Orbital averages
and the secular variation of the orbits

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Orbital averages are employed to compute the secular variation of the elliptical planetary elements in the orbital plane in presence of perturbing forces of various kinds. They are also useful as an aid in the computation of certain complex integrals. An extensive list of computed integrals is given.

Keywords: Unperturbed motion, perturbation equations, third-body perturbations, other force laws, orbital averages, loop integrals.

Introduction

We employ orbital averages for the analytical and numerical determination of the secular part of the variation of the elements a, e, ω (semi-major axis, eccentricity, argument of perihelion) of an elliptic orbit due to perturbing forces. The derivations are developed thorough an averaging process of the first-order equations arising from the method of variation of the arbitrary constants. We shall use the formalism of complex variables, as we consider only the perturbations acting in the orbital plane. As a byproduct of our work we deduce some useful methods for the computation of certain awkward integrals related to the geometry of the ellipse.

I. UNPERTURBED MOTION

We begin with a short review, in the complex notation, of the principal formulas and results of the two-body problem employed in this work.

The position of a planet on the orbital plane we suppose lying on the complex plane, is given by the variable $\boldsymbol{r} = \boldsymbol{r}(t)$, where $\boldsymbol{r} = \boldsymbol{r}(t) = x(t) + iy(t)$. Then $r^2 = \boldsymbol{r}\bar{\boldsymbol{r}}$, $\bar{\boldsymbol{r}}$ being the complex conjugate of $\boldsymbol{r}$. The real and the imaginary parts of a complex number $\boldsymbol{r}$ are denoted by $\mathbb{R}(\boldsymbol{r}) = (\boldsymbol{r}+\bar{\boldsymbol{r}})/2$ and by $\mathbb{I}(\boldsymbol{r}) = i(\bar{\boldsymbol{r}} - \boldsymbol{r})/2$. Then $\mathbb{R}(\boldsymbol{r}) = \mathbb{R}(\bar{\boldsymbol{r}}) = \mathbb{I}(i\boldsymbol{r})$, $\mathbb{I}(\boldsymbol{r}) = -\mathbb{I}(\bar{\boldsymbol{r}}) = -\mathbb{R}(i\boldsymbol{r})$.

In polar coordinates $\boldsymbol{r} = r\,e^{i\theta} = r(\cos\theta + i\sin\theta)$, θ being the true longitude, measured from the arbitrary fixed axis x. The function $\boldsymbol{r}(t)$ will be known as soon as we found the time dependence of θ, so that $\boldsymbol{r}(t) = r[\theta(t)]\exp i\theta(t)$; we have also

$$\dot{\boldsymbol{r}} = \left(\frac{1}{r}\frac{dr}{dt} + i\dot{\theta}\right)\boldsymbol{r}. \tag{1}$$

If we write $\mu = k^2(M+m) \approx k^2 M$, where M is Sun's mass which we take as unity, k is Gauss' gravitational constant, the initial value problem

$$Z \equiv \ddot{\boldsymbol{r}} + \mu\frac{\boldsymbol{r}}{r^3} = 0, \quad \boldsymbol{r}(0) = \boldsymbol{r}_0, \quad \dot{\boldsymbol{r}}(0) = \dot{\boldsymbol{r}}_0, \tag{2}$$

is solved if are known four independent integrals of the motion, that can be found introducing the following integral operations on Z

$$\mathbf{C}(Z) \equiv \int dt\,\mathbb{I}(\bar{\boldsymbol{r}}Z) = 0, \tag{3}$$

$$\mathbf{H}(Z) \equiv \int dt\,\mathbb{R}(\dot{\bar{\boldsymbol{r}}}Z) = 0, \tag{4}$$

$$\mathbf{E}(Z) \equiv \int dt\,\mathbb{I}(\boldsymbol{r}\dot{\bar{\boldsymbol{r}}})\,i\,Z = 0. \tag{5}$$

We so easily obtain three constant functions, two real and one complex. They are

1. **Area integral**: $\mathbf{C}(Z) = 0 \to \mathbb{I}(\dot{\boldsymbol{r}}\bar{\boldsymbol{r}}) = i(\dot{\bar{\boldsymbol{r}}}\boldsymbol{r} - \dot{\boldsymbol{r}}\bar{\boldsymbol{r}})/2 = r^2\dot{\theta} = \mathsf{c} =$ real const. It follows: $dt = r^2 d\theta/\mathsf{c}$.
2. **Energy integral**: $\mathbf{H}(Z) = 0 \to |\dot{\boldsymbol{r}}|^2/2 - \mu/r = \mathsf{h}$, $\mathsf{h} = -\mu/2a$, $\dot{\boldsymbol{r}}\dot{\bar{\boldsymbol{r}}} = \mu\,(2/r - 1/a)$.
3. **Eccentricity vector**: $\mathbf{E}(Z) = 0 \to \dot{\boldsymbol{r}} = (i\mu/\mathsf{c})(\boldsymbol{r}/r + \mathbf{e})$, $\mathbf{e} \equiv \mathrm{e}\exp(i\omega)$, the *eccentricity vector*, is a complex constant, e is the *eccentricity* and ω is the *argument of perihelion.*
4. **Elliptic orbit** in terms of the true longitude: $I(\dot{\boldsymbol{r}}\bar{\boldsymbol{r}}) = \mathsf{c} \to \boldsymbol{r} = r\,e^{i\theta} = (\mathsf{c}^2/\mu)\,e^{i\theta}[1 + \mathrm{e}\cos(\theta - \omega)]^{-1}$, a is the semi-major axis, $\mathsf{c}^2 = \mu a(1 - e^2)$, $f \equiv \theta - \omega$ is the *true anomaly*, $e^{i\theta} = e^{i\omega}e^{if}$, $df = d\theta$. Elliptic orbit in terms of the eccentric anomaly η: $\boldsymbol{r} = \boldsymbol{r}(\eta) = (a\cos\eta + ib\sin\eta - a\mathrm{e})\,e^{i\omega}$ $r = \sqrt{\boldsymbol{r}\bar{\boldsymbol{r}}} = a(1 - \mathrm{e}\cos\eta)$ (origin of coordinates at the *center* of the ellipse), $r d\eta = an\,dt$.
5. **Third law**: $cT = \int_0^T c\,dt = \mathbb{I}\left(\int_0^T \dot{\boldsymbol{r}}\bar{\boldsymbol{r}}\,dt\right) = \mathbb{I}\left(\oint \bar{\boldsymbol{r}}\,d\boldsymbol{r}\right) = \mathbb{I}\left(\int_0^{2\pi}\bar{\boldsymbol{r}}(\eta)\,d\boldsymbol{r}(\eta)\right) = 2\pi ab \implies T^2/a^3 = 4\pi^2/\mu$.
6. **Other definitions**: $b = a\sqrt{(1-\mathrm{e}^2)}$, $n^2 \equiv \mu/a^3$ (n is the *mean motion*), $T = 2\pi/n$ is the *period* of motion. The mean longitude is defined as $\ell \equiv nt$.

II. ORBITAL AVERAGES

In presence of perturbations, each orbital element $E_i \equiv a, \mathsf{c}, \mathbf{e}$, becomes a function of time, and the perturbation equations are first-order equations for the elements of the form

$$\dot{E}_i = g\left(\boldsymbol{r}, \dot{\boldsymbol{r}}, \bar{\mathbf{r}}, \dot{\bar{\mathbf{r}}}, t\right). \tag{6}$$

To obtain the *secular* part of the perturbations of the elements we average with respect to time the right-hand side of the perturbation equations, and obtain so the secular values $\left\langle \dot{E}_i \right\rangle$. For this, we need to know the mean value of some orbital variables and functions over the unperturbed motion, where with the word *orbital* we mean periodicity sharing the same period of the elliptical motion. By definition, the temporal average of a periodic function $g(t)$ over the periodicity interval $(0,T)$ is

$$g(t)_{av} \equiv \langle g(t) \rangle = \frac{1}{T}\int_0^T g(t)\,dt. \tag{7}$$

If $g(t)$ is a total derivative of a periodic function $h(t)$,

$$g(t) = \frac{dh(t)}{dt}, \quad \text{then} \quad g(t)_{av} = \frac{h(T)-h(0)}{T} = 0,$$

because of the periodicity of motion. If g is a constant, then $\langle g \rangle = g$. The temporal averages can be also calculated by means of angular variables in the range $[0, 2\pi]$ employing the following relations:

$$\frac{dt}{T} = \frac{ndt}{2\pi} = \frac{rd\eta}{2\,\pi\,a} = \frac{r^2 df}{2\,\pi\,a\,b} = \frac{d\ell}{2\pi}, \tag{8}$$

applied respectively to the functions $g(t)$, $g(\eta)$, $g(f)$, $g(\ell)$.

In the following we shall give properties, methods of calculation and averages of the orbital functions we need to know. We shall use the notation F_{av} to indicate the average force exerted by the perturbing planet, while the notation $\langle F_{av} \rangle$ is reserved to the successive average with respect to the orbit of the perturbed planet.

III. THE PLANETARY EQUATIONS

When the motion is perturbed, the equation of motion becomes

$$\ddot{\mathbf{r}} = -\mu\frac{\boldsymbol{r}}{r^3} + F, \tag{9}$$

where $F = F(\boldsymbol{r}, \bar{\mathbf{r}}, \dot{\boldsymbol{r}}, \dot{\bar{\mathbf{r}}}, t)$ is the perturbing force in the plane xy. The force F is central if $F = g(r)\,\boldsymbol{r}$, where $g(r)$ is a real function of r. It is assumed that F is of small magnitude as compared to the Keplerian term. Therefore, the planet moves on a weakly perturbed elliptic orbit. The time scales of variation of its elements are a few orders of magnitude longer than the orbital period. Hence, one might perform the averaging of the quantities of interest over fast evolution, the mean anomaly, or any other angular variable according to the relations (8). The planet will move along a variable orbit which at every instant t can be described as an osculating ellipse, in which the orbital elements are supposed slowly changing with the time. Mathematically this concept can be treated with the method of variation of arbitrary constants that can be reduced to action, on the generic integral of the motion $E_i = E_i(\boldsymbol{r}, \bar{\mathbf{r}}, \dot{\boldsymbol{r}}, \dot{\bar{\mathbf{r}}})$, of the differential operator d/dt

$$\dot{E}_i = \frac{\mathsf{d}E_i}{\mathsf{dt}} \equiv \frac{\partial E_i}{\partial \dot{\boldsymbol{r}}}\frac{d\dot{\boldsymbol{r}}}{dt} + \frac{\partial E_i}{\partial \dot{\bar{\mathbf{r}}}}\frac{d\dot{\bar{\mathbf{r}}}}{dt} = F\,\frac{\partial E_i}{\partial \dot{\boldsymbol{r}}} + \bar{F}\,\frac{\partial E_i}{\partial \dot{\bar{\mathbf{r}}}}, \tag{10}$$

which means to consider each element as variable and to perform the ordinary time derivatives of the integrals of the motion with the convention that[1]

$$\frac{d\boldsymbol{r}}{dt} = 0, \qquad \frac{d\dot{\boldsymbol{r}}}{dt} = F, \tag{11}$$

for the perturbed motion, evidencing so only the accelerations produced by the perturbing forces. Thus we find, from the integrals

$$\mathsf{c} = \mathbb{I}(\dot{\boldsymbol{r}}\bar{\mathbf{r}}), \tag{12}$$

$$-\frac{1}{a} = \frac{1}{\mu}\,\dot{\boldsymbol{r}}\dot{\bar{\mathbf{r}}} + \frac{2}{r}, \tag{13}$$

$$\mathbf{e} = -\frac{i\,\mathsf{c}}{\mu}\left(\dot{\boldsymbol{r}} - \frac{\boldsymbol{r}}{r}\right), \tag{14}$$

the following expressions of the planetary equations in the plane

$$\dot{\mathsf{c}} = \mathbb{I}(F\bar{\mathbf{r}}), \tag{15}$$

$$\dot{a} = \frac{2\,a^2}{\mu}\,\mathbb{R}(F\dot{\bar{\mathbf{r}}}), \tag{16}$$

$$\dot{\mathbf{e}} = -\frac{i}{\mu}\left(\mathsf{c}\,F + \dot{\boldsymbol{r}}\,\mathbb{I}(F\bar{\mathbf{r}})\right), \tag{17}$$

were we can put

$$\dot{\boldsymbol{r}} = \frac{i\,n\,a^2}{b}\left(\frac{\boldsymbol{r}}{r} + \mathbf{e}\right), \qquad \dot{\bar{\mathbf{r}}} = -\frac{i\,n\,a^2}{b}\left(\frac{\bar{\mathbf{r}}}{r} + \bar{\mathbf{e}}\right). \tag{18}$$

