ALGEBRAIC STRUCTURES USING SUBSETS

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DEDICATION

GEORGE BOOLE
(2 November 1815 – 8 December 1864)

The English mathematician

George Boole

(2 November 1815 – 8 December 1864),

the founder of the algebraic tradition in logic

is today regarded as the founder of computer science.

Boolean algebra employed the concept of subsets or

symbolic algebra to the field of logic

and revolutionized mathematical logic.

We dedicate this book to George Boole for his

contributions. This is our humble method of paying

homage to his mathematical genius.

PREFACE

The study of subsets and giving algebraic structure to these subsets of a set started in the mid 18th century by George Boole. The first systematic presentation of Boolean algebra emerged in 1860s in papers written by William Jevons and Charles Sanders Peirce. Thus we see if P(X) denotes the collection of all subsets of the set X, then P(X) under the operations of union and intersection is a Boolean algebra.

Next the subsets of a set was used in the construction of topological spaces. We in this book consider subsets of a semigroup or a group or a semiring or a semifield or a ring or a field; if we give the inherited operations of the semigroup or a group or a semiring or a semifield or a ring or a field respectively; the resulting structure is always a semigroup or a semiring or a semifield only. They can never get the structure of a group or a field or a ring. We call these new algebraic structures as subset semigroups or subset semirings or subset semifields. This method gives us infinite number of finite noncommutative semirings.

Using these subset semirings, subset semifields and subset semigroups we can define subset ideal topological spaces and subset set ideal topological spaces. Further using subset semirings and subset semifields we can build new subset topological set ideal spaces which may not be a commutative topological space. This innovative methods gives non commutative new set ideal topological spaces provided the under lying structure used by us is a noncommutative semiring or a noncommutative ring.

Finally we construct a new algebraic structure called the subset semivector spaces. They happen to be very different from usual semivector spaces; for in this situation we see if V is a subset semivector space defined over a non commutative semiring or a noncommutative ring say S, then for s in S and v in V we may not have in general sv = vs. This is one of the marked difference between usual semivector spaces and subset semivector spaces.

Subset topological semivector subspaces and quasi subset topological semivector subspaces are defined and developed.

We thank Dr. K.Kandasamy for proof reading and being extremely supportive.

W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Chapter One

INTRODUCTION

In this book for the first time authors introduce on the subsets of the S where S can be a ring or a semigroup or a field or a semiring or a semifield an operation '+' and ' .', which are inherited operations from these algebraic structures and give a structure to it. It is found that the collection of subsets can maximum be a semiring they can never have a group structure or a field structure or a ring structure.

This study is mainly carried out in this book. The main observations are if $A = \{0, 1, 2\}$ then $A + A = \{0, 1, 2, 3, 4\} \neq A$.

Just like this $A.A = \{1, 2, 0, 4\} \neq A$.

So the usual set theoretic operations are not true in case of the operations on these subsets collections with entries from ring or field or semiring or semifield.

For more about the concept of semigroups and semirings please refer [6, 8].

Finally the book gives the notion of subset semivector spaces of the three types. Using these subset semivector spaces we can build two types of quasi set topological semivector subspaces one with usual union and other with $\cup_N$ and $\cap_N$. We further see $A \cup B \neq A \cup_N B$ in general.

Also $A \cap_N B \neq A \cap B$ for $A, B \in S$. Also $A \cap_N A \neq A$ and $A \cup_N A \neq A$. So T_N is very different from T. This study is interesting and innovative.

We suggest at the end of each chapter several problems for the interested reader to solve. We have also suggested some open problems for researchers.

Further the authors wish to keep on record it was Boole who in 1854 introduced the concept of Boolean algebra which has been basic in the development of computer science. The powerset of X, P(X) gives the Boolean algebra of order $2^{|X|}$.

However both the operations $\cup$ and $\cap$ on P(X) are commutative and idempotent this is not true in general for these subsets.

Chapter Two

SEMIGROUPS USING SUBSETS OF A SET

In this chapter authors for th e first time introduce the new notion of building semigroups using subsets of a ring or a group or a semigroup or a semiring or a field. They are alway s semigroups under $\cup$ and $\cap$ of a powe r set. For t he sake of completeness we just recall the definition of semigroup / semilattice.

DEFINITION 2.1: *Let $S = \{a_1, \ldots, a_n\}$ be a collection of subsets of a ring or a group or a semigroup or a set or a field or a semifield. $o(S)$ can be finite or infinite ($o(S)$ = number of elements in S). Let * be an operation on S so that (S, *) is a semigroup. That is * is an associative closed binary operation. We define (S, *) to be the subset semigroup of S.*

Note 1: We can have more than one operation on S.

Note 2: (S, *) need not be commutative.

Note 3: Depending on the subsets one can have several different semigroups that is $(S, *_1)$, $(S, *_2)$ and so on, where $*_1$ is not the same binary operation as $*_2$.

First we will illustrate this situation by some examples.

Example 2.1: Let X = {1, 2, 3}, P(X) the power set of X. P(X) includes ϕ and X. {P(X), $\cup$} is a commutative semigroup of order 8.

{P(X), $\cap$} is also a commutative semigroup of order 8.

Infact {P(X), $\cap$} and {P(X), $\cup$} are two distinct semigroups which are also semilattices.

Example 2.2: Let X = {a_1, a_2, a_3, a_4, a_5} be a set. P(X) be the power set of X. Clearly number of elements in P(X) = order of P(X) = o (P(X)) = |P(X)| = 2^5.

We see (P(X), $\cap$) is a semigroup which is commutative of finite order. {P(X), $\cup$} is also a semigroup which is commutative of finite order. Both {P(X), $\cup$} and {P(X), $\cap$} are semilattices.

In view of this we just record a well known theorem.

THEOREM 2.1: *Let X = {a_1, …, a_n} be a set. P(X) be the collection of all subsets of X including X and ϕ. {P(X), $\cup$} and {P(X),$\cap$} are both semigroups (semilattices) which is commutative of order 2^n where n = |X| = o(X).*

The proof is direct and hence left as an exercise to the reader.

Now we proceed onto define semigroups on the subsets of groups or semigroups or rings or semifields or fields. For this we make the following definition.

DEFINITION 2.2: *Let X be a group or a semigroup; P(X) be the power set of X. (P(X) need not contain ϕ). Let A, B $\in$ P(X). We define A * B = {a * b | a $\in$ A and b $\in$ B, * the binary operation on X}. {P (X), *} is a semigroup called the subset semigroup of* *the group X or a semigroup and is different from the semigroups {P(X), $\cap$} and {P(X), $\cup$}.*

We will first illustrate this by some examples.

Note P(X) in we need not i nclude ϕ in cas e X has an algebraic structure.

Example 2.3: Let G = {0, 1, 2} be a group under addition modulo 3.

P(G) = {{0}, {1}, {2}, { 0, 1}, {0, 2}, {0, 1, 2} , {2, 1}}. {P(G), +} is a semigroup of subsets of G of order seven given by the following table:

+	{0}	{1}	{2}
{0}	{0}	{1}	{2}
{1}	{1}	{2}	{0}
{2}	{2}	{0}	{1}
{0,1}	{0,1}	{1,2}	{2,0}
{0,2}	{0,2}	{0,1}	{2,1}
{1,2}	{1,2}	{2,0}	{1,0}
{0,1,2}	{0,1,2}	{1,2,0}	{2,0,1}

{0,1}	{0,2}	{1,2}	{1,2,0}
{0,1}	{0,2}	{1,2}	{0,1,2}
{1,2}	{1,0}	{0,2}	{1,0,2}
{2,0}	{2,1}	{0,1}	{1,2,0}
{0,1,2}	{0,1,2}	{1,0,2}	{1,0,2}
{0,1,2}	{0,2,1}	{1,2,0}	{0,1,2}
{1, 0,2}	{1, 0,2}	{1,0,2}	{0,1,2}
{0,1,2}	{0,1,2}	{0,1,2}	{0,1,2}

We see by this method we get a different sem igroup. Thus using a group we get a semigroup of subsets of a group.

Example 2.4: Let Z_3 = {0, 1, 2} be the semigroup u nder product. The subsets of Z_3 are

S = {{0}, {1}, {2}, {1, 0}, {2 , 0}, {1, 2}, {0, 1, 2}} under product is a semigroup given in the following;

×	{0}	{1}	{2}	{0,1}	{0,2}	{1,2}	{0,1,2}
{0}	{0}	{0}	{0}	{0}	{0}	{0}	{0}
{1}	{0}	{1}	{2}	{0, 1}	{0,2}	{1,2}	{0,1,2}
{2}	{0}	{2}	{1}	{0,2}	{0,1}	{2,1}	{0,1,2}
{0,1}	{0}	{1,0}	{0,2}	{0,1}	{0,2}	{0,1,2}	{0,1,2}
{0,2}	{0}	{0,2}	{0,1}	{0,2}	{0,1}	{0,2,1}	{0,1,2}
{1,2}	{0}	{1,2}	{2,1}	{0,1,2}	{0,1,2}	{1,2}	{0,1,2}
{0,1,2}	{0}	{0,1,2}	{0,1,2}	{0,1,2}	{0,1,2}	{0,1,2}	{0,1,2}

We see both the semigroups are distinct.

Now we give more examples.

Example 2.5: Let Z_2 = {0 , 1} be the semigroup u nder product S = {Subsets of Z_2} = {{0}, {1}, {0, 1}}. The table of {S, ×} as follows:

×	{0}	{1}	{0,1}
{0}	{0}	{0}	{0}
{1}	{0}	{1}	{0,1}
{0,1}	{0}	{0,1}	{0,1}

Example 2.6: Let Z_4 = {0, 1, 2, 3} be the semigroup un der product modulo four.

The subsets of Z_4 are S = { {0}, {1}, {2}, {3}, {0,1}, {0,2}, {0,3}, { 1,2}, {1,3}, {2,3} , {1,2,3 }, { 0,1,2}, {0,1,3}, {0,2,3} , {0,1,2,3}}. (S, ×) is a semigroup under product modulo 4 of order 15.

DEFINITION 2.3: *Let S be a collection of all subsets of a semigroup T under product × with zero then S under the same product as that of T is a semigroup with zero divisors if A × B = {0}, A ≠ {0} and B ≠ {0}. If one of A = {0} or B = {0} we do not say A is a zero divisor though A × B = {0}, where A, B ∈ S.*

We will give examples of this.

