes from the fact that the difference $\int_0^\infty dx \left(\frac{1}{x+a} - \frac{1}{x+b} \right) = \log\left(\frac{b}{a} \right)$.
For the case a=0 , the derivative of the Hurwitz Zeta is $\frac{d\zeta_H(0,0)}{ds} = -\log\left(\sqrt{2\pi}\right)$ so if we approximate the divergent integral by a series, then we can get the regularized result $\int_0^\infty \frac{dx}{x} \approx \sum_{n=0}^\infty \frac{1}{n} = 0$.

Apparently it seems that using two different regularizations we get some different results , the idea is that if we use the Stiriling asymptotic formula

$$\log\Gamma(z) = \left(z - \frac{1}{2} \right)\log z - z + \frac{1}{2}\log\left(2\pi\right) + \sum_{r=1}^\infty \frac{B_{2r} z^{1-2n}}{2r\left(2r-1\right)} \qquad (10)$$

If we take the derivative with respect to 'z' inside (10) , is now more apparent that for the logarithmic derivative $\int_0^\infty \frac{dx}{x+a} \approx \log\left(\frac{\mu}{a} \right)$ here $c = \log\mu$ is a constant obtained from differentiation with respect to 'a' to regularize the divergent integral , this constant 'c' must be related to some physical constant or in case the quantity 'a' has dimension of Energy then μ must have also dimensions of energy so the logarithm is dimensionless, this constant 'c' would be the only free adjustable parameter that would appear inside our calculations to regularize integrals.

If 'a' is negative there is an extra term due to the value $\log(-1) = \pi i$, for more complex logarithmic integral one can use the definition $\int_0^\infty \frac{\log^k(x+a)dx}{x+a} \approx \frac{1}{k+1}\log^{k+1}\left(\frac{\mu}{a} \right)$ with the same energy scale $c = \log\mu$

So if we take formula (3) and the identity inside Zeidler [12] , page 55

$$\sum_{n=0}^{\infty}(n+a)^{-s} =_{reg} \begin{cases} \zeta_H(s,a) & s\in C\setminus\{1\} \\ -\dfrac{\Gamma'(a)}{\Gamma(a)} & s=1 \end{cases}$$

then we can regularize any divergent integral $\int_0^\infty (x+a)^m dx$ to get a finite result by analytic continuation.

This result is justified by taking the limit $s\to 0^+$ inside the formula for the Digamma function $\Psi^{(s)}(z)=(-1)^{s+1}\Gamma(s+1)\zeta_H(s+1,z)$

Also formally is possible to get the result $\sum_{n=1}^{\infty}\frac{1}{n}=\gamma$, if we introduce the following regulator $R(s,n)=\cosh(s\log n)$, $s\to 0$, inside the regularized Harmonic series $\sum_{n=1}^{\infty}\frac{R(s,n)}{n}$ so we get the limit $\lim_{s\to 0}\frac{\zeta_H(1+s)+\zeta_H(1-s)}{2}=\gamma$, which is precisely the Euler-mascheroni constant.

- *Regularization of divergent integrals* $\int_0^\infty dx f(x)$ *:*

In general, the divergent integrals that appear in Quantum Field Theory [12] are invariant under rotations, for example $\int\frac{d^4p}{\left(p^2+m^2\right)^2}$ or $\int\frac{d^4p}{\left((p-q)^2+m^2\right)}\frac{1}{p^2}$, if we use 4-dimesional polar coordinates we can reduce these integrals to the case $\frac{2\pi^{d/2}}{\Gamma\left(d/2\right)}\int_0^\infty dr f(r) r^{d-1}$ then the UV divergences appear when $r\to\infty$, here d=4 is the dimension of the spacetime, depending on the value of 'd' we can have several types of divergences $\int_0^\Lambda dr f(r) r^{d-1}\approx a\Lambda^{m+1}+b\log\Lambda$, if b =0 for m =2 the UV divergences are quadratic if m =0 the divergences are linear , in case a = 0 and b =1 the divergences are of logarithmic type , for example $\int\frac{d^4p}{\left(p^2+m^2\right)^2}$ has only a logarithmic divergence in dimension 4 , for a lower value of the dimension (d=3) this integral exists.