In the right-hand side of these equations, in the first-order of approximation, all the elements are considered as constants. The equation for $\dot{\mathsf{c}}$ is useful in the treatment of central perturbing forces, because then $\dot{\mathsf{c}} = \mathbb{I}(F\bar{\mathbf{r}}) = 0$, whereby the first equation for $\dot{\mathbf{e}}$ is simplified to

$$\dot{\mathbf{e}} = \left(\frac{\dot{\mathrm{e}}}{\mathrm{e}} + i\dot{\omega}\right)\mathbf{e} = -\frac{i\,\mathsf{c}}{\mu}\,F \Longrightarrow \frac{\dot{\mathrm{e}}}{\mathrm{e}} + i\dot{\omega} = -\,i\frac{\mathsf{c}}{\mu\,\mathbf{e}}\,F. \tag{19}$$

Equating separately the real and the imaginary part we get

$$\dot{\mathrm{e}} = -\frac{\mathsf{c}\,\mathrm{e}}{\mu}\,\mathbb{R}\left(\frac{i\,F}{\mathbf{e}}\right) = \frac{\mathsf{c}\,\mathrm{e}}{\mu}\,\mathbb{I}\left(\frac{F}{\mathbf{e}}\right), \tag{20}$$

$$\dot{\omega} = -\frac{\mathsf{c}}{\mu}\,\mathbb{I}\left(\frac{iF}{\mathbf{e}}\right) = -\frac{\mathsf{c}}{\mu}\,\mathbb{R}\left(\frac{F}{\mathbf{e}}\right). \tag{21}$$

If the force F is central, then $\langle F \rangle = K\mathbf{e}$, with K real constant. From this we deduce that

$$\langle \dot{\mathrm{e}} \rangle = -\frac{\mathsf{c}\,\mathrm{e}}{\mu}\,\mathbb{I}\left(\frac{\langle F \rangle}{\mathbf{e}}\right) = -\frac{\mathsf{c}\,\mathrm{e}}{\mu}\,\mathbb{I}(K) = 0, \tag{22}$$

so that we can write

$$\langle \dot{\mathbf{e}} \rangle = i\,\langle \dot{\omega} \rangle\,, \qquad \langle \dot{\omega} \rangle = -\frac{\mathsf{c}}{\mu\,\mathbf{e}}\,\langle F \rangle = -\frac{\mathsf{c}}{\mu}\,K. \tag{23}$$

In this circumstance the eccentricity vector $\mathbf{e}$ rotates uniformly about the origin in the $\mathbf{r}$-plane, with a constant length. In general we can write, by averaging,

$$\langle \dot{a} \rangle = \frac{2\,a^2}{\mu\,T}\,\mathbb{R}\left(\oint F\,d\bar{\mathbf{r}}\right) = \frac{n\,a^2}{\pi\,\mu}\,\mathbb{R}\left(\oint F\,d\bar{\mathbf{r}}\right), \tag{24}$$

$$\begin{aligned}\langle \dot{\mathbf{e}} \rangle &= -\frac{i\mathsf{c}}{\mu}\,\langle F \rangle - \frac{i}{\mu\,T}\oint I(F\bar{\mathbf{r}})\,d\boldsymbol{r}\\ &= -\frac{i\mathsf{c}}{\mu}\,\langle F \rangle - \frac{i\,n}{2\pi\mu}\oint I(F\bar{\mathbf{r}})\,d\boldsymbol{r},\end{aligned} \tag{25}$$

in which there appear closed contour or loop integrals[2] over the unperturbed ellipse. From Eq. (24) we see that if F is central then $\langle \dot{a} \rangle$ doesn't have secular terms, because the integral is a pure imaginary number.

IV. SECULAR PERTURBATIONS

A. Third-body perturbations

Suppose that another planet P' is moving on a coplanar orbit around the Sun in the hypothesis that the system P, P' be *non-resonant*, so that the respective mean motions are non-commensurable. If we denote with $\boldsymbol{r}, \boldsymbol{r}'$ the position vectors of P and P', and with μ' the quantity $k^2 m'$, being m' the mass of P', the perturbing force on P is given by[3–5]

$$F = \mu'\left(\frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} - \frac{\mathbf{r}'}{r'^3}\right). \tag{26}$$

The first term is the *direct* force of P' on P, while the second term is the inertial *indirect* force due to the choice of the Sun as origin of the reference frame.[5] The determination of the secular perturbations requires a double averaging process: first, we average the perturbing force with respect to P' and obtain thus F_{av}, after we average respect to P the perturbation equations for $\dot{a}$, $\dot{\mathbf{e}}$ after the substitution of F with F_{av}, obtaining thus $\langle \dot{a} \rangle$, $\langle \dot{\mathbf{e}} \rangle$. This procedure is allowed for the first-order perturbations, because $\boldsymbol{r}$ does not contain terms depending from P'. Notice that the indirect term of the force gives a null contribute to any secular perturbation, since $\left\langle \mathbf{r}' r'^{-3} \right\rangle = 0$.

We have, by definition

$$F_{av} = \frac{\mu'}{2\pi a' b'}\int_0^{2\pi} \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3}\,r'^2\,df' = \frac{\mu'}{T}\int_0^T \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3}\,dt, \tag{27}$$

and it follows *Gauss' theorem*, because with

$$d\mu' = \mu'\,\frac{r'^2\,df'}{2\,\pi\,a'\,b'} = \mu'\,\frac{dt}{T}, \tag{28}$$

we can write

$$F_{av} = \oint \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3}\,d\mu',$$

where the integral is taken along the ellipse of the disturbing body in the direction of motion. So the problem is reduced to that of determining the secular orbital effects of a massive ring whose elementary distribution of mass has the measure given by Eq. (28).[6–8] For the analytical determination of this force, we must approximate the irrational factor

$$\begin{aligned}|\mathbf{r}' - \mathbf{r}|^{-3} &= \left(\mathrm{r}^2 + \mathrm{r}'^2 - 2\mathrm{r}\mathrm{r}'\cos(\theta' - \theta)\right)^{-3/2}\\ &= \mathrm{r}'^{-3}\left(1 + \frac{\mathrm{r}^2}{\mathrm{r}'^2} - 2\frac{\mathrm{r}}{\mathrm{r}'}\cos(\theta' - \theta)\right)^{-3/2}\\ &\equiv \mathrm{r}'^{-3}\Delta_1^{-3/2}, \qquad \mathrm{r}' > \mathrm{r}, \qquad (29)\\ &\equiv \mathrm{r}^{-3}\Delta_2^{-3/2}, \qquad \mathrm{r} > \mathrm{r}', \qquad (30)\end{aligned}$$

where $\Delta_2^{-3/2}$ is as $\Delta_1^{-3/2}$, but with r and r' interchanged.

We develop these expressions in powers of the ratios r/r' or r'/r. This requires, when applied to planetary perturbations, a great number of terms for an acceptable convergence, but the hard work is done by computer algebra. We can also write $\Delta_i^{-3/2} = \Delta_i^{-1}\Delta_i^{-1/2}$ and expand in powers of r/r' or r'/r only the second factor. These two choices, when applied to the same problem, are equivalent, but they codify differently the information on the perturbing force in their succession of terms. For our purposes it is more advantageous to use the first option in the later treatment of an elliptical perturbing orbit, while for the circular orbit we shall use the second one, that has the advantage to give more compact expressions.

B. Average force of a planet on a circular orbit

Let be a planet P' of mass m' in a circular orbit, so that $\mathbf{r}' = a'e^{i\theta'} = a'e^{in't}$, where n' is the constant *mean motion*. Consider the force F^e on the point $\boldsymbol{r} = \mathrm{r}e^{i\theta}$ of the orbit of an internal planet P lying in the same plane. With the notation

$$\alpha = \mathrm{r}/a' < 1, \qquad \Delta_\alpha = 1 + \alpha^2 - 2\alpha\cos\phi, \tag{31}$$

$$\phi = \theta' - \theta, \qquad d\phi = d\theta', \tag{32}$$

we have

$$F^e = \mu'\,\frac{\left(a'e^{i\theta'} - \mathbf{r}\right)}{|a'e^{i\theta'} - \mathbf{r}|^3} = \mu'\left(\frac{e^{i\phi}\,e^{i\theta}}{a'^2} - \frac{\mathbf{r}}{a'^3}\right)\frac{1}{\Delta_\alpha^{3/2}}. \tag{33}$$

Therefore, averaging with respect to P'

$$\begin{aligned}F^e_{av} &= \frac{1}{2\pi a' b'}\int_0^{2\pi} F^{ex}\,\mathrm{r}'^2 d\theta' = \frac{1}{2\pi}\int_0^{2\pi} F^{ex}\,d\phi\\ &= \frac{\mu'}{2\pi}\int_0^{2\pi} \frac{d\phi}{\Delta_\alpha^{3/2}}\left(\frac{\mathbf{r}}{\mathrm{r}}\,\frac{e^{i\phi}}{a'^2} - \frac{\mathbf{r}}{a'^3}\right).\end{aligned} \tag{34}$$

In this integral we write

$$F^e_{av} = \frac{\mu'}{2\pi}\int_0^{2\pi}\frac{d\phi}{\Delta_\alpha}\left(\frac{\mathbf{r}}{r}\frac{e^{i\phi}}{a'^2} - \frac{\mathbf{r}}{a'^3}\right)\frac{1}{\Delta_\alpha^{1/2}}$$
$$\approx \frac{\mu'}{2\pi a'^2}\frac{\mathbf{r}}{r}\int_0^{2\pi}\frac{d\phi}{\Delta_\alpha}\left(e^{i\phi} - \alpha\right)\times$$
$$\times\left(1 + \alpha\cos\phi + \frac{\alpha^2}{4} + \frac{3}{4}\alpha^2\cos 2\phi + \dots\right). \quad (35)$$