Example 2.7: Let
S = {Collection of all subsets of Z_6 barring the empty set} = {{0}, {1}, {2}, {3}, {4} , {5}, { 1, 2}, {1, 3} , {1, 4}, {1, 5}, {0,1}, {0, 2}, {0, 3}, {0, 4}, {0, 5} , {2, 3}, …, {0, 1, 2, 3, 4}, {0, 1, 2, 3, 5}, …, {1, 2, 3, 4, 5}, Z_6} a semigroup under product modulo six.

{Z_6, ×} is a semigroup under product ×.

AB = {2} × {3} = {0},
$A_1 \times B_1$ = {2} × {3, 0} = {0},
$A_2 \times B_2$ = {4} × {3} = {0},
$A_3 \times B_3$ = {0, 2} × {3} = {0},
$A_4 \times B_4$ = {0,2} × {0,3} = {0},
$A_5 \times B_5$ = {0,4} × {3} = {0},
$A_6 \times B_6$ = {0,4} × {0,3} = {0} and
$A_7 \times B_7$ = {4} × {0,3} = {0}.

Thus we have zero divisors in the semigroup S under the product ×.

Example 2.8: Consider
S = {all subsets of Z_5 barring the empty set}; where {Z_5, ×} is a semigroup under product. {S, ×} is a semigroup and {S, ×} has no zero divisors. It is clear from the subsets of S.

We further observe that Z_5 has no proper zero divisors so S also has no zero divisors.

Example 2.9: Let S = {all subsets of Z_{12} barring the empty set}. Z_{12} is a semigroup under product × modulo 12.

(S, ×) is a semigroup of order $2^{12} - 1$.

Further Z_{12} has zero divisors; $\{0, 4, 2, 8, 6, 3, 9, 10\} \subseteq Z_{12}$ contribute to zero divisors in Z_{12}.

Also S has zero divisors given by

$\{0,4\} \times \{3\} = \{0\}$, $\{0,4\} \times \{0,3\} = \{0\}$,
$\{0,4\} \times \{6\} = \{0\}$, $\{0,4\} \times \{0,6\} = \{0\}$,
$\{0,8\} \times \{3\} = \{0\}$, $\{0,8\} \times \{6\} = \{0\}$,
$\{0,8\} \times \{9\} = \{0\}$, $\{0,4,8\} \times \{0,6\} = \{0\}$ and so on.

If Z_n has zero divisors then S the subsets of Z_n has zero divisors.

Example 2.10: Let Z_7 be the semigroup under ×. S the subsets of Z_7 under ×. S has no zero divisors.

In view of all these we have the following theorem the proof of which is left as an exercise to the reader.

THEOREM 2.2: *The subset semigroup {S, ×} has zero divisors if and only if {Z_n, ×} has zero divisors.*

Now we study about units of {S, ×}, S the collection of all subsets of the semigroup of Z_n under product.

Example 2.11: Let
S = {Collection of all subsets of Z_{12} under product modulo 12} be the subset semigroup of the semigroup Z_{12} under product. Consider $\{5\} \times \{5\} = \{1\}$, $\{7\} \times \{7\} = \{1\}$, $\{5\} \times \{5\} = \{1\}$ we see S has units.

Example 2.12: Let
S = {Collection of all subs ets of Z_{13} under product modulo 13} be the subset semigroup of the semigroup Z_{13}. Consider {12} × {12} = 1 and {5} × {8} = 1. S has units.

DEFINITION 2.4: *Let S be the subset semigroup of a semigroup {T, ×} where T has unit 1. Let A, B $\in$ S if A × B = {1} then we say S has units if {A} $\neq$ 1 and {B} $\neq$ 1.*

We will illustrate this situation by some examples.

Example 2.13: Let
S = {Collection of all subsets of a semigroup Z_{15} under product} be the subset semigroup of {Z_{15}, ×}.

{14} × {14} = {1}, {4} × {4} = 1 and
{11} × {11} = {1} are some units in S.

Example 2.14: Let S = {all subsets of Z_{25}} be subset semigroup under product of the semigroup {Z_{25}, ×}.

Consider {24} × {24} = {1},
{2} × {13} = {1}, {17} × {3} = {1} and {21} × {6} = 1 are some of the units of S.

Example 2.15: Let S = {all subsets of Z_{19}} be subset semigroup of the semigroup {Z_{19}, ×}.

{2} × {10} = {1},
{4} × {5} = {1},
{6} × {16} = {1} and so on.

We make the following observations.

1. In S, if an el ement has inverse then th ey are only the singleton sets alone for they only can have inverse.
2. However S can have zero divisors even if S has subsets of order greater than one.

We have the following theorem the proof of which is left as an exercise to the reader.

THEOREM 2.3: *Let S = {all subsets of Z_n} be the subset semigroup of the semigroup {Z_n, ×} under product. All units in S are only singletons.*

We see suppose if S has other than singleton say A = {a, 1} and {b} = B such that ab = 1 then AB = {a, 1} × {b} = {1, b} ≠ {1}.

Hence the claim.

Now we have seen the concept of zero divisors and units in subset semigroup of a semigroup.

We will now proceed onto define idempotents and nilpotents in the subset semigroup.

DEFINITION 2.5: *Let S be a subset semigroup of a semigroup under the operation *. An element A ∈ S is an idempotent if A^2 = A. An element A_1 ∈ S is defined as a nilpotent if A_1^n = (0) for n ≥ 2.*

We will illustrate this situation by some examples.

Example 2.16: Let
S = {Collection of all subsets of a semigroup Z_{12}} be the subset semigroup of {Z_{12}, ×}.

Consider $\{0, 9\}^2 = \{0, 9\}$; $\{0, 6\}^2 = \{0\}$, $\{9\}^2 = \{9\}$.
$\{0, 4, 9\}^2 = \{0, 4, 9\}$ and so on are some of the idempotents and nilpotents of S.

Example 2.17: Let S = {Collection of all subsets of the semigroup Z_{31} under product} be the subsets semigroup of the semigroup {Z_{31}, ×}. Clearly S has no nilpotents and no zero divisors.

Example 2.18: Let S = {Collection of all subsets of the semigroup Z_{20} under product} be the subsets semigroup of the semigroup $(Z_{20}, \times)$. We see $\{0, 10\}^2 = \{0\}$, $\{0, 5\}^2 = \{0, 5\}$, $\{0, 5, 10\}^2 = \{0, 5, 10\}$ and so on.

In view of all these examples we have the following theorem.

THEOREM 2.4: *Let*
S = {Collection of all subsets of the semigroup Z_n} be the subset semigroup of $\{Z_n, \times\}$. S has idempotents and nilpotents elements if and only if n is a composite number.

The proof is direct hence left as an exercise to the reader.

Corollary 2.1: If in the above theorem, n = p, p a prime, S has no idempotent and no nilpotent elements.

Now we proceed onto define subset subsemigroup and subset ideals of a subset semigroup of a semigroup.

DEFINITION 2.6: *Let S = {Collection of all subsets of a semigroup M under product} be the subset semigroup under product of the semigroup M. Let P $\subseteq$ S; if P is also a subset semigroup under the operation of S, we define P to be a subset subsemigroup of S. If for every s $\in$ S and p $\in$ P we have ps and sp are in P then we define the subset subsemigroup P to be a subset ideal of S.*

We will illustrate this situation by some examples.

Example 2.19: Let
S = {Collection of all subsets of the semigroup $\{Z_{12}, \times\}$} be the subset semigroup of the semigroup $\{Z_{12}, \times\}$.

Let $P_1 = \{\{0\}, \{0, 2\}, \{0, 4\}, \{0, 2, 4\}, \{2\}, \{4\}\} \subseteq S$; P_1 is a subset subsemigroup of S.

Consider
$P_2 = \{\{0\}, \{0, 2\}, \{0, 4\}, \{0, 2, 4\}, \{2\}, \{4\}, \{1\}\} \subseteq S$; P_2 is a subset subsemigroup of S. Clearly P_1 is a subset ideal of S but P_2 is not a subset ideal of S. Let $P_3 = \{\{0,3\}, \{3\}, \{0\}\} \subseteq S$. P_3 is a subset ideal of S.

Example 2.20: Let
S = {Collection of all subsets of a semigroup $\{Z_{20}, \times\}$} be a subset semigroup of the semigroup $\{Z_{20}, \times\}$. Let $P_1 = \{\{0\}, \{0, 10\}, \{10\}\} \subseteq S$ be a subset ideal of the semigroup S.

$P_2 = \{\{0\}, \{0,5\}, \{0,10\}, \{0,15\}, \{0,5,10,15\}, \{0,5,15\}\} \subseteq S$ is a subset ideal of the semigroup S.

$P_3 = \{\{0\}, \{0, 4, 8, 16, 12\}\} \subseteq S$ is a subset ideal of a subset semigroup of S.

Now having seen examples of subset ideals and subset subsemigroups we give the following interesting result.

THEOREM 2.5: *Let S = {Collection of all subsets of the semigroup under product} be the subset semigroup. Every subset ideal of a subset semigroup is a subset semigroup of S; but every subset subsemigroup of a subset semigroup in general is not a subset ideal of S.*

The proof is direct and hence left as an exercise to the reader.

Example 2.21: Let
S = {Collection of all subsets of the semigroup $\{Z_{16}, \times\}$} be the subset semigroup.

$P = \{\{0\}, \{4\}, \{8\}, \{12\}, \{0, 8\}, \{0, 12\}, \{0, 4\}, \{1\}\}$ is only a subset subsemigroup and is not a subset ideal of S.

Hence this example proves one part of the theorem.

We see as in case of semigroups with 1 if the subset semigroup S has {1} then the subset ideals of S cannot contain {1}. Further {0} is the trivial subset ideal of every subset semigroup S.

So far we have seen only subset semigroup got from the semigroup $\{Z_n, \times\}$; now we will proceed onto find subset semigroup using non commutative semigroups and subset semigroup of infinite order.

Example 2.22: Let
S = {Collection of all subsets of the semigroup $(Z, \times)$} be the subset semigroup of $(Z, \times)$. S is of infinite order, commutative has no units and idempotents. S has no zero divisors and nilpotent elements. S has several subset subsemigroups and also subset ideals.