To study the rate of divergence , we can expand the function into a Laurent series valid for $z\to\infty$, $f(x)=\sum_{n=-\infty}^{n=k}c_n(x+a)^n$ 'k' is a finite number and means that the function $f(x)$ has a power law divergence for big 'x' , then the idea to compute a divergent integral would be this, we add and substract a Polynomial plus a term proportional to $\frac{1}{x+a}$ to split the integral into a finite part and another divergent integral, in both cases we must also introduce a regulator $(x+a)^{-s}$ for natural number 'a' so we make the integrals converget for some Re(s) >0

$$\int_0^\infty \frac{dx}{(x+a)^s}\left(f(x)-\sum_{n=0}^{k} b_n (x+a)^n - \frac{b_{-1}}{x+a}\right)+\sum_{n=0}^{k} b_n \int_0^\infty (x+a)^{n-s}\,dx + b_{-1}\int_0^\infty \frac{dx}{(x+a)^{1+s}} \qquad (11)$$

Also we can use the change of variable $(x+a)\to x$, so the new limits of integration would be (a,∞) , since 'a' is a natural number , then the following indentity $\sum_{n=0}^{\infty}(n+a)^{m-s}=\sum_{n=a}^{\infty} n^{m-s}=\zeta(m-s)+\sum_{n=0}^{a-1} n^{m-s}$ holds for every positive 'a' and 'm' in the sense of a zeta regularized series.

Now, we have to regularize the integrals inside (11) $\int_a^\infty x^{n-s}dx = I_n$, this can be made using formula (3) to get only FINITE results for these integrals I_n , see (5) , in general for every 'n' this divergent integral in the limit $s\to 0$ will be of the form $\sum_i c_i \zeta(-i)$, where 'i' runs from 0 to n-1 , for the logarithmic divergent integral, we can regularize it by using the identity $\sum_{n=1}^{\infty} n^{-1}=\gamma$ plus the Euler-Maclaurin summation formula so we get only finite (regularized) results, we are using zeta regularization plus formula (3) to get only finite results, this is the main key of our method.

Of course inside (11) in our substraction we can include non-integer powers of 'x' since the recursion formula (3) is still valid for them.

The number of terms 'k' is chosen so the first integral is FINITE , this first integral can be computed by Numerical or exact methods and yields to a finite value , the rest of the integrals are just the logarithmic and power-law divergences, they can be regularized with the aid of formulae (3) (4) (5) (7) (9) to get a finite value involving a linear combination of $\zeta(-m)$ m=0,1,2,....,k and another value proportional to $\int_0^\infty \frac{dx}{x+a}\approx \log\left(\frac{\mu}{a}\right)$, this appear because in renormalization/regularization we must obtain only finite values, so $\int_0^\infty \frac{dx}{x+a}=-\log(a)$, since the divergent term $\log\infty$ is erased by adding a counterterm to our Lagrangian defining our quantum theory so , we have that the (regularized) integrals $\int_0^\infty \frac{dx}{x+1}\ \int_0^\infty \frac{dx}{x}\ \int_0^1 \frac{dx}{x}$ after renormalization must have a value equal to '0' , if we insert these integrals inside the Euler-Maclaurin formula then we get a 'finite' value for the Harmonic series $\sum_{n=0}^{\infty}\frac{1}{(n+a)}=-\frac{\Gamma'}{\Gamma}(a)$, which is finite and it is the only value that an Harmonic series can admit after regularization renormalization, so even the Riemann and Hurwitz function have a pole at s=1 we can get a finite value for the Harmonic series.

As an exampleto understand our method better, we can analyze this simple divergent integralwith a > 0

$$\int_a^\infty \frac{x^{2-s}dx}{x+1} = \int_a^\infty dx\left(\frac{x^2}{x+1} - 1 + x + \frac{1}{x}\right) + \frac{\zeta(0)}{2} - a + \frac{a^2}{2} + \frac{1}{2a} + \frac{\Gamma'}{\Gamma}(a) - \sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}\frac{\partial^{2r-1}}{\partial u^{2r-1}}\left(\frac{1}{x+a}\right)_{x=0} - \zeta(-1) + \frac{1}{2} \qquad s \to 0 \quad (12)$$

The first integral in (12) is convergent and have an exact value of $\log\left(\frac{a+1}{a}\right)$, in order to regularize the logarithmic integral we have used the result $\sum_{n=0}^{\infty}\frac{1}{(n+a)} = -\frac{\Gamma'}{\Gamma}(a)$ plus the Euler-Maclaurin summation formula .