From the formula

$$\int_0^{2\pi}\frac{e^{\pm in\phi}}{\Delta_\alpha} = \int_0^{2\pi}\frac{\cos n\theta}{\Delta_\alpha} = \frac{2\pi\alpha^n}{1-\alpha^2}, \quad (36)$$

with $\alpha < 0,\ n = 0, 1, 2, \dots$ we see that to order α^5 in the numerator only the following terms contribute to the total mean force

$$F^e_{av} = \frac{\mu'}{2\pi a'^2}\frac{\mathbf{r}}{r}\int_0^{2\pi}\frac{d\phi}{\Delta_\alpha}\left[\left(1 - \frac{3}{8}\alpha^2 - \frac{5}{64}\alpha^4\right)\cos\phi\right.$$
$$+\left(\frac{\alpha}{2} - \frac{\alpha^3}{4}\right)\cos 2\phi + \frac{3}{8}\alpha^2\cos 3\phi$$
$$\left. - \frac{\alpha}{2} - \frac{\alpha^3}{16} - \frac{3}{128}\alpha^5\right] \quad (37)$$

so that, after the integration we have

$$F^e_{av} = \frac{\mu'}{a'^2}\left(\frac{\alpha\left(64 + 8\alpha^2 + 3\alpha^4\right)}{128\left(1-\alpha^2\right)}\right)\frac{\mathbf{r}}{r} \quad (38)$$
$$= \frac{\mu'}{128\,a'^2}\left(\frac{75\,\alpha}{(1-\alpha^2)} - 11\,\alpha - 3\,\alpha^3\right)\frac{\mathbf{r}}{r} \quad (39)$$
$$= \frac{\mu'}{128\,a'}\left(\frac{75\,r}{(a'^2 - r^2)} - \frac{11r}{a'^2} - \frac{3r^3}{a'^4}\right)\frac{\mathbf{r}}{r} \quad (40)$$

It easily seen that also to higher approximations F^e_{av} retains the same general structure with the same first term and more terms with higher odd powers of the ratio r/a'. This averaged perturbing force is *central*, so that we can employ the simplified perturbation equation for $\dot\omega$. As value of the radius a' of the circular orbit we take the semi-major axis of the true elliptical orbit of P' because the average value of the modulus of the radius vector r' is a' to order e'. We find

$$\langle\mathbf{r}\rangle = -\frac{3}{2}\,a\,\mathbf{e} \equiv \mathbb{A}\,\mathbf{e}, \quad (41)$$
$$\langle\mathbf{r}r^2\rangle = -\frac{5}{8}\,a^3\left(4 + 3\,\mathrm{e}^2\right)\mathbf{e} \equiv \mathbb{B}\,\mathbf{e}, \quad (42)$$
$$\left\langle\frac{\mathbf{r}}{(a'^2 - r^2)}\right\rangle = \frac{a(1-\mathrm{e}^2)(A-B) - a'(A+B) + 2AB}{2a\,\mathrm{e}^2\,AB}\,\mathbf{e} \quad (43)$$
$$\equiv \mathbb{C}\,\mathbf{e}, \quad (44)$$

where

$$A \equiv \sqrt{(a+a')^2 - a^2\mathrm{e}^2}, \qquad B \equiv \sqrt{(a-a')^2 - a^2\mathrm{e}^2}. \quad (45)$$

A is real for every positive value of a', while B is real for a' outside a definite interval, that, for the Earth, is (0.9833... , 1.0167...), where the formula is meaningless. We have then to order α^5

$$\langle F^e_{av}\rangle = \frac{\mu'}{128}\left(\frac{75}{a'}\,\mathbb{C} - \frac{11}{a'^3}\,\mathbb{A} - \frac{3}{a'^5}\,\mathbb{B}\right)\mathbf{e}, \quad (46)$$

and so

$$\langle\dot\omega\rangle'' = -\kappa\frac{\mathsf{c}}{\mu\,\mathbf{e}}\,\langle F^e_{av}\rangle\,, \quad (47)$$

in which κ is a numerical factor for the conversion from *radian/day* to *arcsec/century* given by

$$\kappa = \frac{365.25\times 100\times 180\times 3600}{\pi} \approx 7.533822048\times 10^9. \quad (48)$$

Then, recalling that $\mu'/\mu = m'$, the centennial precession rate is

$$\langle\dot\omega\rangle'' = -\kappa\frac{m'\mathsf{c}}{128\,a'}\left(\frac{11}{a'}\,\mathbb{A} + \frac{3}{a'^2}\,\mathbb{B} + \frac{75}{a'^4}\,\mathbb{C}\right). \quad (49)$$

If we expand F^e_{av} in powers of α, we have

$$F^e_{av} = \mu'\boldsymbol{r}\left(\frac{1}{2\,a'^3} + \frac{9\,r^2}{16\,a'^5} + \frac{75\,r^4}{128\,a'^7} + \dots\right), \quad (50)$$

the first term it is proportional only to the position vector $\boldsymbol{r}$. Thus an *external* planet exerts an approximately linear *repulsive* force, directed away from the center, on a particle located somewhere near the center of the orbit. We have to push the approximation as far as the term $\boldsymbol{r}r^{10}$ because of the relative greatness of the ratio r/a' for the internal planets Mars, Earth, Venus and Mercury. In the literature[9] was obtained with another method the approximation (see Eq. (38))

$$F^e_{av} = \frac{\mu'}{2a'}\frac{\mathbf{r}}{(a'^2 - r^2)}, \quad (51)$$

for a numerical evaluation of the classical part of the motion of Mercury's perihelion. To this regard we note that the computation can be done analytically since we know that in this case by Eqs. (43), (47) we get

$$\langle\dot\omega\rangle'' = -\kappa\frac{\mathsf{c}}{\mu\,\mathbf{e}}\,\langle F^e_{av}\rangle = -\frac{\kappa\mathsf{c}}{2\mu}\sum_i\frac{\mu'_i}{a'}\,\mathbb{C}_i, \quad (52)$$

with the obvious meaning of the symbolism. We find so the value of 532.53", very near to the exact value obtained by treating the problem in its full generality. Obviously this is only a coincidence, due to an almost exact compensation among the various planets, but it leaves the false impression that the omitted terms do not destroy this excellent agreement. With the more complete expression of F^e_{av} we obtain the value of 553.97". The difference is due to the fact that we have neglected the ellipticity of the orbits of the disturbing planets, other that the mutual inclination of orbits, which in this case is rather significant.

We consider now the situation in which the orbit of P' is internal to that of P. With

$$\Delta_\beta = 1+\beta^2-2\beta\cos\phi, \quad \beta=\alpha^{-1}=a'/r<1, \tag{53}$$

we have, at the order β^3

$$\begin{aligned}
F^i_{av} &= \frac{\mu'}{2\pi}\frac{\mathbf{r}}{r^3}\int_0^{2\pi}\frac{d\phi}{\Delta_\beta^{3/2}}\left(\beta\, e^{i\phi}-1\right)\\
&= \frac{\mu'}{2\pi}\frac{\mathbf{r}}{r^3}\int_0^{2\pi}\frac{d\phi}{\Delta_\beta}\left(\beta\, e^{i\phi}-1\right)\frac{1}{\Delta_\beta^{1/2}}\\
&\approx \frac{\mu'}{2\pi}\frac{\mathbf{r}}{r^3}\int_0^{2\pi}\frac{d\phi}{\Delta_\beta}\left(\beta\, e^{i\phi}-1\right)\left(1+\beta\cos\phi+\frac{\beta^2}{4}\right)\\
&\approx \frac{\mu'}{2\pi}\frac{\mathbf{r}}{r^3}\int_0^{2\pi}\frac{\beta^2/4-1}{\Delta_\beta}\,d\phi = \mu'\,\frac{\mathbf{r}}{r^3}\frac{\beta^2-4}{4(1-\beta^2)}\\
&= -\mu'\,\frac{\mathbf{r}}{r^3}\frac{a'^2-4r^2}{4(a'^2-r^2)} = -\mu'\,\frac{\mathbf{r}}{r^3}\frac{(a'^2-r^2)-3r^2}{4(a'^2-r^2)}\\
&= -\frac{\mu'}{4}\frac{\mathbf{r}}{r^3}+\frac{3}{4}\mu'\frac{\mathbf{r}}{r}\frac{1}{(a'^2-r^2)}.
\end{aligned} \tag{54}$$

By developing in powers of β we find

$$F^i_{av} = -\mu'\frac{\mathbf{r}}{r^3}+\frac{3}{4}\,\mu' a'^2\frac{\mathbf{r}}{r^5}+\dots. \tag{55}$$

Now we have ($m'\equiv\mu'/\mu$)

$$\langle\dot\omega\rangle = -\kappa\frac{\mathsf{c}}{\mu\,\mathbf{e}}\left\langle F^i_{av}\right\rangle = -\kappa\frac{3}{4}\,m'\mathsf{c}\left\langle\frac{e^{i\theta}}{a'^2-r^2}\right\rangle, \tag{56}$$

where

$$\left\langle\frac{e^{i\theta}}{a'^2-r^2}\right\rangle = \frac{a(1-e^2)(A+B)-a'(A-B)}{2a\,a'e^2\,AB}\,\mathbf{e}\equiv\mathbb{D}\,\mathbf{e}, \tag{57}$$

with A, B defined as before, so that

$$\langle\dot\omega\rangle''_{int} = -\kappa\frac{3}{4}\,m'\mathsf{c}\,\mathbb{D}. \tag{58}$$

C. Average force of a planet in an elliptical orbit

Let us suppose the orbit of P' elliptical. Then F_{av} depends also from the mutual geometrical disposition of the two orbits, i.e. from the angular distance $\omega'-\omega$ of the respective perihelia.

If P' is external, with $\psi=f'-f+\omega'-\omega$, $\gamma=r/r'$, we have

$$\begin{aligned}
F^e &= \mu'\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} = \frac{\mu'(\mathbf{r}'-\mathbf{r})}{(r^2+r'^2-2\,rr'\cos\psi)^{3/2}}\\
&= \frac{\mu'(\mathbf{r}'-\mathbf{r})}{r'^3\,(1+\gamma^2-2\gamma\cos\psi)^{3/2}} \approx \frac{\mu'(\mathbf{r}'-\mathbf{r})}{r'^3}\Bigg(1+\frac{9}{4}\gamma^2\\
&\quad+\frac{225}{64}\gamma^4+\dots+3\gamma\cos\psi+\frac{15}{4}\gamma^2\cos 2\psi+\dots\Bigg)\\
&= \mu'(\mathbf{r}'-\mathbf{r})\sum_j H_j\cos j\psi, \qquad j=0,1,2\dots
\end{aligned} \tag{59}$$

where the coefficients H_j are power series in γ, given, with $\nu=3/2$, by

$$H_0 = \frac{1}{r'^3}\left(1+\nu^2\gamma^2+\nu'^2\gamma^4+\dots\right), \tag{60}$$

$$H_1 = \frac{2}{r'^3}\left(\nu\,\gamma+\nu\,\nu'\gamma^3+\nu'\nu''\gamma^5+\dots\right), \tag{61}$$

$$H_2 = \frac{3}{r'^3}\left(\nu'\gamma^2+\nu\,\nu''\gamma^4+\nu'\nu'''\gamma^4+\dots\right), \tag{62}$$

$$H_3 = \dots \tag{63}$$

with

$$\nu' = \frac{\nu\,(\nu+1)}{2!}, \quad \nu'' = \frac{\nu\,(\nu+1)(\nu+2)}{3!},\dots \tag{64}$$