Take P = {all subsets of the set 2Z} $\subseteq$ S; P is a subset ideal of S.

Take P_1 = {all subsets of the set 10Z, {1}} $\subseteq$ S; P_1 is only a subset subsemigroup and is not a subset ideal of S.

Take P_2 = {all subsets of the set 10Z} $\subseteq$ S; P_2 is a subset ideal of S, infact $P_2 \subseteq P$.

Now we have seen infinite subset semigroup.

Example 2.23: Let
S = {set of all subsets of the semigroup $\{Q, \times\}$} be the subset semigroup of $\{Q, \times\}$. S has only subset subsemigroups and has no subset ideals.

T = { set of all subsets of the set Z } $\subseteq$ S is a subset subsemigroup of S and is not a subset ideal of S.

Example 2.24: Let
S = {Collection of all subsets of the semigroup $\{R, \times\}$} be the subset semigroup of the semigroup $\{R, \times\}$.

P = {subsets of the semigroup {Q, ×}} ⊆ S be the subset subsemigroup of S. Clearly P is not a subset ideal of S but only a subset subsemigroup.

We see S has no subset ideals but only subset subsemigroups.

Example 2.25: Let
S = {Collection of all subsets of the semigroup {Q [x], ×}} be the subset semigroup of the semigroup {Q [x], ×}. Clearly S has no subset ideals.

Example 2.26: Let S = {all subsets of the semigroup T = {$M_{2\times2}$ = {A = {a_{ij}} | $a_{ij} \in Z_8$; $1 \leq i, j \leq 2$} under product}. Clearly S is a subset semigroup of the semigroup T. Let P = {all subsets of the subsemigroup
L = {$M_{2\times2}$ = {(m_{ij}) | $m_{ij} \in \{0, 2, 4, 6\}$; $1 \leq i, j \leq 2$} ⊆ T of the semigroup}} be the subset subsemigroup of S which is also a subset ideal of S.

Consider P₁ = {all subsets of the subsemigroup

$$L = \{M_{2\times2} = (m_{ij}) \mid m_{ij} \in \{0, 2, 4, 6\}; 1 \leq i, j \leq 2\} \cup$$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \subseteq S;$$

P_1 is only a subset subsemigroup of S and is not a subset ideal of S.

In case of non commutative semigroups under product we see AB ≠ BA. So even in case of subsets of S. A × B = {ab | a ∈ A and b ∈ B} and B × A = {ba | a ∈ A and b ∈ B} and AB ≠ BA.

Example 2.27: Let S = {Collection of all subsets of the semigroup $M_{3\times3}$ = {M = (a_{ij}) | $a_{ij} \in Z_4$; $1 \leq i, j \leq 3$} under product} be the subset semigroup of the semigroup {$M_{3\times3}$, ×}.

Let P = {Collection of all s ubsets of the subsem igroup $P_{3\times3} = \{B = (p_{ij}) \mid p_{ij} \in \{0, 2\};\ 1 \leq i, j \leq 2\} \subseteq M_{3\times3}\}$; P is a subset subsemigroup of S as well subset ideal of S.

We just show how in general if T is a non comm utative semigroup and
M = {all subsets of $M_{2\times2}$; matrices with entries from Z_4}.

$$\text{Let } A = \left\{\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\right\} \text{ and}$$

$$B = \left\{\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}\right\} \in M.$$

$$AB = \left\{\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}\right\}.$$

$$= \left\{\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix}\right\}.$$

Consider

$$BA = \left\{\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}\right\} \times \left\{\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\right\}$$

$$= \{\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}\}.$$

Clearly $AB \neq BA$.

Thus in case of non com mutative sem igroups we can have the concept of both subs et left ideals of the sem igroup and subset right ideals of a semigroup.

In case of com mutative sem igroups we see the concept of left and right ideals coincide.

The reader is left with the task of finding left an d right subset ideals of a semigroup.

Example 2.28: Let S(4) be a semigroup.

S = {Collection of all subs ets of the sem igroup S(4)} is the subset of semigroup of the symmetric semigroup S(4).

We see

$$T = \{\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\},$$

$$\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\}, \{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}\}, \{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\}, \{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}\},$$

$$\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}\}, \{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}\}, \{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \ldots\}, \ldots\}$$

where T takes its entries from S_4. T is a subset semigroup of the semigroup S(4).

THEOREM 2.6: *Let S = {all subsets of the symmetric semigroup S(n); n < ∞} be the subset semigroup of the symmetric semigroup S(n). S has subset left ideals which are not subset right ideals and vice versa.*

Proof is left as an exercise to the reader.

Now having seen examples of subset semigroups of a semigroup we pass on to study the subset semigroup of a group G.

Example 2.29: Let $G = \{g \mid g^4 = 1\}$ be the cyclic group of order 4. Then S = {Collection of all subsets of G} = $\{\{1\}, \{g\}, \{g^2\}, \{g^3\}, \{1, g\}, \{1, g^2\}, \{1, g^3\}, \{g, g^2\}, \{g, g^3\}, \{g^2, g^3\}, \{1, g, g^2\}, \{1, g, g^3\}, \{1, g^2, g^3\}, \{g, g^2, g^3\}, G\}$.

$\{1, g, g^2\}\ \{g\} = \{g, g^2, g^3\} = \{1, g^2, g^3\}$,
$\{g, g^2, g^3\}\ \{g\} = \{g^2, g^3, 1\}$,
$\{1, g^2, g^3\}\ \{g\} = \{g, g^3, 1\}$,
$\{g, g^2\}\ \{g, g^2\} = \{g^2, 1\}$,
$\{g, g^2\}\ \{g^3\ g^2\} = \{1, g, g^3, 1\} = \{1, g, g^3\}$ and so on.

Thus S need not in general have a group structure.

Example 2.30: Let S = {subsets of a group G = $(Z_5, +)$} be the subset semigroup of the group G.

S = {{0}, {1}, {2}, {3}, {4}, {0, 1}, {0, 2}, {0, 3}, {0, 4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {0, 1, 2}, {0, 1, 3}, {0, 1, 4}, {0, 2, 3}, {0, 2, 4}, {0, 3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}, {0, 1, 2, 3}, {0, 1, 3, 4}, {0, 1, 2 , 4}, {0, 3, 2, 4}, {1, 2, 3, 4, 0}} be the subset semigroup of $\{Z_5, +\}$.

Take {1, 3, 4} + {2, 3, 4} = {3, 0, 1, 4, 1, 2}
{0, 1, 4} + {0, 1, 4} = {0, 3, 1, 2, 4} and so on.

Example 2.31: Let S = {all subsets of S_3} be the subset semigroup of the group S_3. S = {{e}, $\{p_1\}$, $\{p_2\}$, $\{p_3\}$, $\{p_4\}$, $\{p_5\}$, $\{e, p_1\}$, $\{e, p_2\}$, $\{e, p_3\}$, $\{e, p_4\}$, $\{p_1, p_4\}$, $\{e, p_5\}$, $\{p_1, p_3\}$, $\{p_1, p_2\}$, $\{p_1, p_5\}$, $\{p_2, p_3\}$, $\{p_2, p_4\}$, $\{p_2, p_5\}$, $\{p_3, p_4\}$, $\{p_3, p_5\}$, $\{p_4, p_5\}$, $\{e, p_1, p_2\}$, $\{e, p_1, p_3\}$, $\{e, p_1, p_4\}$, $\{e, p_1, p_5\}$, $\{e, p_2, p_3\}$, $\{e, p_2, p_4\}$, $\{e, p_2, p_5\}$, $\{e, p_3, p_4\}$, $\{e, p_3, p_5\}$, $\{e, p_4, p_5\}$, $\{p_1, p_2, p_3\}$, $\{p_2, p_3, p_4\}$, $\{p_2, p_4, p_5\}$, $\{p_1, p_4, p_5\}$, $\{p_1, p_3, p_5\}$, $\{p_1, p_2, p_4\}$, $\{p_1, p_2, p_5\}$, $\{p_1, p_3, p_4\}$, $\{p_2, p_3, p_5\}$, $\{p_3, p_4, p_5\}$, $\{e, p_1, p_2, p_3\}$, …, S_3}.

We see S is only a subset semigroup for every element has no inverse.

$$
\begin{aligned}
\text{Take } \{p_1, p_2, p_3\}^2 &= \{e, p_4, p_5\},\ \{p_1, p_2\} \times \{p_1, p_3\} \\
&= \{e, p_2\, p_1, p_1\, p_3, p_2\, p_3\} \\
&= \{e, p_4, p_5\},\ \{e, p_4, p_5\}\ p_1 \\
&= \{p_1, p_2, p_3\} \text{ and so on.}
\end{aligned}
$$

We see S is a non commutative subset semigroup.
Inview of all this we have the following theorem.

THEOREM 2.7: *Let S = {collection of all subsets of a group G} be the subset semigroup of the group G. S is a commutative subset semigroup if and only if G is commutative.*

Proof is direct hence left as an exercise to the reader.

However these subset semigroups cannot have zero divisors.

Let us study some of examples of subset semigroups of a group.

Example 2.32: Let
S = {Collection of all subsets of the group G = {1, g, g^2}} = {{1}, {g}, {g^2}, {1, g}, {1, g^2}, {g, g^2}, G} be the subset semigroup of the group G.

{g, g^2} × {1, g} = {g, g^2, 1},
{g, g^2}^{2} = {g^2, 1, g},
{1, g} × {1, g} = {1, g, g^2},
{g, g^2} g = {1, g^2},
{g, g^2} g^2 = {g, 1},
{g, g^2} {1, g^2} = {g, g^2, 1, g} = {1, g, g^2},
{1, g^2} × {1, g} = {1, g, g^2} and so on.

Clearly S is only a subset semigroup.

Example 2.33: Let
S = {all subsets of the group G = $D_{2,5}$ = {a,b | a^2 = 1, bab = a, b^5 = 1} be the subset semigroup of the group G. S is non commutative and S is not a subset group.

Now we can study the notion of subset subsemigroups and subset ideals (right or left) of a subset subsemigroup of S of the group G.

Example 2.34: Let S = {all subsets of the group (Z_4, +)} be the subset semigroup of the group (Z_4, +). We see {0, 2} + {0, 2} = {0, 2}, {0, 3} + {0, 1} = {0, 3, 1} and {2} + {0, 2} = {0, 2}, {2} + {2} = {0} and so on.