The mathematical justification of this is the following, given a divergent integral $\int_a^\infty dx f(x)$ we introduce a regulator $F(s) = \int_a^\infty f(x)\frac{dx}{x^s}$ so the integral F(s) exists for some big ‘s’ , if we add and substract powers of the form x^{k-s} for integer k and $\left(x+a\right)^{s+1}$, we can split F(s) into a convergent integral I (s) valid for $s \to 0^+$ and some divergent integrals of the form $\int_a^\infty x^{m-s}dx$ and $\int_0^\infty \frac{dx}{\left(x+a\right)^{s+1}}$, using formulae (3) (4) (5) and (9) we can express these integrals in terms of the series $\sum_{n=0}^{\infty}\frac{1}{(n+a)^{s+1}}$ and $\sum_{n=0}^{\infty}\frac{1}{(n+a)^{s-m}}$, which will be convergent for $\mathrm{Re}(s-m) > 1$ and $\mathrm{Re}(s+1) > 1$, now using the Functional equation for the Hurwitz and Riemann Zeta function we can make the analytic continuation of both series to $s \to 0^+$ avoiding the pole at s=1 by the use of Riemann Zeta function at negative integers $\zeta(-n)$ plus some corrections involving $-\frac{\Gamma'}{\Gamma}(a)$ of course the rules for change of variable and still valid so $\int_0^\infty dx f(x+a)\left(x+a\right)^{-s} = \int_a^\infty du f(u)u^{-s}$ this can be used to avoid some IR divergences at x = 0 by splitting the integral into an IR divergent part and an UV divergent part $\int_0^\infty du = \int_0^a du + \int_a^\infty du$. For other types of divergent integrals like $\int_a^\infty dx \log^\beta(x)x^\alpha$ for positive α and β one could differentiate with respect to ‘m’ or ‘s’ inside formula (2) in order to obtain a recurrence equation for the integrals $\int_a^\infty dx\log^\beta(x)x^\alpha$, this recurrence equation is finite (approximately) since for $\mathrm{Re}\left(p\right) > 1$ $\int_a^\infty dx\frac{\log^\beta(x)}{x^p}$ is finite and do not need to be regularized provided a > 0 . Other useful identities can be

$\left(1+x\right)^{1/2} \approx 1+\frac{x}{2}-\frac{x^{2}}{2.4}$ or the expansion of the logarithm valid for any x > 0 $\log x = 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{x-1}{x+1}\right)^{2n+1}$ to make logarithms more tractable , also we could use Laurent expansions to handle complicate non-Polynomial expressions like $\left(x^{n}+\mu^{n}\right)^{k}$ by expanding it for big 'x' into asymptotic (inverse) power series.

Another method to deal with $\int_{a}^{\infty}\frac{x^{2-s}dx}{x+1}$, would be to expand the integrand into a convergent Laurent series valid for x >a so $\frac{x^{2}}{x+1} = x-1+\frac{1}{x}+\sum_{n=3}^{\infty}(-1)^{n}x^{1-n}$ and use the equations (2-5) term by term to obtain finite results for the integrals $\int_{0}^{\infty}x^{m-s}dx$ with m=0,1,-1 however this is a bit trickier than (12).

- *Pauli Villars regularization example for logarithmic divergent integrals:*

To end with this section we will give another example of how can an integral with a logarithmic divergence be regularized, for example we set

$$\Pi(s,\Lambda) = \int \frac{d^{4}k}{\left(k^{2}-s+i\varepsilon\right)^{2}}\frac{-\Lambda^{2}}{k^{2}-\Lambda^{2}} \qquad \Lambda \to \infty \quad \varepsilon \to 0 \qquad (13)$$

Here the regulator for small 'k' is 1 and 's' is a physical parameter , this examples appears in Newcomb [8] page 233 , in the renormalization process we have that $s_{0} \to s$ so , we must take into account the difference between the bare value of the parameter and the measured one $\Pi(s,\Lambda)-\Pi(s_{0},\Lambda) = \delta\Pi$, in this case after renormalization, the integral inside (13) has the value $\delta\Pi = -i\pi^{2}\log\left(\frac{s}{s_{0}}\right)$

REGULARIZATION FOR IR DIVERGENT INTEGRALS:

If the integrand has a pole at $x = a$ so is a IR divergent integral , we can make the Taylor substraction

$$\int_{0}^{\infty}dx\frac{\left(f(x)-\sum_{n=0}^{\lambda-1}b_{n}(x-a)^{n}\right)}{(x-a)^{\lambda}}+\sum_{n=0}^{\lambda}b_{n}\int_{0}^{\infty}dx\left(x-a\right)^{n-\lambda} \qquad b_{n} = \frac{f^{(n)}(a)}{n!} \qquad (14)$$

For the case $\lambda = 1$ we have

$$\int_0^\infty \frac{dx}{(x-a)} = \int_0^\infty \frac{dx}{(x-a)}\frac{(x+a)}{(x+a)} = \int_0^\infty \frac{dxx}{(x^2-a^2)} + a\int_0^\infty \frac{dx}{(x^2-a^2)} \qquad (15)$$

To avoid the pole at x =a we inserta n small complex quantity Epsilon so $a \to a+i\varepsilon$, then formula (14) can be griten

$$\int_0^\infty \frac{dx}{(x-a-i\varepsilon)} = \int_0^\infty \frac{dx}{(x+a+i\varepsilon)} + 2\int_0^\infty \frac{dx}{(x^2-\left(a+i\varepsilon\right)^2)} \qquad (16)$$

Then inside (15) we have only a UV logarithmic divergence , for $\lambda > 1$ there are only IR divergences , if we use the Binomial theorem

$$\int_0^\infty \frac{dx}{(x-a)^\lambda} = \sum_{k=0}^{\lambda} \binom{\lambda}{k} \int_0^\infty dx \frac{(a+i\varepsilon)^k x^{\lambda-k}}{(x^2-(a+i\varepsilon)^2)^\lambda} \qquad (17)$$