By developing the above expression of F^e after the substitution

$$\cos j\psi = \frac{\mathbf{r}'^j\,\bar{\mathbf{r}}^j+\bar{\mathbf{r}}'^j\,\mathbf{r}^j}{2r'^j r^j}, \tag{65}$$

we can write the force as the sum

$$\frac{F^e}{\mu'} = F^0+F^1+F^2+F^3+\dots, \tag{66}$$

where

$$F^0 = \sum_j h^0_j\,\mathbf{r}r^{2j}\frac{1}{r'^{2j+3}}, \qquad F^1 = \sum_j h^1_j\,r^{2j}\frac{\mathbf{r}'}{r'^{2j+3}}, \tag{67}$$

$$F^2 = \sum_j h^2_j\,\mathbf{r}^2r^{2j}\frac{\bar{\mathbf{r}}'}{r'^{2j+5}}, \qquad F^3 = \sum_j h^3_j\,\bar{\mathbf{r}}r^{2j}\frac{\mathbf{r}'^2}{r'^{2j+5}}, \tag{68}$$

$$F^4 = \sum_j h^4_j\,\mathbf{r}^3r^{2j}\frac{\bar{\mathbf{r}}'^2}{r'^{2j+7}}, \qquad F^5 = \sum_j h^5_j\,\mathbf{r}^2r^{2j}\frac{\mathbf{r}'^3}{r'^{2j+7}}, \tag{69}$$

and so on. The coefficients h^i_j $(i,h=0,1,2\dots)$ are rational numbers. After the average with respect to P', F^e_{av} will have terms proportional to

$$\frac{\bar{\mathbf{e}}'^k}{a'^{2j+n-k}}\,\mathbf{r}^i r^{2j}, \qquad \frac{\mathbf{e}'^k}{a'^{2j+n-k}}\,\bar{\mathbf{r}}^i r^{2j}.$$

where $n=3,5,7\dots$. From this we are lead to the following conclusions as regard to F^e_{av}:

- All terms go to zero as the ratio $\frac{a^{i+2j}}{a'^{2j+n-k}}$ for $i,j,k,n\to\infty$.
- F^0_{av} is a force of the central type that coincides (with $e'=0$) with the expression of F^e_{av} already found for the circular orbit.
- F^1_{av} gives the contributes of order e', while all the successive terms give contributes containing the product of powers of e, e'.
- F^2_{av} and F^3_{av}, F^4_{av} and F^5_{av}, and in general F^i_{av} and F^{i+1}_{av} give contributes of the same order of greatness, so that they must be calculated always together.

As regard to $\langle \dot{a} \rangle$, the demonstration that at the first order this element does not have secular terms arising from the planetary perturbing force here is immediate only for the central part F^0_{av}. The other forces F^i_{av} will give little positive and negative contributes, that in the long run counteract themselves.

Now we insert F^e_{av} in the perturbation equations and we get, for the secular part,

$$\langle \dot{a} \rangle = \frac{a^2 n}{\pi\,\mu}\,\mathbb{R}\left(\oint F^e_{av}\,d\bar{\mathbf{r}}\right), \tag{70}$$

$$\langle \dot{\mathbf{e}} \rangle = -\frac{i\mathsf{c}}{\mu}\,\langle F^e_{av} \rangle - \frac{i\,n}{2\,\pi\,\mu}\oint \mathbb{I}(F^e_{av}\bar{\mathbf{r}})\,d\boldsymbol{r}, \tag{71}$$

and, dividing by $\mathbf{e}$,

$$\frac{\langle \dot{e} \rangle}{e} + i\,\langle \dot{\omega} \rangle = -\frac{i\mathsf{c}}{\mu\,\mathbf{e}}\,\langle F^e_{av} \rangle - \frac{i\,n}{2\,\pi\,\mu\,\mathbf{e}}\oint I(F^e_{av}\bar{\mathbf{r}})\,d\boldsymbol{r}, \tag{72}$$

so that we must find $\langle F^e_{av} \rangle$ and loop integrals of the form

$$\oint \boldsymbol{r}^i r^{2j} d\bar{\boldsymbol{r}}, \qquad \oint \bar{\boldsymbol{r}}^i r^{2j} d\bar{\boldsymbol{r}}, \tag{73}$$

$$\oint \mathbb{I}\left(\boldsymbol{r}^{i-1} r^{2j+2}\right) d\boldsymbol{r}, \qquad \oint \mathbb{I}\left(\bar{\boldsymbol{r}}^{i+1} r^{2j}\right) d\boldsymbol{r}. \tag{74}$$

Considerations of the same type can be made when the orbit of P' is internal. Then we have the force

$$\frac{F^i}{\mu} = F^O + F^I + F^{II} + F^{III} + \ldots \tag{75}$$

$$F^O = \sum_j k^o_j \frac{\mathbf{r}}{r^{2j+3}}\,r'^{2j}, \qquad F^I = \sum_j k^I_j \frac{1}{r^{2j+3}}\,\mathbf{r}' r'^{2j}, \tag{76}$$

$$F^{II} = \sum_j k^{II}_j \frac{1}{r^{2j+3}}\,\mathbf{r}' r'^{2j}, \qquad F^{III} = \sum_j k^{III}_j \frac{\mathbf{r}^2}{r^{2j+5}}\,\bar{\mathbf{r}}' r'^{2j}, \tag{77}$$

$$F^{IV} = \sum_j k^{IV}_j \frac{\bar{\mathbf{r}}}{r^{2j+5}}\,\mathbf{r}'^2 r'^{2j}, \qquad F^{V} = \sum_j k^{V}_j \frac{\mathbf{r}^2}{r^{2j+7}}\,\mathbf{r}'^3 r'^{2j}, \tag{78}$$

and so on, where k^N_j are rational numbers. Then the typical terms of F^i_{av} are

$$\mathbf{e}'^k a'^{2j+k}\frac{\boldsymbol{r}^i}{r^{2j+n-i}}, \qquad \bar{\mathbf{e}}'^k a'^{2j+k}\frac{\bar{\boldsymbol{r}}^i}{r^{2j+n-i}}. \tag{79}$$

- All terms go to zero as the ratio $\frac{a'^{i+2j}}{a^{2j+n-k}}$ for for $i, j, k, n \to \infty$.
- F^O_{av} is a force of the central type that coincides (with $e' = 0$) with the expression of F^i_{av} already found for the circular orbit.
- F^I_{av} gives the contributes of order e$'$, while all the successive terms give contributes containing the product of powers of e, e$'$.
- F^{II}_{av} and F^{III}_{av}, F^{IV}_{av} and F^{V}_{av}, and in general F^{N}_{av} and F^{NI}_{av} give contributes of the same order of greatness, so that they must be calculated always together.

At last, we must find $\langle F^i_{av} \rangle$ and loop integrals of the form

$$\oint \frac{\boldsymbol{r}^i\,d\bar{\boldsymbol{r}}}{r^{2j+n-1}}, \qquad \oint \frac{\bar{\boldsymbol{r}}^i\,d\bar{\boldsymbol{r}}}{r^{2j+n-1}}, \tag{80}$$

$$\oint \mathbb{I}\left(\frac{\boldsymbol{r}^{i-1}}{r^{2j+n-3}}\right) d\boldsymbol{r}, \qquad \oint \mathbb{I}\left(\frac{\boldsymbol{r}^{i+1}}{r^{2j+n-1}}\right) d\boldsymbol{r}. \tag{81}$$

V. APPLICATIONS

After the formal development of the planetary perturbing force, we are left with the practical application of the formulas. For the secular perturbations in the planetary problem,[10,11] we have some possibilities, each depending from the concrete problem at hand. So, after the choice of the order of approximation, we can proceed first symbolically, and then we shall have the characteristic structure of each of the particular terms considered, and after the successive numerical determination we shall have the contribution to the secular variation of the same term. All this it is possible because we are in a linear environment: first-order perturbations, integrations of a sum of elementary terms, real and imaginary parts determinations. We can verify the work done with a direct determination of $\langle \dot{a} \rangle$ and $\langle \dot{\mathbf{e}} \rangle$. For this it is required the numerical computation of the double integrals

$$\frac{a\,m'}{2\pi^2 a' b\, b'}\int_0^{2\pi}\int_0^{2\pi} \mathbb{R}(F\dot{\bar{\mathbf{r}}})\,r^2 r'^2 df\,df', \tag{82}$$

$$\frac{i\,m'}{4\,\pi^2 a\,a' b\,b'}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{\dot{\mathbf{r}}\,\mathbb{I}(\mathbf{r}'\bar{\mathbf{r}})}{|\mathbf{r}'-\mathbf{r}|^3} + \frac{\mathsf{c}\,(\mathbf{r}'-\mathbf{r})}{|\mathbf{r}'-\mathbf{r}|^3}\right) r^2 r'^2 df\,df', \tag{83}$$

where

$$\dot{\boldsymbol{r}} = \frac{i\mu}{\mathsf{c}}\left(\frac{\boldsymbol{r}}{r} + \mathbf{e}\right). \tag{84}$$

At last we get, for a century,

$$\dot{a}_{sec} = 36525 \cdot \langle\langle \dot{a} \rangle\rangle\,, \tag{85}$$

$$\dot{e}_{sec} = 36525 \cdot e \cdot \mathbb{R}\left\langle \frac{\dot{\mathbf{e}}}{\mathbf{e}} \right\rangle, \tag{86}$$

$$\dot{\omega}''_{sec} = \kappa \cdot \mathbb{I}\left\langle \frac{\dot{\mathbf{e}}}{\mathbf{e}} \right\rangle. \tag{87}$$

For a numerical verification of the method applied to the motion of the perihelion, we preferred to consider the Earth instead of the usual Mercury, because the coplanarity of the orbits involved is best verified for the former planet. The results for the Earth to order α^5 are given in the following tables, where we have employed the planetary elements given in the Appendix for the epoch January 1, 2000. In the second table, relative to the Earth's perihelion in arcsec/century, the first column is referred to the full classical approximation, while the others give respectively the contributions, computed by our method, of the circular (by Eq. (50)) and elliptical parts, and their sum. The diction "Theory" is referred to complete published calculations, which however make reference to a slightly different epoch.[12]

Planet	Theory (A)	$e = 0$	$e \neq 0$	Total (B)	(B-A)
Mercury	-13.75	3.30	-15.01	-11.71	2.04
Venus	345.49	508.73	-151.97	356.76	+11.27
Mars	97.69	21.05	75.52	96.57	-1.12
Jupiter	696.85	709.79	-12.95	696.84	-0.01
Saturn	18.74	32.82	-12.85	19.97	+1.23
Uranus	0.57	0.60	-0.03	0.57	0.00
Neptune	0.18	0.18	0.00	0.18	0.00
Total	1145.77	1276.47	-117.29	1159.18	+13.41

Secular motion of the Earth perihelion

We have a rather good agreement between the two sets of results. The discrepancies are mainly due to: 1) to having neglected the non-coplanarity of planetary orbits, 2) to having used elements related to different epochs, 3) the order of approximation considered. As a further example we consider the secular advance of the perihelion of Mars. Here "Theory" refers to the computation of Doolittle.[13]

Planet	Theory (A)	$e = 0$	$e \neq 0$	Total (B)	(B-A)
Mercury	0.62	0.64	-18.01	-14.71	-0.96
Venus	49.48	47.52	-2.2	45.32	-2.2
Earth	229.03	208.73	6.03	214.76	-14.27
Jupiter	1247.24	1468.28	-214.38	1253.90	+6.66
,Saturn	66.77	63.29	3.40	66.69	-0.08
Uranus	1.20	1.13	0.07	1.20	0.00
Neptune	0.34	0.34	0.00	0.34	0.00
Total	1594.67	1789.90	-207.75	1582.15	-12.52

Secular motion of the Mars perihelion

with the same restrictions as in the previous table. In conclusion, are worth noting the significant corrections to the secular motion of the perihelion due to the presence of ellipticity of the perturbing planets and to the neglect of the relative inclinations of the orbits. It is also evident that Venus and the Earth, for their respective proximity to Mercury and Mars, would require the introduction of more higher-order terms in the development of their disturbing force.