One can find subset ideals and subset subsemigroups of this also.

It is left as an exercise as it is a matter of routine.

Now we proceed onto define the notion of Smarandache subset semigroup of a subset semigroup S.

DEFINITION 2.7: *Let S be a subset semigroup of the semigroup M (or a group G). Let A ⊆ S; if A is group under the operations* *of S over M (or G); we define S to be a subset Smarandache semigroup of M (or of G) (Smarandache subset semigroup).*

We will illustrate this situation by some examples.

Example 2.35: Let
S = {Collection of all subsets of the sem igroup $\{Z_5, \times\}$} be the subset semigroup of $\{Z_5, \times\}$. Take P = {{1}, {2}, {3}, {4}} $\subseteq$ S, P is a group, hence S is a subset Smarandache semigroup.

Example 2.36: Let
S = {Collection of all subsets of the group G = S_4} be the subset semigroup of the group G. Take A = {{g} | g $\in$ G = S_4} $\subseteq$ S. A is a group; so S is a subset Smarandache semigroup.

Inview of this we have the following theorem.

THEOREM 2.8: *Let*
S = {Collection of all subsets of the group G} be the subset semigroup of the group G. S is a subset Smarandache semigroup of G.

Proof is direct and hence left as an exercise to the reader.

Now we can give exam ples of S marandache subset subsemigroup and Sm arandache subset ideal of a subset semigroup.

Before we p roceed onto give exa mples w e just g ive the following theorem the proof of which is direct.

THEOREM 2.9: *Let S be a subset semigroup of a group G. If S has a subset subsemigroup P($\subseteq$ S) which is a subset Smarandache subsemigroup of S then S is a subset Smarandache semigroup.*

The proof is direct and hence left as an exercise to the reader.

Example 2.37: Let S = {subsets collection of the group S_5} be the subset semigroup of the group S_5.

P = {all subsets of the subgroup A_5} $\subseteq$ S; P is a subset Smarandache subsemigroup of S as A = {{g} | g $\in$ A_5} is a group in P.

Infact S has several subset subgroups and S itself is a subset Smarandache semigroup of the group G.

Example 2.38: Let
S = {Collection of all subsets of a semigroup S(3)} be the subset semigroup of the symmetric semigroup S(3). S is a subset Smarandache subsemigroup of the semigroup S(3).

Let
P = {Collection of all subsets of the semigroup $S_3 \subseteq S(3)$} $\subseteq$ S be the subset subsemigroup of S. Take T = {{g} | g $\in$ S_3} $\subseteq$ P $\subseteq$ S; T is a group hence P is a Smarandache subset subsemigroup of S.

Infact S itself is a subset Smarandache semigroup of the semigroup S(3).

Example 2.39: Let
S = {collection of all subsets of the semigroup {Z_{48}, ×}} be the subset semigroup of the semigroup {Z_{48}, ×}.

Take P = {Collection of all subsets of the semigroup {0}, {1}, {47}} $\subseteq$ S; P is a Smarandache subset subsemigroup of S of the semigroup {Z_{48}, ×}; for A = {{1}, {47}} $\subseteq$ P is a group.

Inview of all these we have the following interesting result the proof of which is left as an exercise to the reader.

THEOREM 2.10: *Let*
S = {Collection of all subsets of the semigroup P} be the subset semigroup of the semigroup P. S is a Smarandache subset semigroup if and only if P is a Smarandache semigroup.

Proof: Clearly if P is a Smaradache semigroup then P contains a subset $A \subseteq P$; such that A is a group under the operations of S.

Thus $M = \{\{a\} \mid a \in A\} \subseteq S$ is a group hence the claim.

If P has no subgroups then we cannot find any subgroup from the subsets of P so S cannot be a Smarandache subset semigroup.

Now we know if G or P the group or the semigroup is of order n then S = {the collection of all subsets of S} is of order 2^n-1.

We study several of the extended classical theorems for subset semigroups of a semigroup or a group.

Recall a finite S-semigroup S is a Smarandache Lagrange semigroup if the order of every subgroup of S divides the order of S.

We see most of the subset semigroups S of the finite semigroup P or group G are not Smarandache Lagrange subset semigroups for the reason being $o(S) = 2^{|P|} - 1$ or $o(S) = 2^{|G|} - 1$.

However some of them can be Smarandache subset weakly semigroups for we may have a subgroup which divides order of S.

We will study some examples characterize those subset semigroups which are neither Smarandache Langrange or Smarandache weakly Lagrange.

Example 2.40: Let S be a Smarandache subset semigroup of the semigroup P or a group G of order 5 or 7, (i.e., $|P| = 5$ or 7 or $|G| = 5$ or 7). S is not Smarandache weakly Lagrange subset semigroup.

In view of this we propose the following simple problems.

Problem 2.1: Does there exists a finite Smarandache subset Lagrange semigroup?

Problem 2.2: Does there exist a finite Smarandache subset Lagrange weakly semigroup?

Example 2.41: Let
S = {collection of all subsets of the semigroup P = $\{Z_{10}, \times\}$} be the subset semigroup of S of order $2^{10} - 1 = 1023$.

The subgroup of S are $A_1 = \{\{1\}, \{9\}\}$.

Clearly $|A_1| \nmid 1023$ so S is not a Smarandache Lagrange weakly subset semigroup.

Example 2.42: Let
S = {Collection of all subsets of the group G = $\{g \mid g^8 = 1\}$} be the subset semigroup of the group G.

The subgroups of S are $A_1 = \{\{1\}, \{g^4\}\}$ and $A_2 = \{\{1\}, \{g^2\}, \{g^4\}, \{g^6\}\}$. We see $|S| = 2^8 - 1$ and clearly $o(A_1) \nmid o(S)$ and $o(A_2) \nmid o(S)$. So S is not a subset Smarandache weakly Lagrange subsemigroup of G.

Example 2.43: Let
S = {Collection of all subsets of the semigroup P = $\{Z_6, \times\}$} be the set semigroup of the semigroup P. $|S|=2^6-1 = 63$.

$A_1 = \{\{1\}, \{5\}\}$ is a subgroup of S. Clearly $|A_1| \nmid o(S)$. So S is not a Smarandache subset weakly Lagrange semigroup.

Now we give some examples of non commutative subset semigroups of a semigroup (or a group).

Example 2.44: Let
S = {Collection of all subsets of the semigroup S(3)} be the subset semigroup of S(3). Clearly $o(S) = 2^{|S(3)|} - 1 = 2^{27} - 1$.

The subset subgroup of S is $A_1 = \{\{1\}, \{p_1\} \{p_2\}, \{p_3\}, \{p_4\}, \{p_5\}\} \subseteq S$. Clearly $o(A_1) \nmid o(S)$. Consider $A_2 = \{\{1\},\{p_2\}\} \subseteq S$, is subset subgroup of S and we see $o(A_2) \nmid o(S)$.

Take $A_3 = \{\{1\}, \{p_4\}, \{p_5\}\} \subseteq S$, A_3 is a subgroup of S are we see $|A_3| \nmid o(S)$.

Thus S is not even a Smarandache weakly Lagrange subset semigroup.

Example 2.45: Let S = {Collection of all subsets of the group A_4} be the non commutative subset semigroup of the group A_4.

$o(S) = 2^{12} - 1 = 4095.$

Consider the subset subgroup

$$P_1 = \{\left\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}\right\}\} \subseteq S.$$

P_1 is a group in S and $o(P_1) = 3$ and 3/4095. Thus S is a Smarandache weakly Lagrange subset semigroup of the group G but S is not a Smarandache Lagrange subset semigroup; for take $P_2 = \{\{g\} \mid g \in A_4\} \subseteq S$; P_2 is group and $o(P_2) = 12$ but $12 \nmid 4095$ hence the claim.

Now we see the subset semigroup can be infinite complex semigroup / group or a finite complex modulo integers group / semigroup.

Example 2.46: Let
S = {Collection of all subsets of C under '+'} be the subset semigroup of the group C. Clearly C has subset subgroups say (Z, +), (Q, +), (R, +) and so on. So S is a Smarandache subset semigroup of the group C.

Example 2.47: Let
S = {Collection of subsets of the semigroup $C(Z_3)$ under product} be the subset semigroup of the complex modulo integers of $C(Z_3)$. S is a Smarandache subset semigroup but is not a Smarandache Lagrange subset semigroup of $\{C(Z_3), \times\}$.

Example 2.48: Let
S = {Collection of all subsets of $C(Z_n)$ under $\times$} be the subset semigroup of complex modulo integers S is a Smarandache subset semigroup of $C(Z_n)$.

THEOREM 2.11: *Let*
S = {Collection of all subsets of the semigroup $\{C(Z_n), \times\}$ be the subset semigroup of $\{C(Z_n), \times\}$.

(i) S is a Smarandache subset semigroup of $\{C(Z_n), \times\}$.
(ii) S is not a Smarandache subset Lagrange semigroup of $\{C(Z_n), \times\}$.

The proof is direct, hence left as an exercise to the reader.

Next we proceed onto give examples of subset semigroup of dual numbers, special dual like numbers and their mixed structure.

Example 2.49: Let S = { Collection of all subsets of $C(Z_{12})$} be the subset semigroup of complex modulo integers S. S is a Smarandache subset semigroup of S.

Suppose we have S to be a subset semigroup over the semigroup P (or group G) then there exist $T \subseteq S$ such that $T \cong P$ (as a semigroup) or $T \cong G$ as a group. That is $B = \{\{g\} \mid g \in P\}$ is such that $B \cong P$ as a semigroup.

Take $D = \{\{g\} \mid g \in G\} \subset S$. Clearly $D \cong G$ as a group. Thus the basic structure over which we build a subset semigroup contains an isomorphic copy of that structure.

Example 2.50: Let S = {Collection of all subset of the dual number semigroup Z(g) under product} be the subset semigroup of Z(g). S has ideals and zero divisors. $P = \{\{ng\} \mid n \in Z\} \subseteq S$ is a nilpotent subset subsem igroup of S as $ab = 0$ for all a, b $\in$ P.