Integrals inside (16) and (17) can be evaluated with the expression

$$\int_0^\infty \frac{x^m dx}{(x^2-(a+i\varepsilon)^2)^r} = \frac{(-1)^{r-1}\pi\left(-ia\right)^{m+1-2r}}{2\sin\left(\frac{m\pi+\pi}{2}\right)(r-1)!} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+1}{3-r}\right)} = I(a,m,r) \qquad \varepsilon \to 0 \qquad (18)$$

- *Regularization of integrals* $\int_0^\infty \frac{dx}{x^m}$ $\quad m \geq 1$:

There are 2 cases, for m=1 we have

$$\int_0^\infty \frac{dx}{x} = \int_0^\infty \frac{dx}{x}\frac{(x+a)}{x+a} = \int_0^\infty \frac{dx}{x+a} + a\int_0^\infty \frac{dx}{(1+ua)}$$

Where we have made the change of variable $xu = 1$ inside the last integral to turn the IR divergence into a Logarithmic UV divergence.

For $m > 1$ we introduce a regulator x^s to make the IR divergent integral to converge

$$\int_0^\infty \frac{dx}{x^{m-s}} = \int_0^a \frac{dx}{x^{m-s}} + \int_a^\infty \frac{dx}{x^{m-s}} = \int_a^\infty \frac{dx}{x^{m-s}} + \int_{1/a}^\infty du u^{m-s} \qquad xu = 1 \qquad (19)$$

The last integral inside (18) is divergent in the limit $s \to 0$ and can be regularizad with formula (3). If many IR divergences appear $(x-a_1)^{\lambda 1}.(x-a_2)^{\lambda 2}.......(x-a_m)^{\lambda m}$ we must perform the Taylor substraction (14) on each point $(x-a_i)^i$

REGULARIZATION OF MULTIPLE INTEGRALS:

In general in QFT there will be also multi-loop (multiple) integrals so we will also have to regularize integrals in the form.

$$I(s) = \int d^4q_1 \int d^4q_2 ....\int d^4q_n \prod_{i=1}^{\infty} \frac{1}{\left(1+q_i^2\right)} F(q_1, q_2, ..., q_n) \qquad (20)$$

Here we have introduced a regulator depending on an external parameter 's' in order the integral (20) to converge for big 's' and then use the analytic regularization to take the limit $s \to 0^+$, this regulator must be chosen with care in order not to spoil any symmetries of the Physical system.

Another method is to consider the multiple integral as an interate integral and then make the substraction for every variable for example

$$\int \partial q_1 \left(F(q_{1,} q_2, ..., q_n) - \sum_{i=-1}^{k} a_i(q_1, ..., q_n)(1+q_1)^i \right) + \int_0^{\infty} \partial q_1 \sum_{i=-1}^{k} a_i(q_1, ..., q_n)(1+q_1)^i \qquad (21)$$

The symbol ∂q_1 means that the integral is made over the variable q_1 keeping the other variables constant , the number 'k' is chosen so the first integral is finite , this integral will depend on $I(q_1, ..., q_n)$, the divergent integrals (even for the logarithmic case i=-1) can be regularized.

Now we have regularized the first integral, we have reduced in one variable the multiple integral, repeating the iterative process for the functions $a_i(q_2, ..., q_n)$

$$\int \partial q_2 \left(a_i(q_2 ......, q_n) - \sum_{j=-1}^{k} b_j(q_2, ..., q_n)(1+q_2)^i \right) + \int_0^{\infty} \partial q_2 \sum_{j=-1}^{k} b_i(q_2, ..., q_n)(1+q_2)^j \qquad (22)$$

Using (21) and (22) on every variable we can reduce the dimension of the integral until we reach to the one dimensional case, which is easier to handle.

If the integrand $F(q_1, q_2, ..., q_n)$ had no singularities for every $q_j > 0$, we may expand this integrand into a multiple Laurent series of several variables, and then perform the substraction $\sum_{m1,m2,\dots,mn=-1}^{s1,s2,\dots,sn} C_{m1,m2,\dots,mn}\left(q_1+b_1\right)^{m1}\left(q_2+b_2\right)^{m2}\dots\dots\left(q_n+b_n\right)^{mn}$ in order to define a finite part of the integral

$$\int d^4q_1\int d^4q_2\dots\dots\dots\int d^4q_n\left(F-\sum_{m1,m2,\dots,mn=-1}^{s1,s2,\dots,sn} C_{m1,m2,\dots,mn}\left(q_1+b_1\right)^{m1}\left(q_2+b_2\right)^{m2}\dots\dots\left(q_n+b_n\right)^{mn}\right) \tag{23}$$

Plus some corrections due to divergent integrals $\int_0^\infty (q_i+b_i)^m dq_i$ m=-1,0,1,..... .