A. Other force laws

We examine now the effects of some other force laws,[14,15] beginning from:

General relativity (**GR**). The general relativistic correction to the gravitational law in the first approximation can be obtained introducing modifications to the classical equation of the motion. If we put $\beta = |\dot{\boldsymbol{r}}|/c$, where c is the speed of light in vacuum (173.144 AU/day), and set $\gamma \equiv \sqrt{1-\beta^2}$, **GR** requires the following corrections to t, m, r:[16]

$$t \to t_0\gamma^{-1} \approx t_0\left(1+\frac{1}{2}\beta^2\right) = t\left(1+\frac{\alpha}{r}\right), \tag{88}$$

$$\mu \to \mu_0\gamma^{-1} \approx \mu_0\left(1+\frac{1}{2}\beta^2\right) = \mu_0\left(1+\frac{\alpha}{r}\right), \tag{89}$$

$$r \to \dot{r}_0\gamma \approx r_0\left(1-\frac{1}{2}\beta^2\right) = r_0\left(1-\frac{\alpha}{r}\right), \tag{90}$$

to the order β^2, where the last is a pure general relativistic effect because involves the *radial* distance r, and where we have put $\beta^2 \equiv |\dot{\boldsymbol{r}}|^2/c^2 = (2\mu)/(c^2) = 2\alpha/r$, with $\alpha \equiv \mu/c^2$ defining the *gravitational radius* of the mass M.

We make these substitutions in the equation of motion by dropping the zero suffixes

$$\ddot{\mathbf{r}} + \mu\frac{\boldsymbol{r}}{r^3} \to \frac{\ddot{\mathbf{r}}}{\left(1+\frac{\alpha}{r}\right)^2} + \frac{\mu\left(1+\frac{\alpha}{r}\right)\boldsymbol{r}}{r^3\left(1-\frac{\alpha}{r}\right)^3} = 0, \tag{91}$$

and, to the $O(\alpha)$ we have:

$$\ddot{\mathbf{r}} + \frac{\mu\boldsymbol{r}}{r^3}\left(1+\frac{2\alpha}{r}\right)\left(1+\frac{\alpha}{r}\right)\left(1+\frac{3\alpha}{r}\right) = 0, \tag{92}$$

$$\ddot{\mathbf{r}} + \frac{\mu\boldsymbol{r}}{r^3} = -\frac{6\mu\alpha\boldsymbol{r}}{r^4} = F. \tag{93}$$

The perturbing force is central, with $\langle \boldsymbol{r} r^{-4}\rangle = \mathbf{e}/(2\,b^3)$, so that

$$\begin{aligned}\langle\dot{\omega}\rangle'' &= -\kappa\,\frac{\mathsf{c}}{\mu\,\mathbf{e}}\,\langle F\rangle = \kappa\,\frac{3\,\alpha\,\mathsf{c}}{b^3} = \kappa\frac{3\,\alpha\,n\,a\,b}{c^2b^3}\\ &= \kappa\frac{3\,\alpha\,n}{c^2\,a(1-\mathrm{e}^2)}\end{aligned} \tag{94}$$

For the centennial motion we have

$$\Delta\omega'' = \kappa\int_0^T \frac{3\,\alpha\,n}{c^2a(1-\mathrm{e}^2)}\,dt = \kappa\frac{6\,\pi\,\alpha}{c^2a(1-\mathrm{e}^2)}. \tag{95}$$

We see from the above developments that the variations with the speed of the planet of time, mass and radial distance give a respective contribution of 1/3, 1/6 and 1/2 to the perihelion precession.

Almost inverse-square law. Let us suppose that the gravitational law goes as $r^{-(2+\epsilon)}$, where $0 < \epsilon \ll 1$. Then by expanding in powers of ϵ

$$\frac{1}{r^{(2+\epsilon)}} \approx \frac{1}{r^2} - \frac{\epsilon Ln(r)}{r^2} + \ldots, \tag{96}$$

and we have the perturbing central force

$$F = -\mu\,\frac{\epsilon Ln(r)}{r^2}\,\frac{\mathbf{r}}{r}. \tag{97}$$

The secular perihelion's motion is given by

$$\begin{aligned}\omega''_{av} &= -\kappa\frac{\mathsf{c}}{\mu\,\mathbf{e}}\,\langle F\rangle = \kappa\frac{\epsilon nab}{\mathbf{e}}\left\langle\frac{Ln(r)}{r^2}\,e^{i\theta}\right\rangle,\\ \langle\dot{\omega}\rangle'' &= \kappa\frac{\epsilon}{2\pi ab}\,\frac{nab\,e^{i\omega}}{\mathbf{e}}\int_0^{2\pi}\frac{Ln(r)}{r^2}r^2e^{if}\,df &(98)\\ &= \kappa\frac{n\epsilon}{2\pi\mathrm{e}}\int_0^{2\pi}Ln(r)\,e^{if}\,df &(99)\end{aligned}$$

Now

$$Ln(r) \approx Ln(a) + \mathrm{e}\cos f - \mathrm{e}^2\left(\frac{5}{4} + \frac{\cos 2f}{4}\right) + ..., \tag{100}$$

so

$$\langle\dot\omega\rangle'' \approx \kappa n\epsilon\left(\frac{1}{2} + \frac{1}{8}\mathrm{e}^2 + \frac{1}{16}\mathrm{e}^4 + \frac{5}{128}\mathrm{e}^6 + \frac{7}{256}\mathrm{e}^8 + ...\right). \tag{101}$$

Let us calculate ϵ for a tentative explanation of the non-classical perihelion shift of Mercury. We find

$$42.95'' = \epsilon \cdot 2.71937 \cdot 10^8 \implies \epsilon = 1.579 \cdot 10^{-7}. \tag{102}$$

Weber's Law. Now we briefly study a proposed alternative to the Newton's law, Weber's law,[17] and apply it to Mercury perihelion. This law it is interesting, other than for historical reasons, also because it introduces additional terms, containing the temporal derivatives of r, to the inverse-square law. By equating the two expressions of $\dot{\boldsymbol{r}}$ we find

$$\frac{dr}{dt} + i\frac{\mathsf{c}}{r} = \frac{i\mu}{\mathsf{c}} + \frac{i\mu}{\mathsf{c}}\,\mathbf{e}\,e^{-i\theta}, \tag{103}$$

and, taking the real part

$$\frac{dr}{dt} = \frac{\mu}{\mathsf{c}}\,\mathbb{R}\left(i\,\mathbf{e}\,e^{-i\theta}\right) = \frac{\mu}{\mathsf{c}}\,\mathrm{e}\sin(\theta-\omega) = \frac{\mu}{\mathsf{c}}\,\mathrm{e}\sin f, \tag{104}$$

$$\frac{d^2r}{dt^2} = \frac{\mu}{r^2}\,\mathrm{e}\cos(\theta-\omega) = \frac{\mu}{r^2}\,\mathrm{e}\cos f, \tag{105}$$

because, from the area integral

$$\frac{d}{dt} = \frac{\mathsf{c}}{r^2}\frac{d}{df}. \tag{106}$$

Weber's law is

$$F = -\mu\left[1 + \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 - \frac{2}{c^2}\,r\,\frac{d^2r}{dt^2}\right]\frac{\boldsymbol{r}}{r^3}, \tag{107}$$

where c is the speed of light. Substituting for the derivative and introducing the true anomaly $f = \theta - \omega$ we have

$$F = -\frac{\mathbf{e}\,\mu^2}{c^2r^2}\left(\frac{\mathrm{e}\mu}{2\mathsf{c}^2} - \frac{2}{r}\cos f + \frac{\mathrm{e}\mu}{2\mathsf{c}^2}\cos 2f\right)e^{if}, \tag{108}$$

and we find

$$\begin{aligned} F_{av} &= -\frac{\mathbf{e}}{2\pi ab}\int_0^{2\pi}\frac{\mu^2}{c^2}\left(\frac{\mathrm{e}\mu}{2\mathsf{c}^2} - \frac{2}{r}\cos f + \frac{\mathrm{e}\mu}{2\mathsf{c}^2}\cos 2f\right)e^{if}df \\ &= -\frac{\mu^2}{c^2b^3}\,\mathbf{e}, \end{aligned} \tag{109}$$

a complex constant. The secular perihelion motion is given by

$$\langle\dot\omega\rangle'' = -\kappa\frac{\mathsf{c}}{\mu\,\mathbf{e}}\langle F_{av}\rangle = \kappa\frac{n\,\mu}{c^2a(1-\mathrm{e}^2)}, \tag{110}$$

and numerically for Mercury

$$\Delta\omega'' = \kappa\frac{n\,\mu}{c^2a(1-\mathrm{e}^2)} = 14.32''. \tag{111}$$

B. The lunar apse

As last application of the method of the averages, we apply them to the derivation at order m^3 of the part of the motion of the lunar perigee independent from the eccentricity, where $m = n'/n$ is the ratio of the mean motions of the Sun and the Moon, in the hypothesis of the main lunar problem (the Sun in a circular orbit in the same plane of Moon's orbit). The perturbing force

$$F = \mu'\left(\frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} - \frac{\mathbf{r}'}{r'^3}\right), \tag{112}$$

with $\mathbf{r}' = a'e^{i\theta'} = a'e^{i\ell'}$ becomes, with $\mu'/a'^3 = n'^3$ and neglecting the solar parallax,

$$\begin{aligned} F &= n'^2(\mathbf{r}' - \mathbf{r})\left(1 + 3\frac{r}{a'}\cos(\theta'-\theta) + \dots\right) - n'^2\mathbf{r}' \\ &\approx -n'^2\mathbf{r} + 3\,n'^2\mathbf{r}'\frac{r}{a'}\cos(\theta'-\theta) \\ &= -n'^2\mathbf{r} + \frac{3}{2}n'^2e^{i\theta'}r\left(e^{i(\theta'-\theta)} + e^{-i(\theta'-\theta)}\right) \\ &= \frac{1}{2}n'^2\mathbf{r} + \frac{3}{2}n'^2\bar{\mathbf{r}}e^{2i\theta'} = \frac{1}{2}n'^2\mathbf{r} + \frac{3}{2}n'^2\bar{\mathbf{r}}e^{2i\ell'}. \end{aligned} \tag{113}$$

In the unperturbed motion at order e, we have for the Moon's orbit in terms of the mean longitude ℓ

$$\mathbf{r}_0 = ae^{i\ell} - \frac{3}{2}a\,\mathbf{e} + \frac{1}{2}a\,\bar{\mathbf{e}}\,e^{2i\ell}. \tag{114}$$

In the first approximation, by solving the perturbation equations, we find the *evection*, given by the following terms[18]

$$\delta\mathbf{r} = -\frac{45}{16}a\,\bar{\mathbf{e}}\,m\,e^{2i\ell'} + \frac{15}{16}a\,\mathbf{e}\,m\,e^{2(\ell-\ell')}. \tag{115}$$

In the second approximation we put $\mathbf{r} = \mathbf{r}_0 + \delta\mathbf{r}$ in the equation

$$\dot{\mathbf{e}} = -\frac{i}{\mu}\left\{\mathbb{I}(\dot{\mathbf{r}}\bar{\mathbf{r}})F + \dot{\mathbf{r}}\,\mathbb{I}(F\bar{\mathbf{r}})\right\}, \tag{116}$$

with $\dot{\mathbf{r}} = n d\mathbf{r}/d\ell$. We cannot use F_{av} in this equation, because in $\mathbf{r}, \dot{\mathbf{r}}$ are present terms containing ℓ'. We find, at order e, the following constant terms

$$-\frac{i}{n^2a^3}\,\mathbb{I}(\dot{\mathbf{r}}\bar{\mathbf{r}})F = i\left(\frac{3}{4}m^2 + \frac{135}{32}m^3\right)n\,\mathbf{e}, \tag{117}$$

$$-\frac{i}{n^2a^3}\,\dot{\mathbf{r}}\,\mathbb{I}(F\bar{\mathbf{r}}) = i\frac{45}{16}m^3n\,\mathbf{e}, \tag{118}$$

and periodical functions of ℓ, ℓ' that give a zero contribution to the double average. Thus

$$\langle\langle\dot{\mathbf{e}}\rangle\rangle = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\dot{\mathbf{e}}\,d\ell'\,d\ell = i\left(\frac{3}{4}m^2 + \frac{225}{32}m^3\right)n\,\mathbf{e}, \tag{119}$$

so that from Eq. (23)

$$\langle\dot{\mathrm{e}}\rangle = 0, \qquad \langle\dot\omega\rangle = \left(\frac{3}{4}m^2 + \frac{225}{32}m^3\right)n. \tag{120}$$

The m^3 term is that found for the first time numerically by Clairaut and algebraically by D'Alembert in their respective theories of lunar motion, and this solved an apparent problematic aspect of the lunar orbit evidenced since the publication of the *Principia* of Newton.