Example 2.51: Let S = {Collection of all subsets of the semigroup $Z_{10}(g) = \{a + bg \mid a, b \in Z_{10}, g^2 = 0\}$ under product} be the subset sem igroup of $Z_{10}(g)$. S has nilpotent subset subsemigroup.

In view of these examples we have the following theorem.

THEOREM 2.12: *Let S = {Collection of subsets of the semigroup $Z_n(g)$ of dual numbers under product} be the subset semigroup of $Z_n(g)$. S has a nilpotent semigroup of order n.*

Proof: Follows from the simple number theoretic techniques.

We see $P = \{\{0\}, \{g\}, \{2g\}, \ldots, \{(n-1)g\}\} \subseteq S$ is such that $P^2 = \{0\}$, hence the claim.

Now we proceed onto give examples of subset semigroup of special dual like number semigroup.

Example 2.52: Let S = {Collection of all subsets of the semigroup $Z_n(g_1)$ where $g_1^2 = g_1$ and $Z_n(g_1) = \{a + bg_1 \mid a, b \in Z_n\}$ under product} be the subset semigroup of special dual lik e number under product.

S has idem potents and zero divisors. S has subset ideals and subset subsemigroups.

Example 2.53: Let S = {Collection of all subsets of the special quasi dual num ber semigroup, $Z_6(g_2)$} be the subset sem igroup of the special quasi dual number semigroup $Z_6(g_2)$ where $g_2^2 = -g_2$.

S has zero divisors idem potents and units. Infact S is a Smarandache subset semigroup which is not a Smarandache Lagrange subset semigroup.

Example 2.54: Let S = {Collection of all subsets of the mixed dual number semigroup $Z_{18}(g, g_1) = \{a_1 + a_2g + a_3g_1 \mid a_i \in Z_{18}, g^2 = 0, g_1^2 = g_1, g_1g_2 = gg_1 = 0, 1 \leq i \leq 3\}$ under product} be the subset semigroup of $Z_{18}(g, g_1)$. S has units, zero divisors, zero square subset subsemigroup and S is a Smarnadache subset semigroup which is not a Smarandache Lagrange subset semigroup.

P = {{g}, {0}, {2g}, …, {17g}} is the zero square subset subsemigroup.

Example 2.55: Let S = {Collection of all subsets of the dual number semigroup of dimension three given by $Z_7(g_1, g_2, g_3) = \{a_1 + a_1g_1 + a_3g_2 + a_4g_3 \mid a_i \in Z_7, g_1^2 = g_2^2 = g_3^2 = gg_2 = g_2g_3 = g_3g_1 = g_1g_3 = g_3g_2 = g_2g_1 = 0, 1 \leq i \leq 4\}$ under product} be the subset semigroup of $Z_7(g_1, g_2, g_3)$. S under product has zero divisors, zero subset subsemigroups and S is a Smarandache subset semigroup which is not a Smarandache Lagrange subset semigroup.

Example 2.56: Let S = {Collection of all subsets of the dual number semigroup T of dimension five; that is $T = \{Z(g_1, g_2, g_3, g_4, g_5) \mid g_ig_j = 0, 1 \leq i, j \leq 5\} = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_4 \mid a_i \in Z, 1 \leq i \leq 6\}, \times\}\}$ be the subset semigroup of T.

T has zero square subset subsemigroups, zero divisors and no units or idempotents. Infact T is a Smarandache subset semigroup of infinite order.

Example 2.57: Let $M_{5\times 5} = \{M = (m_{ij}) \mid m_{ij} \in Z_{10}(g, g_1) = a + bg + cg_1 \mid a, b, c \in Z_{10},\ g_1^2 = g^2 = 0\ g_1 g = gg_1 = 0\}$ be a sem igroup of higher dimensional dual number under matrix product.

S = {Collection of all subsets of M} is the subset semigroup. S is non commutative and of finite order.

Example 2.58: Let S = {subset collection of t he semigroup $M_{2\times 2} = \{M = (a_{ij}) \mid a_{ij} \in Q(g_1)$ where $g_1^2 = 0,\ 1 \leq i, j \leq 2\}\}$ be the subset semigroup of the semigroup $Q(g_1)$ under matrix product. S is a non commutative subset semigroup of infin ite order. S contains zero square subsemigroups.

Example 2.59: Let S = {Collection of all subs ets of the semigroup $T = \{C(g_1, g_2, g_3) \mid g_1^2 = 0 = g_i g_j\ 1 \leq i, j \leq 3\}$ under product} be the subset semigroup of the semigroup T. S has zero square subsemigroups.

Example 2.60: Let S = { Collection of all subsets of sem igroup $T = C(g_1, g_2) = \{a_1 + a_2 g_1 + a_3 g_2 \mid a_i \in C;\ 1 \leq i \leq 3\}$ under '+'} be the subset subsemigroup of T under '+'.

Example 2.61: Let S = {Collection of all subsets of $M_{3\times 3} = \{(a_{ij}) \mid a_{ij} \in C(g_1, g_2, g_3);\ g_1^2 = 0,\ g_2^2 = g_2,\ g_3^2 = 0,\ g_i g_j = 0 = g_j g_i,\ 1 \leq i, j \leq 3\}\}$ be a subset sem igroup of $M_{3\times 3}$ under product; S is commutative.

Now we can also define set ideals of subset sem igroups as in case of semigroups.

DEFINITION 2.8: *Let S be a subset semigroup of the semigroup (or a group). $P \subseteq S$ be a subset of the subset semigroup. Let $A \subseteq S$, we say A is a set subset ideal over P of S if ap, pa $\in$ A for every a $\in$ A and p $\in$ P.*

We will illustrate this situation by some examples.

Example 2.62: Let S = { Collection of all subsets of Z_6} be the subset semigroup of Z_6 under ×.

Take P = {{ 0), {0, 3}} ⊆ S, a subset subsemigroup of S. Take A_1 = {{0}, {2}} ⊆ S; A_1 is a set subset ideal of S over P of S.

Take A_2 = {{0}, {4}} ⊆ S as a set subset ideal over P of S. A_3 = {{0}, { 0, 2}} ⊆ S is a set subset ideal over P of S. Take A_4 = {{0}, {0, 4}} ⊆ S to be a set subset ideal over P of S and so on.

Example 2.63: Let S = {Collection of all subsets of the semigroup Z_8 under product} be the subset semigroup of S. P = {{0}, {4}} ⊆ S is a subset subsemigroup in S. A_1 = {{0}, {2}} ⊆ S is a set subset ideal of S over the subset subsemigroup P of S.

Example 2.64: Let
S = {Collection of all subsets of the semigroup T=(Z_{12}, ×)} be the subset semigroup of the semigroup T.

P = {{0}, {4}, {8}} ⊆ S be the subset subsemigroup of S. A_1 = {{0}, {3}}, A_2 = {{0}, {6}}, A_3 = {{0}, {6}, {3}}, A_4 = {{0}, {0,3}} , A_5 = {{0}, {0, 6}} , A_6 = {{0}, {0, 3}, {3}} , A_7 = {{0}, {0,3}, {6}} and so on are all set subset ideals of S over P of the subset semigroup S.

Example 2.65: Let
S = {Collection of all subsets of the semigroup T = { Z_{10}, ×}} be the subset semigroup of T.

Let P = {{0}, {2}, {0, 2}, {4}, {0, 4}, {0, 6}, {6}, {8}, {0, 8}} ⊆ S be a subset subsemigroup of S. A_1 = {{0}, {5}}⊆S is a set subset ideal of S over A_1. A_2 = {{0}, {0, 5}} ⊆ S is a set subset ideal of S over the subset subsemigroup P of S.

Example 2.66: Let
S = {Collection of all subsets of the semigroup S(3)} be the subset semigroup of S. P={{e}, {p_1}} be a subset subsemigroup of S. A = {{e}, {p_2}, {p_5}, {p_4}} $\subseteq$ S is a set ideal subset of S over the subset subsemigroup P of S.

Example 2.67: Let S = {Collection of all subsets of the semigroup T = {Z_6, $\times$}} be the subset semigroup of the semigroup T.

P = {{0}, {0, 3}} is the subset subsemigroup of S. A_1 = {{0}}, A_2 = {{0}, { 0, 3}}, A_3 = {{0}, {2}}, A_4 = {{0}, {4}}, A_5 = { {0}, {0, 2}}, A_6 = {{0}, { 0, 4}}, A_7 = {{0}, {1}, {0, 3}}, A_8 = {{0}, {0, 1}, {0, 3}}, A_9 = {{0}, {5, 0}} and so on are all set subset ideals of S over P.

As in case of set ideals of a semigroup we can also in case of set subset ideals define a topology which we call as set subset ideal topological space analogous to set ideal topological space. Study in this direction is similar to set ideal topological spaces hence left as an exercise to the reader.

We however give some examples of a set subset ideal topological space of a subset semigroup defined over a subset subsemigroup.

Example 2.68: Let
S = {Collection of all subset of a semigroup B = {Z_4, $\times$}} be the subset of the semigroup. Let P = {{0}, {2}} be a subset subsemigroup of S over P.

Let T = {Collection of all set subset ideals of S over P}

= {{0}, {{0}, {0, 2}}, {{0}, {2}}, {{0}, {1}, {2} }, {{0}, {1}, {0, 2}, {2}}, {{0}, {0, 1}, {0, 2}}, {{0}, {3}, {2}}, {{0}, {0, 3}, {0, 2}}, {{0}, {0, 1}, {0, 3}, {0, 2}}, {{0}, {0, 2, 3}, {0, 2}} and so on} be the set subset ideal topological space of the subset semigroup S over P.

We can take $P_1 = \{\{0\}, \{0, 2\}\}$ and find the related set subset ideal topological space.

It is pertinent to keep on record that this study of set subset ideal topological space is a matter of routine.

Further we see in case the set subset ideal topological space has finite number of elements we can find the related lattice.

Finally we can also have for set subset ideal topological space the notion of set subset ideal topological subspaces.

We just note this sort of defining set subset ideal topological spaces increase the number of finite topological space for this also depends on the subset subsemigroup on which it is defined.

Example 2.69: Let
S = {Collection of all subsets of the semigroup $\{Z_3, \times\}$ where $Z_3 = \{0,1,2\}\} = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$.