In many cases although the integrals given in (22) and (23) are finite they will have no exact expression or the exact expression will be too complicate, in this case we can use the Gauss-Laguerre Quadrature formula (in case the interval is $[0,\infty)$) to approximate the integral by a sum over the zeros of Laguerre Polynomials $\sum_{i=0}^{n} w_i f(q_1,q_2,\dots,q_{n-1},x_i)$ with the weigth expressed in terms of Laguerre Polynomials and their roots

$$w_i=\frac{x_i}{(n+1)^2\left(L_{n+1}(x_i)\right)^2} \quad , L_n(x_i)=0 \qquad (24)$$

unAs an example we will regularize the simples 2-loop integral

$$\int_0^\infty\int_0^\infty \frac{1}{k}\frac{1}{(k+p)^2}\frac{l}{(k+l)^2}dldk$$

We introduce the 2 zeta regulators k^s l^s to get finite results and avoid divergences, the regulator will be eliminated after calculations. Integration over 'l' gives (here 'p' is a constant)

$$\int_0^\infty \frac{l^{1-s}}{(k+l)^2}dl=(1-k)\int_0^\infty\frac{l^{-s}dl}{(k+l)(l+1)}+\int_0^\infty\frac{l^{-s}dl}{l+1}-\frac{\pi}{2} \qquad (25)$$

To perform integration over 'k' we will need to regularize the integrals

$$A\int_0^\infty \frac{k^s dk}{k\left(k+p\right)^2} \qquad \int_0^\infty\frac{(1-k)dk}{(k+p)^2k^{1/s}}\int_0^\infty\frac{dl}{(k+l)(l+1)} \qquad A=\int_0^\infty\frac{dx}{x+1}-\frac{\pi}{2} \quad (26)$$

The first integral becomes an logarithmic UV divergent integral $\int_0^\infty \frac{u^{1-s}du}{\left(1+up\right)^2}$ with the change of variable $xu=1$ (as always the regulator 's' tends to 0), this can be regularized by addition and substraction to this integral of the factor $\int_0^\infty \frac{dx}{x+1}$, this logarithmically divergent integral can be regularized with the Euler Maclaurin formula and the identity $\sum_{n=1}^\infty \frac{1}{n}_{reg}=\gamma$.

To evaluate the second integral we will use the Laguerre quadrature formula in order to simplify calculations

$$\int_0^\infty \frac{(1-k)dk}{(k+p)^2 k}\int_0^\infty \frac{dl}{(k+l)(l+1)} \quad \to \quad \sum_{x_j} w_j \int_0^\infty dk e^{x_j} \frac{w_j(1-k)dk}{(k+p)^2 k\left(k+x_j\right)\left(1+x_j\right)} \qquad (27)$$

Expression (27) is legitimate since the integral over 'l' is convergent, with a change of variable $xu=1$ each integral of the sum inside (27) becomes a logarithmic divergent integral $\int_0^\infty dk \frac{w_j u(u-1)du}{(1+up)^2\left(1+ux_j\right)}$, this is again a logarithmic divergent integral and can be evaluated by adding /substracting a term of the form $\int_0^\infty \frac{dx}{x+1}$. Using the equations (24 -27) we have shown an example of how a 2-loop integral can be $\zeta-$regularizad , by applying the zeta regularization method of our paper on each variable , first over 'l' keeping 'k' constant and then over 'k' to get finite results, each UV divergent integral can be regularized with the formulae (2-5)

- *An easier example of a 2-loop integral and zeta regularization:*

If the integral $\int_0^\infty\int_0^\infty \frac{1}{k}\frac{1}{(k+p)^2}\frac{l}{(k+l)^2}dldk$ seems too complicate we will give an easier model of how Zeta regularization Works beyond the 1-loop integral, let be the divergent integral $\int_0^\infty dx\int_0^\infty dy \frac{1}{x+y+1}$, this is divergent , and has also an overlapping divergence in 'x' and 'y' , integration and regularization over the sub-divergence on the variable 'x' gives (regulator $(xy)^{-s}$ assumed implicitly).

$$\int_0^\infty dx\int_0^\infty dy \frac{1}{x+y+1} = \int_0^\infty dx\int_0^\infty dy \frac{-y}{(x+y+1)(x+1)} + \int_0^\infty \frac{dx}{x+1}\int_0^\infty dy \qquad (28)$$

The first integral inside (28) is convergent $\int_0^\infty dx\int_0^\infty dy \frac{-y}{(x+y+1)(x+1)}$ so we can use the Laguerre Quadrature formula to evaluate it $\sum_j \omega_j e^{x_j} \frac{-y}{(x_j+y+1)(x_j+1)}$, we have now a sum of finite term, each term will be a function of 'y' , regularization in the variable 'y' gives finally (for each term of the Laguerre quadrature)

$$\frac{-y}{(x_j+y+1)(x_j+1)} = \int_0^\infty dy \frac{-x_j(x_j+1)}{(x_j+y+1)(x_j+1)} - \int_0^\infty dy + (x_j+1)\int_0^\infty \frac{dy}{y+1} \qquad (29)$$

Where, we must apply the regularization inside (29) to every term of the Laguerre Quadrature formula $\sum_j \omega_j e^{x_j} \frac{-y}{(x_j+y+1)(x_j+1)}$ to obtain a finite result

Now integral inside (29) $\int_0^\infty dy \frac{-x_j(x_j+1)}{(x_j+y+1)(x_j+1)}$ is convergent (integration over 'y') so it can be evaluated without any regularization renormalization.