VI. ORBITAL AVERAGES - B

A. Analytical Methods

Some averages are immediate. So

$$\langle \dot{\mathbf{r}} \rangle = \frac{1}{T}\int_0^T \frac{d\mathbf{r}}{dt}\,dt = \frac{1}{T}\oint d\mathbf{r} = 0, \tag{121}$$

because this is a contour integral over a closed orbit. In general for the analyticity of the integrands we have $\langle \mathbf{r}^n \dot{\mathbf{r}} \rangle = 0,\ n \geq 0$. From the orbital expression of $\dot{\mathbf{r}}$ we have at once

$$\left\langle \frac{\mathbf{r}}{r} \right\rangle = \langle e^{i\theta} \rangle = e^{i\omega}\left\langle e^{if} \right\rangle = -\,\mathbf{e} \rightarrow \left\langle e^{if} \right\rangle = -\mathrm{e}, \tag{122}$$

$$\left\langle \frac{\mathbf{r}^n}{r} \right\rangle = -\left\langle \mathbf{r}^{n-1} \right\rangle \mathbf{e}, \tag{123}$$

$$\left\langle \frac{\mathbf{r}^n}{r^{n+2}} \right\rangle = \frac{e^{in\omega}}{2\pi ab}\int_0^{2\pi} e^{inf}\,df = 0, \quad n = 1, 2 \ldots . \tag{124}$$

Many averages we must compute are of the type

$$\left\langle \mathbf{r}^m r^{\pm n} \right\rangle \equiv \langle m, \pm n \rangle\,, \tag{125}$$

with $m, n \geq 0$ positive integers. We can limit ourselves to take $m \geq 0$, because every combination of $\mathbf{r}$, r can be reduced to one of the precedent type employing the equality $\mathbf{r}^{-1} = \bar{\mathbf{r}}/r^2$:

$$\mathbf{r}^{-m} r^n = \bar{\mathbf{r}}^m \mathbf{r}^{-m} \bar{\mathbf{r}}^{-m} r^{\pm n} = \bar{\mathbf{r}}^m r^{\pm n - 2m} = \overline{\mathbf{r}^m r^{\pm n-2m}}. \tag{126}$$

An important property of these average is that they are of the form

$$\left\langle \mathbf{r}^m r^{\pm n} \right\rangle = K\mathbf{e}^m, \tag{127}$$

where K is real, because we have

$$\left\langle \mathbf{r}^m r^{\pm n} \right\rangle = \frac{\mathrm{e}\, e^{im\omega}}{2\pi ab\,\mathrm{e}}\int_{-\pi}^{\pi} g(r)\, e^{\pm imf} df, \tag{128}$$

where $g(r)$ is a periodical even function of f, and the integral of the imaginary part of $g(r)\, e^{\pm imf}$ is zero, because this function is odd. In particular, if F is of *central* type, $F = g(r)\mathbf{r}$, we have $\langle F \rangle = K\mathbf{e}$.

It is also evident that in general

$$\left\langle \mathbf{r}^m r^{\pm n} \right\rangle = e^{im\omega}\left\langle e^{imf} r^{m \pm n} \right\rangle \sim \mathbf{e}^m a^{m \pm n}. \tag{129}$$

The calculations of an average will be done by adopting the more convenient variable for the situation at hand.

If $n = 0$, it will be used the eccentric anomaly

$$\langle m, 0 \rangle = \frac{a^m e^{im\omega}}{2\pi}\int_0^{2\pi} (1 - \mathrm{e}\cos\eta)(\cos\eta - \mathrm{e} + i\sqrt{1-\mathrm{e}^2}\sin\eta)^m d\eta.$$

If $m = 0$, it will be convenient employ the eccentric anomaly for $n > 0$, and the true anomaly for $|n| < 2$. So we have, for $(0, n)$ and for $(0, -n)$

$$\langle r^n \rangle = \frac{1}{2\pi a}\int_0^{2\pi} r^{n+1} d\eta = \frac{a^n}{2\pi}\int_0^{2\pi} (1 - \mathrm{e}\cos\eta)^{n+1} d\eta,$$

$$\left\langle r^{-n} \right\rangle = \frac{1}{2\pi ab}\int_0^{2\pi} \frac{df}{r^{n-2}} = \frac{a^{n-3}}{2\pi b^{2n-3}}\int_0^{2\pi} (1 + \mathrm{e}\cos f)^{n-2} df.$$

For $m + n + 2 > 0$ the computation with respect to the true anomaly requires the integral

$$\frac{e^{im\omega}}{2\pi ab}\int_0^{2\pi} e^{imf} r^{m+n+2} df, \tag{130}$$

with $r = a(1 - \mathrm{e}^2)/(1 + e\cos f)$, that it can be done by means of the repeated use of Cauchy integral formula[2] for the derivatives. If we substitute in the integral

$$e^{if} \rightarrow s, \quad \mathrm{e}\cos f \rightarrow \frac{\mathrm{e}(s^2+1)}{2s}, \quad df \rightarrow \frac{ds}{is}, \tag{131}$$

we obtain

$$\frac{1}{2\pi iab}\left[\frac{2a(1-\mathrm{e}^2)}{\mathrm{e}}\right]^{m+n+2} \int_{|s|=1} \frac{s^{2m+n+1}}{[(s-p)(s-q)]^{m+n+2}} \tag{132}$$

where $|s| = 1$ is the unitary circle centered at the origin and where

$$p = \frac{\sqrt{1-\mathrm{e}^2} - 1}{\mathrm{e}}, \qquad q = -\frac{\sqrt{1-\mathrm{e}^2} + 1}{\mathrm{e}},$$

$$(p - q) = \frac{2\sqrt{1-\mathrm{e}^2}}{\mathrm{e}}. \tag{133}$$

are the solutions of the equation in s

$$s^2 + \frac{2}{\mathrm{e}}\,s + \frac{1}{\mathrm{e}} = 0. \tag{134}$$

The pole within the circle $|s| = 1$ is p, of order $m+n+2$. Then by Cauchy integral formula

$$\int_{|s|=1} \frac{f(s)\,ds}{(s-p)^k} = \frac{2\pi i\, f^{(k-1)}(p)}{(k-1)!}, \tag{135}$$

we can write

$$(m, n) = \frac{1}{ab}\left[\frac{2a(1-\mathrm{e}^2)}{\mathrm{e}}\right]^{m+n+2} \frac{f^{(m+n+1)}(p)}{(m+n+1)!}, \tag{136}$$

with

$$f(s) = \frac{s^{2m+n+1}}{(s-q)^{m+n+2}}. \tag{137}$$

This formula fails for $m+n+1<0$, and in this circumstance we resort to the calculation of

$$\frac{e^{in\omega}}{2\pi ab}\int_0^{2\pi} df\, e^{inf}\left[\frac{1+\mathrm{e}\cos f}{a(1-\mathrm{e}^2)}\right]^{(m+n+2)}, \tag{138}$$

for $m-n+2>0$. Some results:

$$\langle n,-n\rangle = \frac{\left(\sqrt{1-\mathrm{e}^2}-1\right)^n\left(n\sqrt{1-\mathrm{e}^2}+1\right)}{\mathrm{e}^{2n}}\,\mathbf{e}^n, \tag{139}$$

$$\langle n,-(n+1)\rangle = \frac{\left(\sqrt{1-\mathrm{e}^2}-1\right)^n}{a\mathrm{e}^{2n}}\,\mathbf{e}^n, \tag{140}$$

$$\langle n,-(n+2)\rangle = \frac{e^{in\omega}}{2\pi ab}\int_0^{2\pi} e^{inf}df = \begin{cases}\frac{1}{ab} & n=0\\ 0 & n>0,\end{cases} \tag{141}$$

(142)

$$\langle 1,-4\rangle = \frac{\mathbf{e}}{2b^3}, \tag{143}$$

$$\langle 1,-5\rangle = \frac{a\,\mathbf{e}}{b^5}, \tag{144}$$

$$\langle 1,-6\rangle = \frac{(12+3\mathrm{e}^2)a^2\mathbf{e}}{8b^7}, \tag{145}$$

$$\langle 1,-7\rangle = \frac{(4+3\mathrm{e}^2)a^3\mathbf{e}}{2b^9}, \tag{146}$$

$$\langle 0,-1\rangle = \frac{1}{a} \tag{147}$$

$$\langle 0,-2\rangle = \frac{1}{a^2(1-\mathrm{e}^2)^{1/2}}, \tag{148}$$

$$\langle 0,-3\rangle = \frac{1}{a^3(1-\mathrm{e}^2)^{3/2}}, \tag{149}$$

$$\langle 0,-4\rangle = \frac{\mathrm{e}^2+2}{2a^4(1-\mathrm{e}^2)^{5/2}}, \tag{150}$$

$$\langle 0,-5\rangle = \frac{3\mathrm{e}^2+2}{2a^5(1-\mathrm{e}^2)^{7/2}}, \tag{151}$$

$$\langle 0,-7\rangle = \frac{15\mathrm{e}^4+40\mathrm{e}^2+8}{8a^7(1-\mathrm{e}^2)^{11/2}}, \tag{152}$$

$$\langle 0,-9\rangle = \frac{35\mathrm{e}^6+210\mathrm{e}^4+168\mathrm{e}^2+16}{16a^9(1-\mathrm{e}^2)^{15/2}}, \tag{153}$$

$$\langle 0,-11\rangle = \frac{315\mathrm{e}^8+3360\mathrm{e}^6+6048\mathrm{e}^4+2304\mathrm{e}^2+128}{128a^11(1-\mathrm{e}^2)^{19/2}}, \tag{154}$$

$$\langle \mathbf{r}\rangle = \left\langle r e^{i\theta}\right\rangle = \left\langle r e^{if}\right\rangle e^{i\omega}, \tag{155}$$

$$\langle r\cos f\rangle = \langle r\cos(\theta-\omega)\rangle = \mathbb{R}\left(\langle\mathbf{r}\rangle\, e^{-i\omega}\right) = -\frac{3}{2}\,a\,\mathrm{e}, \tag{156}$$

$$\langle r\sin f\rangle = \langle r\sin(\theta-\omega)\rangle = \mathbb{I}\left(\langle\mathbf{r}\rangle\, e^{-i\omega}\right) = 0. \tag{157}$$