Let $P = \{\{0\}, \{1\}\} \subseteq S$ be a subset subsemigroup of S. Let T = {Collection of all set subset ideals of S over the subset semigroup P}

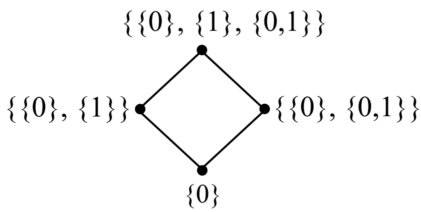
$= \{\{0\}, \{\{0\}, \{1\}\}, \{\{0\}, \{2\}\}, \{\{0\}, \{0, 1\}\}, \{\{0\}, \{0, 2\}\}, \{\{0\}, \{1, 2\}\}, \{\{0\}, \{1, 2, 0\}\}, \{\{0\}, \{1, 2\}, \{1\}\}, \{\{0\}, \{1\}, \{2\}\}, \{\{0\}, \{1\}, \{0, 1\}\}, \{\{0\}, \{1\}, \{0, 2\}\}, \{\{0\}, \{1\}, \{1, 2, 0\}\}, \{\{0\}, \{2\}, \{0, 1\}\}, \{\{0\}, \{2\}, \{0, 2\}\}, \{\{0\}, \{2\}, \{0, 1, 2\}\}, \{\{0\}, \{0, 1\}, \{0, 2\}\}, \{\{0\}, \{0, 1\}, \{0, 1, 2\}\}, \{\{0\}, \{0, 2\}, \{0, 1, 2\}\}, \{\{0\}, \{1\}, \{2\}, \{0, 1\}\}, \{\{0\}, \{1\}, \{2\}, \{0, 2\}\}, \{\{0\}, \{1\}, \{0, 1\}, \{0, 2\}\}, \{\{0\}, \{1\}, \{0, 1\}, \{0, 1, 2\}\} \ldots\}$.

T is a set subset ideal topological space of S over the semigroup P.

Example 2.70: Let S = {Collection of all subsets of the semigroup $(Z_2, \times) = \{\{0, 1\}, \times\}\} = \{\{0\}, \{1\}, \{0, 1\}\}$. This has $P = \{\{0\}, \{1\}\} \subseteq S$ to be a subset subsemigroup of S.

Let T = {Collection of all set subset ideals of S over the subset subsemigroup P of S} = {{0}, { 0}, {1}}, {{0}, {0, 1}}, {{0}, {1}, {0, 1}}} be a set subset ideal topological space of S over P.

The lattice associated with S is



Now we proceed on to the give some more properties of subset semigroups.

Before enumerating these properties we wish to state even if the set S = {Collection of all subsets of a semigroup or group say of order 3}, then also for the subset subsemigroup {{0}, {1}} = P we have a very large collection of set subset ideals; we see if T denotes the collection of all set subset ideals of S over the subset subsemigroup P;

then o(T) = ${}_7C_1 + {}_7C_2 + {}_7C_3 + {}_7C_4 + {}_7C_5 + {}_7C_6 + {}_7C_7$.

So T is a set subset ideal topological space of a fairly large size. Thus if we change the subset subsemigroup we may have a smaller set subset ideal topological space of S.
P = {{0}, {0,1}, {0,2}} $\subseteq$ S is the subset subsemigroup of S.

Let T = {{0}, {{0}, {0, 2} , {0, 1}}, {{0}, {1}, {2}}, {{0}, {0, 1}, {0, 2}, {1}, {2}}} be a topological space of lesser order.

Take P_1 = { {0}, {1}, {2}}, we see the set subset ideal topological space associated with the subset subsemigroup is also small.

Recall if S is a finite S-subsemigroup. We define a $\in$ A (A subset of S) to be Smarandache Cauchy element of S if

$a^r = 1$ ($r > 1$) and 1 is a unit of A and r divides the order of S otherwise a is not a Smarandache Cauchy element of S.

We have the same definition associated with the Smarandache subset semigroup also though the concept of subset semigroup is new.

We will illustrate this situation by some examples.

Example 2.71: Let
S = {Collection of all subsets of the semigroup $\{Z_6, \times\}$} be the subset semigroup of the semigroup $\{Z_6, \times\}$.

Clearly S is a Smarandache subset semigroup.

Take A = {{1}, {5}} $\subseteq$ S, A is a subset subgroup of S. |A| = 2 and {5} $\in$ S such that $\{5\}^2 = \{1\}$; however $2 \nmid 2^6 - 1$.

So {5} is not a S-Cauchy element of S.

Example 2.72: Let S = {Collection of subsets of P = $\{\{Z_5, \times\}\}$ be the subset semigroup of P. $o(S) = 2^5 - 1$.

Clearly A = {{1}, {2},{3},{4}} $\subseteq$ S is a subgroup of S. Now {4} $\in$ S, $\{4\}^2 = \{1\}$ but $2 \nmid 2^5-1$ so {4} is not a Smarandache Cauchy element of S.

Consider {2} $\in$ S we see $\{2\}^4 = \{1\}$ but $4 \nmid 2^5-1$ so {2} is not a S-Cauchy element of S.

Thus the subset semigroup has no Smarandache Cauchy elements.

Inview of this we have the following theorem.

THEOREM 2.13: *Let*
S = {Collection of all subsets of a semigroup $\{Z_n, \times\}$} be the subset semigroup of the semigroup $\{Z_n, \times\}$. We have $a \in A \subseteq S$

a subgroup of S such that a^2 = identity in A, is never a S-Cauchy element of S.

Proof: We see $\{Z_n, \times\}$ is a S-semigroup for all n as A = {1, n–1} is group under product.

Further S = {Collection of all subsets of $\{Z_n, \times\}$} is a subset semigroup which is always a S-subset semigroup as A = {{1}, {n–1}} ⊆ S is such that a = {n–1} ∈ A is such that $a^2 = \{n-1\}^2 = \{1\}$ that $2 \nmid (o(S))$ as $o(S) = 2^n - 1$.

Hence the claim.

Corollary 2.2: If S = {Collection of all subsets of the semigroup $\{Z_p, \times\}$, p a prime} be the subset subsemigroup of S. A = {{1}, {2}, {3}, …, { p–1}} ⊆ S is a subgroup of S. There exists elements in A which are not S-Cauchy elements of S.

We see all elements a ∈ A such that $\{a\}^{2m} = \{1\}$ (m ≥ 1) are not Smarandache Cauchy elements of A.

Now we see the properties in case of the symmetric semigroup S(n).

Example 2.73: Let S = {Collection of all subsets of the semigroup S(4)} be the subset semigroup. A = {{g} | g ∈ S_4} ⊆ S is a group in S.

We see no element a ∈ A such that a^n = (e); n even is a S-Cauchy element of S.

This follows from the simple fact $2^n - 1$ is always an odd number so it is impossible for any a ∈ A which is of even power to divide $2^n - 1$ which is the o(S).

In view of this we have the following theorem.

THEOREM 2.14: *Let S = {Collection of all subsets of a semigroup P of finite order} be the subset semigroup of P. Suppose S is a S-subset semigroup. $A \subseteq S$ be a group of S. Every $a \in A$ such that a^m (m even) are not S-Cauchy elements of S.*

Proof: Follows from the simple fact $o(S) = 2^n–1$ is an odd number.

We can define Smarandache p-Sylow subgroups of a subset semigroup in an analogous way as S is only a semigroup.

We first make the following observations from the following example.

Example 2.74: Let S = {Collection of all subsets of a semigroup $P = (Z_{13}, \times)$} be the subset semigroup of P. We see $A = \{\{g\} \mid g \in Z_{13} \setminus \{0\}\} \subseteq S$ is a group. So S is a S-subset semigroup of P.

We see S has no Smarandache 2-Sylow subgroup for $o(S) = 2^{13} – 1$.

Thus we see this can be extended to a case of any general subset semigroup S.

Example 2.75: Let S = {Collection of all subsets of the semigroup $P = Z_n$ with $|P| = n$} be the subset semigroup. We see S is a finite S-subset semigroup. $o(S) = 2^n–1$. S has no Smarandache 2-Sylow subgroup.

THEOREM 2.15: *Let S = {Collection of all subsets of the finite semigroup P} be the subset semigroup of order $2^{|P|}–1$. Clearly S has no Smarandache 2-Sylow subgroups.*

How to find or overcome all these problems? These problems may be over come but we may have to face other new problems. In view of all these now we make a new definition called power set semigroup S^P with various types of operations like $\cup$, $\cap$ or the operation of the semigroup over which S^P is built.

Throughout this book S^P will denote the power set semigroup of a semigroup that is $\phi \in S^P$. When it is a just a set we see the power set semigroup S includes the empty set ϕ and S^P is of order 2^n if n is the number of elements in the set. We have only two types of operations viz., $\cup$ and $\cap$ in both the cases $\{S^P, \cup\}$ and $\{S^P, \cap\}$ are semilattices of order 2^n.

This we have already discussed in the earlier part of this chapter.

Now we study only power set semigroup S^P of a semigroup M.

DEFINITION 2.9: *Let S^P = {Collection of all subsets of a semigroup T including the empty set ϕ}. S^P is a power set semigroup with $A\phi = \phi A = \phi$ for all $A \in S^P$.*

We give some examples before we make more conditions of them.

Example 2.76: Let
S^P = {Collection of all subsets of $\{Z_3, \times\}$ together with ϕ} be the power set semigroup of the semigroup $\{Z_3, \times\}$.

$S^P = \{\{\phi\}, \{1\}, \{0\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$.
We see S^P is a semigroup
$\{\phi\}\ A = \{\phi\}$. $\phi\{0\} = \phi$,
$\{0\}A = \{0\}$ $(A \neq \phi)$
and $\{1\}\ A = A$ for all $A \in S^P$. $o(S^P) = 2^3 = 8$.

Example 2.77: Let
S^P = {Collection of all subsets of the semigroup $\{Z_{20}, \times\}$ be the power set semigroup.

$|S^P| = 2^{20}$.

We make the following observations.

(i) Clearly $S \subseteq S^P$ and S is the hyper subset subsemigroup of S^P.
(ii) $o(S) = 2^n - 1$ and $o(S^P) = 2^n$.
(iii) By inducting ϕ in S we see other operations like $\cup$ and $\cap$ can also be given on S^P.

Now we see the power set semigroup S^P is a Smarandache power set semigroup if the semigroup T using which S^P is built is a S-semigroup.