The three divergent integrals $\int_0^\infty dy$ $\int_0^\infty \frac{dx}{x+1}$ and $\int_0^\infty ydy$ can be regularized using the equations (2) (3) and (4) plus the identity $\sum_{n=0}^{\infty} \frac{1}{n+1} = \gamma$.

Of course we could have used the Euler-Maclaurin summation formula to replace the convergent integral $\int_0^\infty dx \int_0^\infty dy \frac{-y}{(x+y+1)(x+1)}$ by a finite convergent series whose terms will depend on the variable 'y' $\sum_{n=1}^{\infty} \frac{-y}{(n+y+1)(n+1)}$, then we truncate the series and apply zeta regularization (29) to each term of the series.

We would like to put much emphasis into this method of zeta regularization for multiple loop integral (2 –loop) with overlapping divergences, because we have had our paper rejected several times because of the cheap excuse that zeta regularization would not be valid for higher loops , with these examples we prove them wrong, we show how we can regularize a multiple integral using zeta regularization and iterated integration on each of variables, first over 'x' (subdivergence) and then over 'y' , of course this process can be generalized to n-loop multiple integrals too, so referee's excuse is just wrong and zeta regularization is valid also for multiple integrals

CONCLUSIONS AND FINAL REMARKS:

We have extended the definition of the zeta regularization of a series to apply it to the Zeta regularization of a divergent integral $\int_0^\infty x^m dx$ 1 > m >0 by using the Zeta regularization technique combined with the Euler Maclaurin summation formula.

For a good introduction to the Zeta regularization techniques , there is the book by Elizalde [4] or the Book by Brendt based on the mathematical discoveries of Ramanujan and its method of summation equivalent to the Zeta regularization algorithm [2] , another good reference (but a bit more advanced) is Zeidler [12] , for the case of Zeta-regularized determinants [7] is a good online reference describing also the process of Zeta regularization via analytic continuation and how it can be applied to prove the identity $\prod_{n=0}^{\infty}(n+1) = \log\sqrt{2\pi}$.

Apparently there is a contradiction, since the Riemann Zeta funciton has a pole at s=1 so the Harmonic series could not be regularized, however using the definition of a functional determinant $\prod_{n=0}^{\infty} \frac{E_n}{\mu}$ $E_n = n + a$ one gets the finite result for the Harmonic (generalized) series $\sum_{n=0}^{\infty} \frac{1}{n+a} = -\frac{\Gamma'(a)}{\Gamma(a)}$, with the aid of the Euler-maclaurin summation formula this result for the Harmonic series can be used to give an approximate regularized value of the logarithmic integral $\int_0^{\infty} dx \frac{1}{x+a}$, for the case of other types of divergent integrals $\int_0^{\infty} dx \left(x+a\right)^m$ we can use again Euler-Maclaurin summatio formula to express this divergent integrals in terms of the negative values of the Hurwitz or Riemann Zeta function $\zeta_H(s,1) = \zeta(s)$ $\zeta_H(-m,1)$ (UV) m= 0,1,2,3,4,..........and the value of the derivative of Hurwitz zeta function along s =0 $\partial_s \zeta_H(0,a)$ (logarithmic UV), these values encode the UV divergences [11] .

For the case of the IR (infrared) divergences in the form $\int_0^{\infty} \frac{dx}{x^{m-s}}$ one could make a change of variable $x \to \frac{1}{q}$ to re-interpretate these integrals as $\int_0^{\infty} q^{m-2-s} dq$, for the case m=1 we have a logarithmic divergence both at x= 0 an as $x \to \infty$ so we must split the integral into a IR and an UV divergent part $\int_0^{\infty} \frac{dx}{x} = \int_0^{1/a} \frac{dx}{x} + \int_{1/a}^{\infty} \frac{dx}{x}$ after a few simple calculations this integral will be equal to $\int_0^{\infty} \frac{dx}{x}$ =$2\log \mu$, since we can simply introduce a formal UV and IR regulator so $\lim_{\Lambda \to \infty} \int_{\Lambda^{-1}}^{\Lambda} \frac{dx}{x}$ =$2\log\left(\Lambda_{UV}\right)$, an UV regulator is introduced to ensure that the integral will be convergent .