Sometimes it is possible to obtain the same result more easily by means of a clever use of already known relations. So, from the immediate averages

$$\left\langle\frac{1}{r}\right\rangle = \frac{1}{2\pi a}\int_0^{2\pi} d\eta = \frac{1}{a}, \tag{158}$$

$$\left\langle\frac{\dot{\mathbf{r}}}{\mathbf{r}}\right\rangle = \frac{1}{T}\int_0^T \frac{d\mathbf{r}}{dt}\frac{dt}{\mathbf{r}} = \frac{1}{T}\oint\frac{d\mathbf{r}}{\mathbf{r}} = \frac{2\pi i}{T} = ni, \tag{159}$$

we have from the orbital expression of $\dot{\boldsymbol{r}}$

$$\left\langle\frac{1}{\mathbf{r}}\right\rangle = -\frac{1}{\mathbf{e}}\left\langle\frac{1}{r}\right\rangle - \frac{i\mathsf{c}}{\mu\mathbf{e}}\left\langle\frac{\dot{\mathbf{r}}}{\mathbf{r}}\right\rangle = -\frac{1}{a\,\mathbf{e}} + \frac{\mathsf{c}n}{\mu\mathbf{e}} = \frac{\sqrt{1-\mathrm{e}^2}-1}{a\,\mathbf{e}}. \tag{160}$$

From the expression

$$\langle \mathbf{r}^m r^n\rangle = \frac{e^{i\omega}}{2\pi\,a}\int_0^{2\pi}\left[\left(a\cos\eta+ia\sqrt{1-e^2}\sin\eta-a\,e\right)\right]^m \times \times\,[a(1-e\cos\eta)]^{n+1}\,d\eta \tag{161}$$

with $m,n\geq 0$, we find

$$(0,1) = \frac{a\,(2+\mathrm{e}^2)}{2}, \tag{162}$$

$$(0,2) = \frac{a^2\,(2+3\mathrm{e}^2)}{2}, \tag{163}$$

$$(1,0) = -\frac{3}{2}\,a\,\mathbf{e}, \tag{164}$$

$$(1,1) = -\frac{a^2(4+\mathrm{e}^2)}{2}\,\mathbf{e}, \tag{165}$$

$$(1,2) = -\frac{5a^3(3\mathrm{e}^2+4)}{8}\,\mathbf{e}, \tag{166}$$

$$(1,3) = -\frac{3a^4(8+12\mathrm{e}^2+\mathrm{e}^4)}{8}\,\mathbf{e}, \tag{167}$$

$$(1,4) = -\frac{7a^5(5\mathrm{e}^4+20\mathrm{e}^2+8)}{16}\,\mathbf{e}, \tag{168}$$

$$(1,6) = -\frac{9a^7(35\mathrm{e}^6+280\mathrm{e}^4+336\mathrm{e}^2+64)}{128}\,\mathbf{e}, \tag{169}$$

$$(1,8) = -\frac{11a^9(63\mathrm{e}^8+840\mathrm{e}^6+2016\mathrm{e}^4+1152\mathrm{e}^2+128)}{256}\,\mathbf{e}. \tag{170}$$

$$(1,-1) = -\mathbf{e} \tag{171}$$

$$(1,-2) = \frac{\sqrt{1-\mathrm{e}^2}-1}{ae^2}\,\mathbf{e}, \tag{172}$$

$$(1,-3) = 0, \tag{173}$$

$$(1,-4) = \frac{\mathbf{e}}{2\,b^3}, \tag{174}$$

$$(1,-5) = \frac{a\,\mathbf{e}}{b^5}, \tag{175}$$

$$(1,-6) = \frac{3\,a^2(4+\mathrm{e}^2)\,\mathbf{e}}{8\,b^7}, \tag{176}$$

$$(1,-7) = \frac{a^3\,(4+3\,\mathrm{e}^2)\,\mathbf{e}}{2\,b^9}, \tag{177}$$

$$(1,-9) = \frac{3\,a^5\,(5\,\mathrm{e}^4+20\,\mathrm{e}^2+8)\,\mathbf{e}}{8\,b^{13}}, \tag{178}$$

$$(1,-11) = \frac{a^7\,(35\,\mathrm{e}^6+280\,\mathrm{e}^4+336\,\mathrm{e}^2+64)\,\mathbf{e}}{16\,b^{17}}, \tag{179}$$

$$(2,\ \ 0) = \frac{5}{2}\,a\,\mathbf{e}^2, \tag{180}$$

$$(2,-5) = 0, \tag{181}$$

$$(2,-7) = \frac{3\,\mathbf{e}^2}{4a^5(1-\mathrm{e}^2)^{7/2}}, \tag{182}$$

$$(2,-9) = \frac{5(\mathrm{e}^2+2)\mathbf{e}^2}{4a^7(1-\mathrm{e}^2)^{11/2}}. \tag{183}$$

$$\left\langle \frac{e^{i\theta}}{a'+\mathrm{r}} \right\rangle = \frac{a(1-\mathrm{e}^2) - A + a'}{a\,\mathrm{e}^2\,A}\,\mathbf{e}, \tag{184}$$

$$\left\langle \frac{e^{i\theta}}{a'-\mathrm{r}} \right\rangle = \frac{a(1-\mathrm{e}^2) + B - a'}{a\,\mathrm{e}^2\,B}\,\mathbf{e}, \tag{185}$$

$$\left\langle \frac{\mathbf{r}}{a'^2-\mathrm{r}^2} \right\rangle = \frac{1}{2}\left\langle\left(\frac{e^{i\theta}}{a'-\mathrm{r}} - \frac{e^{i\theta}}{a'+\mathrm{r}}\right)\right\rangle \tag{186}$$

$$= \frac{a(1-\mathrm{e}^2)(A-B) - a'(A+B) + 2AB}{2a\,\mathrm{e}^2\,AB}\,\mathbf{e}, \tag{187}$$

$$\left\langle \frac{e^{i\theta}}{a'^2-\mathrm{r}^2} \right\rangle = \frac{1}{2a'}\left\langle\left(\frac{e^{i\theta}}{a'-\mathrm{r}} + \frac{e^{i\theta}}{a'+\mathrm{r}}\right)\right\rangle \tag{188}$$

$$= \frac{a(1-\mathrm{e}^2)(A+B) - a'(A-B)}{2a\,a'\mathrm{e}^2\,AB}\,\mathbf{e}, \tag{189}$$

with

$$A \equiv \sqrt{(a+a')^2 - a^2\mathrm{e}^2}, \qquad B \equiv \sqrt{(a-a')^2 - a^2\mathrm{e}^2}. \tag{190}$$

Loop integrals of the type $\oint f(\mathbf{r},\bar{\mathbf{r}})\,d\mathbf{r}$ and $\oint f(\mathbf{r},\bar{\mathbf{r}})\,d\mathrm{r}$ over an ellipse in the complex plane are present when the expression to be averaged contains $\dot{\mathbf{r}}$ or $\dot{\bar{\mathbf{r}}}$. This is accomplished by using the identity $\dot{\mathbf{r}}dt = d\mathbf{r}$ in the averages $\langle f(\mathbf{r},\bar{\mathbf{r}})\,\dot{\mathbf{r}}\,\rangle$ and $\langle f(\mathbf{r},\bar{\mathbf{r}})\,d\mathrm{r}/dt\rangle$, together with the expressions

$$T = \frac{2\pi}{n}, \quad \dot{\mathbf{r}} = \frac{i\mu}{\mathsf{c}}\frac{\mathbf{r}}{\mathrm{r}} + \frac{i\mu}{\mathsf{c}}\mathbf{e}, \quad \dot{\mathbf{r}}\,\dot{\bar{\mathbf{r}}} = \mu\left(\frac{2}{\mathrm{r}} + \frac{1}{a}\right), \tag{191}$$

$$\frac{d\mathrm{r}}{dt} = \frac{1}{2\mathrm{r}}\left(\mathbf{r}\,\dot{\bar{\mathbf{r}}} + \bar{\mathbf{r}}\,\dot{\mathbf{r}}\right) = \frac{\mathbb{R}(\bar{\mathbf{r}}\,d\mathbf{r})}{\mathrm{r}dt} = -\frac{i\mathsf{c}}{\mathrm{r}} + \frac{i\mu}{\mathsf{c}} + \frac{i\mu}{\mathsf{c}}\frac{\bar{\mathbf{r}}}{\mathrm{r}}\,\mathbf{e}. \tag{192}$$

These are easily computed when is is possible to use, in the integrand, the eccentric anomaly in orbit's parametrization. We put

$$\mathbf{r} = (a\cos\eta + ib\sin\eta - ae)\,e^{i\omega}, \tag{193}$$

$$\mathrm{r} = a(1 - e\cos\eta), \tag{194}$$

$$d\mathbf{r} = -(a\sin\eta - ib\cos\eta)\,e^{i\omega}d\eta. \tag{195}$$

It is immediate to see that, when the force F is central, the integral $\oint F\,d\bar{\mathbf{r}}$ is a pure imaginary number, because

$$\oint F\,d\bar{\mathbf{r}} = \oint g(\mathrm{r})\,\mathbf{r}\,d\bar{\mathbf{r}} = \int_0^{2\pi} g(\mathrm{r})\left[\frac{(b^2-a^2)}{2}\,\sin 2\eta - a^2\mathrm{e}\sin\eta - i(ab\mathrm{e}\cos\eta + a\,b)\right]d\eta, \tag{196}$$

and, since $g(\mathrm{r})$ is an even function of η, the terms containing $\sin\eta$, $\sin 2\eta$ are zero in the integration. In general we have, with n relative integer,

$$\oint \mathbf{r}^m\mathrm{r}^n d\mathbf{r} \sim a^{m+n+1}\mathbf{e}^{m+1} \qquad m \geq 0, \tag{197}$$

$$\oint \mathbf{r}^m\mathrm{r}^n d\bar{\mathbf{r}} \sim a^{m+n+1}\mathbf{e}^{m-1} \qquad m \geq 1. \tag{198}$$

Some examples:

$$\left\langle \mathbf{r}\frac{d\mathrm{r}}{dt} \right\rangle = \frac{1}{T}\oint \mathbf{r}\,d\mathrm{r} = \frac{1}{2T}\oint \frac{\mathbf{r}^2}{\mathrm{r}}\,d\bar{\mathbf{r}} + \frac{1}{2T}\oint \mathrm{r}\,d\mathbf{r}, \tag{199}$$

$$\oint \mathbf{r}\,d\mathrm{r} = \pi\,i\,a\,b\,\mathbf{e}, \qquad \oint \mathrm{r}\,d\mathbf{r} = -\pi\,i\,a\,b\,\mathbf{e}, \tag{200}$$

$$\oint \frac{\mathbf{r}^2}{\mathrm{r}}\,d\bar{\mathbf{r}} = 2\oint \mathbf{r}\,d\mathrm{r} - \oint \mathrm{r}\,d\mathbf{r} = 3\,\pi\,i\,a\,b\,\mathbf{e}, \tag{201}$$

and

$$\left\langle \mathbf{r}\frac{d\mathrm{r}}{dt} \right\rangle = \frac{1}{2}\,n\,i\,a\,b\,\mathbf{e}. \tag{202}$$