Now if we take the S-power semigroup S then $o(S^P) = 2^n$.

When $o(S^P) = 2^n$ we cannot have any Smarandache p-Sylow subgroups for S^P; $p > 2$ (p a prime or p a power of a prime).

Secondly S^P cannot be Smarandache Lagrange power set semigroup for we may have subgroups of order other than powers of two.

All these will be illustrated by some examples.

Example 2.78: Let
S^P = {Collection of all subsets of the semigroup $\{Z_{11}, \times\}$} be the power set semigroup of the semigroup $\{Z_{11}, \times\}$. $o(S^P) = 2^{11}$.

Now the subgroups of S^P are
$A_1 = \{\{1\}, \{1, 0\}\}$ and
$A_2 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}\}$.

Clearly $o(A_1) / S^{|P|} = 2^{11}$ but $o(A_2) \not| \; 2^{11}$ as $10 \not| \; 2^{11}$.

So S^P cannot be S-Lagrange it can only be S-weakly Lagrange.

We see A_2 has $\{2\}$ but $\{2\}^{10} = \{1\}$ but $10 \nmid 2^{11}$ so $\{2\}$ is a S-Cauchy element of S^P.

We see $\{4\} \in A_2$ and $\{4\}^5 = \{1\}$ but $5 \nmid 2^{11}$ so $\{4\}$ is not a S-Cauchy element of S^P.

Thus if at all S^P has any S-Cauchy element it must be of order 2^s ($s \leq n$).

Now we can as in case of usual subset semigroup build in case of power set semigroup the set power set ideal and over a power set subsemigroup. Using the set power set ideals of S^P over any power set subsemigroup construct set ideal power set semigroup topological spaces and study them.

This in turn increases the number of topological spaces of finite order.

Now we proceed onto present a few problems for the reader.

Problems:

1. Find the subset semigroup of the semigroup $(Z_{30}, \times)$.

2. Find the subset semigroup S of the semigroup $(Z_{37}, \times)$.

   (i) Can S have ideals?
   (ii) Does S contain zero divisors?
   (iii) Find the number of elements in S.

3. Let M be the subset semigroup of the symmetric semigroup S(5).

   (i) Find the order of M.
   (ii) Give a subset subsemigroup of M which is not an ideal.
   (iii) Can M have idempotents?

(iv) Find a su bset left ideal of M which is not a su bset right ideal of M and vice versa.

4. Let S be the subset semigroup of $\{Z_5 \times Z_5, \times\}$.

(i) Find the number of elements in S.
(ii) Can S have zero divisors?
(iii) Can S have idempotents?
(iv) Give a subset subsemigroup of S which is not a subset ideal of S.

5. Let $P = \left\{\left.\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right| x_i \in Z_{12}; 1 \leq i \leq 3\right\}$

be a semigroup under natural product $\times_n$.

(i) Find the subset semigroup S of P.
(ii) Find o(S).
(iii) Prove S has zero divisors.
(iv) Prove S has nilpotents.
(v) Find idempotents of S.

6. Find the diff erence between the subset sem igroup of a semigroup and the subset semigroup of a group.

7. Can a subset semigroup of a group be a group?

8. Find som e interesting p roperties enjo yed b y su bset semigroup of a semigroup.

9. Does there exist a S marandache subset weakly Lagrange semigroup of a group or a semigroup?

10. Does there exist a Smarandache subset Lagrange semigroup of a group or semigroup?

11. Let S = {Collection of all subsets of the semigroup $M_{3\times3} = \{A = (a_{ij}) \mid a_{ij} \in C(Z_{10}), 1 \leq i, j \leq 3\}$ under product} be the subset semigroup of $M_{3\times3}$ under matrix product.

    (i) Find all right subset ideals of S.
    (ii) Find all those subset subsemigroups which are not subset ideals of S.
    (iii) Can S be Smarandache Lagrange subset semigroup?
    (iv) Can S be atleast Smarandache weakly Lagrange subset semigroup?

12. Let S be the subset semigroup of the semigroup $P = \{Z_{12}, \times\}$.

    (i) Find idempotents and nilpotents of S.
    (ii) Can S have units?
    (iii) Is S a Smarandache subset semigroup?
    (iv) Does S contain subset subsemigroup which is not a subset ideal?

13. Does there exists a S-Lagrange subset semigroup for a suitable semigroup P or a group G?

14. Does there exists S-Cauchy elements of order p, p > 2 for the semigroup P = S(6)?

15. Let P = S(7) be the symmetric semigroup. S = {Collection of all subsets of P} be the subset semigroup of the semigroup P.

    (i) Find o(S).
    (ii) Is S a S-subset semigroup?
    (iii) Find 2 right ideals which are not left ideals.
    (iv) Is S a S-Lagrange subset semigroup?
    (v) Does S contain S-Cauchy elements?
    (vi) Can S have S-p-Sylow subgroups?

16. Let S = {Collection of all subsets of the semigroup $M_{3\times3} = \{(m_{ij}) = M \mid m_{ij} \in Z_{10}(g);\ 1 \leq i, j \leq 3, g^2 = 0\}$ under product} be the subset semigroup of $M_{3\times3}$.

    (i) Is S a Smarandache subset semigroup?
    (ii) Find 3 subset subsemigroups which are not subset ideals of S.
    (iii) Give 2 subset right ideals which are not subset left ideals of S.
    (iv) Find all the subgroups of S.
    (v) Is S a S-Lagrange subset subsemigroup?

17. Let S = {Collection of all subsets of $C(Z_{14})$} be the subset semigroup of the semigroup $C(Z_{14})$ under $\times$.

    (i) Find zero divisors of S.
    (ii) Prove S is a S-subset semigroup.
    (iii) Find idempotents of S.
    (iv) Can S have S-Cauchy elements?

18. Let S = {collection of all subsets of the semigroup $P = \{C(Z_{10})(g_1, g_2, g_3, g_4) = a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_5 \mid a_i \in C(Z_{10}),\ g_1^2 = 0;\ g_2^2 = g_2,\ g_3^2 = -g_3,\ g_4^2 = g_4,\ g_ig_j = g_jg_i = 0,\ 1 \leq i, j \leq 4\ (i \neq j)\}$ under product} be the subset semigroup of $C(Z_{10})(g_1, g_2, g_3, g_4)$.

    (i) Find o(S).
    (ii) Is S a S-subset semigroup?
    (iii) Can S have zero square subset subsemigroups?
    (iv) Can S have S-Cauchy elements?
    (v) Can S have S-p-Sylow subgroups?

19. Let S = {subsets of the group $G = D_{2,7}$} be the subset semigroup of the group G.

    (i) Can S have idempotents?
    (ii) Is S a S-subset semigroup?
    (iii) Can S be S-Lagrange subset semigroup?
    (iv) Can S have S-Cauchy elements?

(v) Can S be a group?
(vi) Find subset ideals in S.
(vii) Show S has subset subsemigroups which are not set ideals?

20. Let S = {Collection of all subsets of the group Z_{15}, +} be the subset semigroup of the group $\{Z_{15}, +\}$.

(i) Find at least two subset subsemigroups which are not subset ideals of S.
(ii) Is S a S-subset semigroup?
(iii) Can S have S-Cauchy elements?

21. Obtain some interesting and special features enjoyed by the subset semigroups S of the group S_n.

22. Can we say if S_1 is a subset semigroup of S(n), S in problem 21 is a subset subsemigroup of S_1?

23. Can we have based on problems (21) and (22) the notion of Cayleys theorem for S-subset semigroup?

24. Does there exist a S-subset semigroup of finite order which satisfies the S-Lagrange theorem?

25. Does there exist a S-subset semigroup of finite order which does not satisfy S-weakly Lagranges theorem?

26. Is it possible to have a finite S-subset semigroup S with A as a subgroup of S, so that every element $a \in A$ is a S-Cauchy element of S?

27. Does there exist in a finite S-subset semigroup with a proper sub group A such that no $a \in A$ is a S-Cauchy element of S?

28. Does there exist in a finite S-subset semigroup S, a S-p-Sylow subset subgroup?

29. Let S = {Collection of all subsets of the semigroup T = $\{Z_{10}, \times\}$} be the subset semigroup of the semigroup T.

(i) Let P_1 = { {0}, {5}} be a subset subsemigroup. T_1 = {Collection of all set subset ideals of S over P_1}.

(a) Find o(T).
(b) Prove T_1 is a set subset ideal topological space of S over P_1.

(ii) Take P_2 = { {0}, {2}, {4}, {6}, {8}} be a subset subsemigroup.
T_2 = {Collection of set subset ideals of S over P_2}.
Study (a) and (b) for T_2.

(iii) Find the total number of subset subsemigroups in S.

(iv) How many set subset ideal topological spaces over these subset subsemigroups are distinct?

30. Let S_1 = {Collection of all subsets of the group S_7} be the subset semigroup of the group S_7.
Study the problems mentioned in problem 29 for this S_1.

31. Let S = {Collection of all subsets of the semigroup, T = {Z(g), ×} $g^2 = 0$} be the subset semigroup of the semigroup T.

(i) Prove S has infinite number of subset subsemigroups.
(ii) Prove using S we have infinite number of distinct set subset ideal topological spaces.

32. Let S = {collection of all subsets of the semigroup $M_{2\times2} = \{M = (a_{ij}) \mid a_{ij} \in C(Z_5), 1 \leq i, j \leq 2\}$ be the subset semigroup of the semigroup $\{M_{2\times2}, \times\}$.

(i) Find the number of subset subsemigroups of S.

(ii) Find the number of Smarandache subset subsemigroups of S.
(iii) Find the number of set subset ideal topological spaces of S over these subset subsemigroups of S.

33. Define and develop the concept of Smarandache quasi set subset ideal of a subset semigroup S.

34. Give examples of S-quasi set subset ideal of the subset semigroup S.

35. Define strong set subset ideal of a subset semigroup S built over $\{Z_{18}, \times\}$.

36. Develop properties discussed in problems (33), (34) and (35) in case of subset semigroup of the semigroup S(20).

37. Let
$S = \{$Collection of all subsets of the semigroup $\{Z_{12}, \times\}\}$ be the subset semigroup of the semigroup $\{Z_{12}, \times\}$.

(i) Show S has S-subset subsemigroups.