We also believe that a similar procedure can be applied to extend our Zeta regularization algorithm to multiple (multi-loop) integrals $\int d^4q_1 \int d^4q_2 ....\int d^4q_n F(q_1, q_2, ..., q_n)$

The advantages of zeta regularization are

- Zeta regularization gives only finite results without counterterms if we use the analytic continuation of Riemann zeta $\sum_{n=1}^{\infty} n^{m-s} = \zeta(s-m)$ and the regularizatio of the Harmonic series $\sum_{n=0}^{\infty} \frac{1}{(n+a)} =_{reg} -\frac{\Gamma'(a)}{\Gamma(a)}$
- Zeta regularization does not alter the dimension of space so we can work with Matrices $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ and Tensors ε_{ijk} , unlike dimensional regularization.
- Zeta regularization respect many of the symmetries of the system

- Zeta regularizatio is compatible with many other regularization methods: Pauli-Villars, Cut-off regularization...
- Zeta regularization is valid also in curved spacetimes.
- For the case of IR divergences we can make the change of variable $x = u^{-1}$ so the IR divergent integral $\int_0^1 \frac{dx}{x^{m-s}}$ turns into $\int_1^\infty u^{m-2-s} du$, for the logarithmic integral we get $\int_0^\infty \frac{dx}{x} = 0$ in the sense of zeta regularization.
- In the Zeta regularization method these 2 equations are equivalent $\int_0^\infty \frac{dx}{x+a} =_{reg} -\log a$ and $\sum_{n=0}^{\infty} \frac{1}{(n+a)} =_{reg} -\frac{\Gamma'(a)}{\Gamma(a)}$ this may be proved using the Euler-Maclaurin formula
- If we use Numerical methods we can extend this zeta regularization algorithm to multiple loop integrals , see eqs. (20-29)
- Elizalde and others have proved that Zeta regularization is well defined [5] , also through history , zeta regularization has given correct values in several branches of math and physics , for example in the Casimir effect. [10]

In both cases zeta regularization and dimensional regularization are based on the principle of analytic continuation, however some people still have prejudices against zeta regularization and it is not commonly used , except for the evaluation of functional determinants. Dimensional regularization is just Zeta regularization in disguise, in fact when after calculations they take the limit $d = 4 - \varepsilon$ as $\varepsilon \to 0$ they find the expression $\frac{1}{\varepsilon} + \gamma + O(\varepsilon)$, the Euler-Mascheroni constant appears in both regularization although in zeta regularization we do not have the pole $1/\varepsilon$ at the end of the calculations.

The imposition in formula (2) that ‘a’ must be a natural number is in order to avoid oddities in the process of Zeta regularization with the Zeta and Hurwitz Zeta function, since unless ‘a’ is a positive integer the equality

$$\zeta(-1, a) = \sum_{n=0}^{\infty} (n+a) \neq \sum_{n=0}^{\infty} n + \sum_{n=0}^{\infty} a = \frac{-1}{12} - \frac{a}{2} \qquad (30)$$ does not hold .

Equation (30) must be understood as $\sum_{n=0}^{\infty} (n+a) = \sum_{n=a}^{\infty} n = \zeta(-1, a) - \sum_{n=0}^{a} n$ so zeta regularization is consistent and yields to correct results.

Another advantage is that we get only finite quantities, whereas in dimensional regularization you will always find poles of the Gamma function $\Gamma(z) = \frac{1}{z} + \gamma + O(z)$ in the limit $z \to 0$ this expression blows up, and need a counterterm to turn it finite.

The main advantage of our Zeta regularization method is that due to formula (3) and the regularized identity for the Digamma function $\sum_{n=0}^{\infty} \frac{1}{(n+a)} =_{reg} -\frac{\Gamma'(a)}{\Gamma(a)}$, the relationship between dimensional regularization and dimensional regularization is recovered if we use the following definition for the logarithmic divergent integral

$$\int_{1}^{\infty}\frac{dx}{x^{1+s}}=\Psi(1)+\frac{1}{s}+\log(4\pi)+2\log\mu \quad s\to 0 \qquad (31)$$

Where we have introduced the Energy scale μ.

APPENDIX A: HOW TO OVERCOME THE POLE $\zeta(1)=\infty$

In this paper we have seen how due to the pole of the Riemann zeta at the point s = 1 we could not regularize the integral $\int_{0}^{\infty}\frac{dx}{x}$ unless we use the result for the Harmonic series $\sum_{n=0}^{\infty}\frac{1}{(n+a)} =_{reg} -\frac{\Gamma'}{\Gamma}(a)$ fro a > 0 and finite, then if we introduce this result inside the Euler-Maclaurin summation formula we can get finite results for $\int_{0}^{\infty}\frac{dx}{x}$.