Curve rectification. From

$$\frac{|d\mathbf{r}|}{dt} = \sqrt{\frac{2\mu}{\mathrm{r}} - \frac{\mu}{a}}, \tag{203}$$

$$\oint |d\mathbf{r}| = \frac{2\pi}{n}\,\frac{\sqrt{\mu}}{2\pi a^3}\int_0^{2\pi}\sqrt{2\,a\mathrm{r} - \mathrm{r}^2}\,d\eta = a\int_0^{2\pi}\sqrt{1 - \mathrm{e}^2\cos^2\eta}\,d\eta = \text{ellipse lenght.} \tag{204}$$

Some other integrals:

$$\oint \frac{d\mathbf{r}}{\mathrm{r}} = \frac{2\pi i(\sqrt{1-\mathrm{e}^2} + \mathrm{e}^2 - 1)}{\mathrm{e}^2\sqrt{1-\mathrm{e}^2}}\,\mathbf{e}, \tag{205}$$

$$\oint \frac{d\mathbf{r}}{\mathrm{r}^n} = \frac{n\pi i}{[a(1-\mathrm{e}^2)]^{n-1}}\,\mathbf{e}, \qquad n = 2, 3. \tag{206}$$

$$\oint \frac{\mathbf{r}^n}{\mathrm{r}^n}\,d\mathbf{r} = (-\mathbf{e})^{n+1}\,\frac{2n\pi i a(1-\sqrt{1-\mathrm{e}^2})^n((1-\mathrm{e}^2)^{3/2} + \mathrm{e}^2 - 1)}{\mathrm{e}^{2n+2}\sqrt{1-\mathrm{e}^2}}. \tag{207}$$

In these expressions we may assume that $\omega = 0$, $\mathbf{e} = e$, i.e. that the semi-major axis of the ellipse lies on the real axis.

$$\oint \mathbf{r}\mathrm{r}\,d\bar{\mathbf{r}} = \pi\,i\,a^2b\left(\mathrm{e}^2 - 2\right), \tag{208}$$

$$\oint \mathbf{r}\mathrm{r}^2 d\bar{\mathbf{r}} = \pi\,i\,a^3b\left(\mathrm{e}^2 - 2\right), \tag{209}$$

$$\oint \mathbf{r}\mathrm{r}^4 d\bar{\mathbf{r}} = \frac{1}{4}\,\pi\,i\,a^5b\left(9\mathrm{e}^4 - 8\mathrm{e}^2 - 8\right), \tag{210}$$

$$\oint \mathbf{r}\mathrm{r}^6 d\bar{\mathbf{r}} = \frac{1}{8}\,\pi\,i\,a^7b\left(25\mathrm{e}^6 + 30\mathrm{e}^4 - 72\mathrm{e}^2 - 16\right), \tag{211}$$

$$\oint \mathbf{r}^2 d\bar{\mathbf{r}} = -4\,\pi\,i\,a^2b\,\mathbf{e}, \tag{212}$$

$$\oint \mathbf{r}^3 d\bar{\mathbf{r}} = -\frac{15}{2}\,\pi\,i\,a^3b\,\mathbf{e}^2, \tag{213}$$

$$\oint \mathbf{r}^4 d\bar{\mathbf{r}} = -14\,\pi\,i\,a^4b\,\mathbf{e}^3, \tag{214}$$

$$\oint \mathbf{r}^5 d\bar{\mathbf{r}} = -\frac{105}{4}\,\pi\,i\,a^5b\,\mathbf{e}^4, \tag{215}$$

$$\oint \mathrm{r}^2 d\bar{\mathbf{r}} = 2\pi i a^2b\,\bar{\mathbf{e}}, \tag{216}$$

$$\oint \mathrm{r}^4 d\bar{\mathbf{r}} = \pi i a^4b\,(3\mathrm{e}^2 + 4)\,\bar{\mathbf{e}}, \tag{217}$$

$$\oint \mathrm{r}^n d\mathrm{r} = 0 \;\text{ because }\; \int_0^T \mathrm{r}^n\dot{\mathrm{r}}dt = \left[\frac{\mathrm{r}^{n+1}}{n+1}\right]_0^T = 0. \tag{218}$$

Loop integrals of the form

$$\oint \frac{\boldsymbol{r}^n d\boldsymbol{r}}{\mathrm{r}^m}, \qquad \oint \frac{\boldsymbol{r}^n d\bar{\boldsymbol{r}}}{\mathrm{r}^m}, \tag{219}$$

can easily be transformed to an orbital average in the true anomaly we already have considered. For example,

$$\begin{aligned}\oint \frac{\boldsymbol{r}^n}{\mathrm{r}^m}\, d\boldsymbol{r} &= \int_0^T \frac{\boldsymbol{r}^n}{\mathrm{r}^m}\, \dot{\boldsymbol{r}}\, dt = \frac{1}{c}\int_0^{2\pi} \frac{\boldsymbol{r}^n \dot{\boldsymbol{r}}}{\mathrm{r}^{m-2}}\, df \\ &= \frac{i\mu}{c^2}\int_0^{2\pi} \frac{\boldsymbol{r}^n}{\mathrm{r}^{m-2}}\left(\frac{\boldsymbol{r}}{\mathrm{r}} + \mathbf{e}\right) df \\ &= \frac{i\mu}{c^2}\int_0^{2\pi} \frac{\boldsymbol{r}^{n+1}}{\mathrm{r}^{m-1}}\, df + \frac{i\,\mathbf{e}\mu}{c^2}\int_0^{2\pi} \frac{\boldsymbol{r}^n}{\mathrm{r}^{m-2}}\, df.\end{aligned} \tag{220}$$

The independent variable f of each of these integrals can be transformed, if more convenient, in η by means of the relation $df = (b/\mathrm{r})\, d\eta$. This is an interesting by-product of the calculus of orbital averages: the possibility of alternate computations of certain integrals. Since an angular average of a finite two-body expression can be done in two ways, with f, η as independent variables, we can choice the most convenient for the calculation, and automatically we have the result of the corresponding integral in the other variable. We denote as *dual* the two correlated expressions. In practice, the method is the following: given the integral of an expression derived from a two-body orbital function, we can interpret it as an average in f or η, and after the computation, made with the more convenient variable, we have also the value of the dual integral in η or f obtained by using the relations

$$df = \frac{b}{\mathrm{r}}\, d\eta, \qquad d\eta = \frac{\mathrm{r}}{b}\, df. \tag{221}$$

We give here only the simplest example, which non requires any integration at all. With $\alpha > \beta$, $\beta/\alpha = e < 1$

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{(\alpha + \beta\cos\theta)^2} &= \frac{1}{\alpha^2}\int_0^{2\pi} \frac{d\theta}{(1 + e\cos\theta)^2} \\ = \frac{1}{\alpha^2 a^2 (1-e^2)^2}\int_0^{2\pi} \mathrm{r}^2 d\theta &= \frac{2\pi a^2\sqrt{1-e^2}}{\alpha^2(1-e^2)^2} \;\langle 1\rangle \\ = \frac{2\pi}{\alpha^2(1-e^2)^{3/2}} &= \frac{2\pi\alpha}{\sqrt{(\alpha^2-\beta^2)^3}}.\end{aligned} \tag{222}$$

Last, the Green's theorem on the complex plane

$$\iint_D \frac{\partial f}{\partial \bar{\boldsymbol{r}}}\, dx\, dy = \frac{1}{2\,i}\oint_{\partial D} f(\boldsymbol{r}, \bar{\boldsymbol{r}})\, d\boldsymbol{r}, \tag{223}$$

where D is the simply connected domain bounded by the unperturbed elliptic orbit ∂D, gives immediately, for each computed value of a line integral, the value of a surface integral over the domain and viceversa, given a function $g(\boldsymbol{r}, \bar{\boldsymbol{r}})$, we first integrate it

$$\int g(\boldsymbol{r}, \bar{\boldsymbol{r}})\, d\bar{\boldsymbol{r}} = f(\boldsymbol{r}, \bar{\boldsymbol{r}}), \tag{224}$$

and after we calculate the line integral of f.

Some examples:

$$\iint_D dx\, dy = \iint_D \frac{\partial \bar{\mathbf{r}}}{\partial \bar{\mathbf{r}}}\, dx\, dy = \frac{1}{2i}\oint_{\partial D} \bar{\mathbf{r}}\, d\boldsymbol{r} = \pi\, a\, b, \tag{225}$$

$$\iint_D \frac{\boldsymbol{r}}{\mathrm{r}^3}\, dx\, dy = -\oint \frac{d\boldsymbol{r}}{\mathrm{r}} = \frac{\pi\,(1 - \mathrm{e}^2 - \sqrt{1-\mathrm{e}^2})}{\mathrm{e}\sqrt{1-\mathrm{e}^2}}. \tag{226}$$

All these results are purely geometrical, without reference to the underlying dynamical problem we employ as solution device.

VII. DISCUSSION

We have given a rather consistent number of examples of application of the method of the orbital averages, so it is now possible to draw conclusions about its pros and cons. From the theoretical point of view, it reduces the secular perturbation of two gravitationally interacting bodies in complex geometric situations to a succession of ever smaller simple forces each of which provides a contribution that can be computed exactly. Having available a literal expression deepens our knowledge about the various factors that help produce the final result. An important point to emphasize is that, with the symbolic formulas, the work can be done once and for all because in actual cases will suffice replacing the various symbols with the numerical values of the orbital parameters. We have also harnessed the power of complex analysis to obtain our results in an elegant way. This also paradoxically constitutes the major drawback of the method: it works under conditions of coplanarity or, at most, in situations of almost coplanarity, but we believe that in its scope it allows to obtain interesting results, especially in the calculation of the effects of gravitational forces arising from alternative theories to general relativity. In conclusion, we think that this method represent a useful working tool that can provide valuable services to the researcher in a wide field of study.

VIII. APPENDIX

Tables of the planetary orbital elements.[?] The figures are rounded to the fourth decimal.

Jan 1, 2000	a	b	e	ω (*rad.*)
Mercury	0.3871	0.3788	0.2056	1.351870079
Venus	0.7233	0.7233	0.0067	2.295683576
Earth	1.0000	0.9999	0.0167	1.796767421
Mars	1.5237	1.5170	0.0934	5.865019079
Jupiter	5.2034	5.1973	0.0484	0.257503259
Saturn	9.5371	9.5231	0.0541	1.613241687
Uranus	19.1913	19.1699	0.0472	2.983888891
Neptune	30.0690	30.0679	0.0086	0.784898126

Jan 1, 2000	$m' = \mu'/\mu$	c	n
Mercury	$1.67 \cdot 10^{-7}$	0.010473	$7.142 \cdot 10^{-2}$
Venus	$2.44 \cdot 10^{-6}$	0.014629	$2.796 \cdot 10^{-2}$
Earth	$3.01 \cdot 10^{-6}$	0.017199	$1.720 \cdot 10^{-2}$
Mars	$3.31 \cdot 10^{-7}$	0.021142	$9.143 \cdot 10^{-3}$
Jupiter	$9.59 \cdot 10^{-4}$	0.039236	$1.470 \cdot 10^{-3}$
Saturn	$2.87 \cdot 10^{-4}$	0.053046	$5.88 \cdot 10^{-4}$
Uranus	$4.37 \cdot 10^{-5}$	0.075179	$2.08 \cdot 10^{-4}$
Neptune	$5.18 \cdot 10^{-5}$	0.094527	$1.05 \cdot 10^{-4}$

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