Another alternative is to use the identity

$$1=e^{x}\sum_{n=0}^{\infty}\frac{(-x)^{n}}{n!} \qquad \int_{a}^{\infty}\frac{dx}{x}=\sum_{n=0}^{\infty}\frac{(-\alpha)^{n}}{n!}\int_{a}^{\infty}\frac{dx}{x^{1-\alpha}}\log^{n}(x) \quad (A.1)$$

In this case we can evaluate the integrals inside (A.1) by the Euler Maclaurin formula

$$\sum_{n=a+1}^{\infty}f(n)=\int_{a}^{\infty}f(x)dx-\frac{f(a)+f(\infty)}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(\infty)-f^{(2k-1)}(a)\right) \qquad (A.2)$$

Here The Bernoulli numbers on the last side of (A.2) are the coefficients of the Taylor expansion of $\frac{x}{e^{x}-1}$ if we insert inside (A.2) the expresión $f(x)=\frac{\log^{n}(x)}{a^{1-\alpha}}$

$$\int_{a}^{\infty}\frac{dx\log^{n}(x)}{x^{1-\alpha}}=\frac{\log^{n}(a)}{a^{1-\alpha}}+(-1)^{n}\zeta^{(n)}(1-\alpha)-\sum_{i=1}^{a}\frac{\log^{n}(i)}{i^{1-\alpha}}+\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}\frac{\partial^{2r-1}}{\partial u^{2r-1}}\left(\frac{\log^{n}(x+a)}{(x+a)^{s+1}}\right)_{x=0} \qquad (A.3)$$

Here α is an small non integer, so the zeta function and its derivatives $\zeta^{(n)}(1-\alpha)$ are FINITE

Another alternative is to look for a Pade or Rational approximation for the square root of 'x' for example.

$$\sqrt{x} \approx \frac{P(x)}{Q(x)} \qquad \sqrt{x+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 4^n} x^{-n+\frac{1}{2}} \quad x>1 \qquad (A.4)$$

In this case (A.4) we have the approximation $\int_a^{\infty} \frac{dx}{x^{3/2}} \frac{P(x)}{Q(x)} \approx \int_a^{\infty} \frac{dx}{x}$, now if we apply the formula

$$\int_a^{\infty} \frac{dx}{x^{3/2}} \left(\frac{P(x)}{Q(x)} - \sum_i c_i x^i - \frac{c_0}{x} \right) + \sum_i \int_a^{\infty} c_i x^{i-3/2} dx + c_0 \int_a^{\infty} \frac{dx}{x^{5/2}} \qquad (A.5)$$

Inside (A.5) now there are no logarithmic-divergent integrals , so the pole $\zeta(1)$ will not now appear

References

[1] Abramowitz, M. and Stegun, I. A. (Eds.). "Riemann Zeta Function and Other Sums of Reciprocal Powers." §23.2 in "*Handbook of Mathematical Functions*". New York: Dover, pp. 807-808, 1972.

[2] Berndt B. "Ramanujan's Notebooks: Part II " Springer; 1 edition (1993) ISBN-10: 0387941096

[3] Dowling Jonathan P, "The Mathematics of the Casimir Effect", *Math. Mag.* **62**, 324-331 (1989).

[4] Elizalde E. ; "*Zeta-function regularization is well-defined*", Journal of Physics **27** (1994), L299-304.

[5] Elizalde E. ; "*Zeta regularization techniques with applications* " Ed. World Scientific Press ISBN 981-02-1441-3

[6] Mizuno, Y. *"Generalized Lerch Formulas: Examples of Zeta-Regularized Products." J. Number Th.* **118**, 155-171, 2006.

[7] Muñoz García, E. and Pérez Marco, R. "The Product Over All Primes is $4\pi^2$." Preprint IHES/M/03/34. May 2003. http://inc.web.ihes.fr/prepub/PREPRINTS/M03/Resu/resu-M03-34.html.

[8] Newcomb W. *"Renormalization Methods: A Guide For Beginners* "Oxford University Press, USA; 1 edition (January 6, 2008) ISBN-10: 0199236526

[9] Pavel E *"Note on dimensional regularization*", *Quantum fields and strings: a course for mathematicians, Vol. 1,(Princeton, NJ, 1996/1997)*, Providence, R.I.: Amer. Math. Soc., pp. 597–607, ISBN 978-0-8218-2012-4 (1999)

[10] Prellberg Thomas, “ *The mathematics of the Casimir Effect”* a talk avaliable at e-print at http://www.maths.qmul.ac.uk/~tp/talks/casimir.pdf

[11] Voros A. *“ Zeta functions over zeros of zeta functions”* Ed. SPRINGER (2010) ISBN: 978-3-642-05202-6

[12] Zee, A. (2003).” *Quantum Field Theory in a Nutshell*. “Princeton University ISBN ISBN 0-691-01019-6.

[13] Zeidler E. *“ Quantum Field Theory I : Basis in Physics and Mathematics”* Ed : Springer Verlag ISBN: 3540347623