(ii) For the subset subsemigroup
$T = \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}\}$ find the associated set subset ideal of S over the subset semigroup T of S.

38. Let $S^P = \{$Collection of all subsets of a semigroup $T = \{Z_{40}, \times\}$ including $\phi\}$ be the power set semigroup of T.

(i) Show S^P cannot have S-p-Sylow subgroups $p \geq 3$.
(ii) Show S^P can only be a S-weakly Lagrange subset semigroup.
(iii) Prove S^P cannot have S-Cauchy element of order greater than or equal to 3.

39. Let
S^P={Collection of all subsets of the group S_5 including ϕ} be the subset semigroup of the group S_5.

(i) Find for the subset subsemigroup

$$P_1 = \{\left\{\begin{pmatrix}1&2&3&4&5\\1&2&3&4&5\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4&5\\1&1&3&4&5\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4&5\\2&2&3&4&5\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4&5\\2&1&3&4&5\end{pmatrix}\right\}\} \subseteq S^P;$$

of S the set subset ideal topolo gical space of S associated with P_1.

(ii) Find some S-subset subsemigroups which are not S-subset ideals.

40. Let S^P={Collection of all subsets of the group $D_{2,9}$ together with ϕ} be the subset sem igroup of the group D_{29}.

(i) Find S^P-subset ideals of S^P.

(ii) For the subset subsem igroup P_1 = {{1}, {a}, {a,1}} of S^P find th e set subset ideal of S^P over the subset subsemigroup P_1 of S.

41. Let
S^P = {Collection of all subsets of the semigroup {Z_{18}, ×}} be the subset semigroup of the semigroup {Z_{18}, ×}.

(i) Let P_1 = {{0}, {3}, {6}, {9}, {12}, {15}} be a subset subsemigroup of S. Find the set subset ideal topological space of S^P.

(ii) Show S^P can only be a S-weakly Lagrange subgroup.

42. Obtain some interesting properties about power set subset semigroups.

43. Distinguish the power set subset semigroup and the subset semigroup for any semigroup P.

44. Let S^P = {Collection of all subsets of the group $G = \langle g \mid g^{12} = 1\rangle$ together with ϕ}.
Find the total number of set power set ideal topological spaces of S^P.

45. Let S^P = {Collection of all subsets of the semigroup

$$P = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \, a_i \in Z_6(g),\ 1 \leq i \leq 3,\ g^2 = 0\right\}$$ under natural

product $\times_n$} be the power set semigroup of the semigroup P.

(i) Find the total number of power set subsemigroups.
(ii) Find the total number of distinct set ideal power set topological spaces.

46. Let S^P = {Collection of all subsets of the semigroup

$$P = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \text{ where } a_i \in C(Z_5),\ 1 \leq i \leq 6\right\}$$

under the natural product $\times_n$} be the power set semigroup of the semigroup P.

(i) Find all power set subsemigroup of S^P.
(ii) Find all set power set ideals of S^P.

Chapter Three

SUBSET SEMIRINGS

In this chapter we define the new notion of subset semirings that is we take subsets of a set or a ring or a field or a semiring or a semifield and using the operations of the ring or field or the semiring or the semifield on these subsets we define semiring structure.

Here we define, describe and develop such algebraic structures. We see these new structures can maximum be a semiring. It is not possible to get a field or a ring using subsets.

We would just proceed onto give definition of these concepts.

DEFINITION 3.1: *Let S = {Collection of all subsets of a set X = {1, 2, …, n} together with X and ϕ}. We know {S = P(X), $\cup$, $\cap$} is a semiring or Boolean algebra or a distributive lattice.*

Now we replace in the above definition X by a ring or semiring or a field or a semifield and study the algebraic structure enjoyed by S where S does not include ϕ the empty set then S is a subset semiring.

We will illustrate this by some examples.

Example 3.1: Let S = {set of all subsets of the ring Z_2}. We see S is a semigroup under '+' and S is again a semigroup under ×.

We will verify the distributive laws on S where
S = {{0}, {1}, {0, 1}}.

+	{0}	{1}	{0, 1}
{0}	{0}	{1}	{0, 1}
{1}	{1}	{0}	{1,0}
{0,1}	{0,1}	{1,0}	{0,1}

(S, +) is a semigroup.

×	{0}	{1}	{0, 1}
{0}	{0}	{0}	{0}
{1}	{0}	{1}	{1,0}
{0,1}	{0}	{0,1}	{0,1}

{S, ×} is also a semigroup.

Consider {0, 1} × ({1} + {0}) = {0, 1} × {0, 1}
= {0, 1} + {0, 1} = {0, 1}.

{0, 1} × {1} + {0, 1} × {0} = {0, 1}.

{0} ({0, 1} + {0}) = {1} × {0, 1} = {0, 1}

{1} × {0, 1} + {1} × {0} = {0, 1}

{0} ({0, 1} + {1}) = {0} × {0, 1}

{0} × {0, 1} + {0} × {1} = {0}

$\{0, 1\} \times (\{0, 1\} + \{0\}) = \{0, 1\} \times \{0, 1\} = \{0, 1\}$

$\{0, 1\} \times \{0, 1\} + \{0, 1\} \times \{0\} = \{0, 1\}$.

We see $\{S, +, \times\}$ is a subset semiring.

Example 3.2: Let S = {Collection of all subsets of th e ring Z_3} be the subset semiring of order $2^3 - 1$.

$S = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{1, 2, 0\} = Z_3\}$.

The tables of S for × and + is as follows:

×	{0}	{1}	{2}	{0, 1}
{0}	{0}	{0}	{0}	{0}
{1}	{0}	{1}	{2}	{0, 1}
{2}	{0}	{2}	{1}	{0, 2}
{0, 1}	{0}	{0, 1}	{0, 2}	{0, 1}
{0, 2}	{0}	{0, 2}	{0, 1}	{0, 2}
{2, 1}	{0}	{1, 2}	{2, 1}	{0, 1, 2}
{1, 2, 0}	{0}	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}

{0, 2}	{1, 2}	{0, 1, 2}
{0}	{0}	{0}
{0, 2}	{1, 2}	{0, 1, 2}
{0, 1}	{2, 1}	{0, 1, 2}
{0, 2}	{0, 1, 2}	{0, 1, 2}
{0, 1}	{0,1,2}	{0, 1, 2}
{0, 2, 1}	{1, 2}	{0, 1, 2}
{1, 2, 0}	{1, 2, 0}	{0, 1, 2}

Clearly $\{S, +\}$ is only a semigroup under ×.

Now we find the table {S, +}

+	{0}	{1}	{2}	{0,1}	{0,2}	{1,2}	{0, 1, 2}
{0}	{0}	{1}	{2}	{0,1}	{0,2}	{1,2}	{0, 1, 2}
{1}	{1}	{2}	{0}	{1,2}	{1,0}	{2,0}	{0, 1, 2}
{2}	{2}	{0}	{1}	{2,0}	{2,1}	{0,1}	{0, 1, 2}
{0, 1}	{0, 1}	{1,2}	{2,0}	{0,1,2}	{0,1,2}	{0,1,2}	{0, 1, 2}
{0, 2}	{0, 2}	{1, 0}	{2,1}	{0,1,2}	{2,0,1}	{1,0,2}	{0, 1, 2}
{1, 2}	{1, 2}	{2,0}	{0,1}	{0,1,2}	{0,1,2}	{0,1,2}	{0, 1, 2}
{0,1, 2}	{0,1, 2}	{1,0,2}	{0,1,2}	{0,1,2}	{0,1,2}	{1,2,0}	{0, 1, 2}

We see (S, +) is semigroup with a special property.

{0} acts as the additive identity and {0,1,2} is such that

{0,1,2} + A = {0,1,2} for all A $\in$ S.

We see (S, +) is semigroup with a special property.

{0} acts as the additive identity and {0, 1, 2} is such that {0, 1, 2} + A = {0, 1, 2} for all A $\in$ S.

Thus {S, +, $\times$} is a co mmutative semiring of order 7. It is the subset semiring of the ring Z_3.

We see both Z_2 and Z_3 in examples 3.1 and 3.2 are fields yet the subsets a re only sem irings. We se e these se mirings infact are semifields of finite order.

Thus using subset sem irings we are in a position t o get a class of finite sem irings of odd order. This also is a solution to the problem proposed in [8].

Example 3.3: Let S = {Collection of all subsets of th e ring Z_4} = {{0}, {1}, {0, 1}, {2}, { 3}, {0, 2}, {0, 3}, {1, 2}, {1, 3}, {2, 3}, {0, 1, 2}, {0, 1, 3}, {0, 2, 3}, {1, 2, 3}, {0, 1, 2, 3},} be the subset semiring of order 15.

Clearly S is not a subset semifield as S has zero divisors. For take {0, 2} × {2} = { 0}. S is only a commutative subset semiring as Z_4 is a commutative ring.

Example 3.4: Consider

S = {Collect ion of all subsets of the ring Z_6}, S is a subset semiring of order $2^6 - 1$.

We see S i s only a subset se miring wh ich is not a subset semifield.

For we see S has zero divisors. Take x = {0, 2} and y = {3} in S we see x × y = {0}. Let A = {0, 2, 4} and B = {0, 3} be in S. A × B = {0}.

{0, 2} × {0, 3} = {0} and so on.

In view of all these ex amples we can make a si mple observation which is as follows:

THEOREM 3.1: *Let*
S = {Collection of all subsets of the ring Z_n} be the subset semiring.

(i) S is a subset semifield if n is a prime.
(ii) S is just a subset semiring if n is not a prime.
(iii) S has nontrivial zero divisors if n is not a prime.

The proof is direct and is left as an exercise to the reader.

Now we give examples of non commutative subset semirings.

Example 3.5: Let $R = Z_2S_6$ be the group ring of the group S_6 over the ring Z_2.

S = {Collection of all subsets of the group ring $Z_2 S_6$} be the subset semiring of Z_2S_6. Clearly Z_2S_6 is non commutative and has zero divisors and idempotents.

Consider $A = \{0, 1+p_1\} \in S$ we see $A^2 = \{0\}$ so S has zero divisors.

Consider $B = \{0, 1+p_2\} \in S$ we see $B^2 = \{0\}$.

Take $X = \{0, 1 + p_4 + p_5\} \in S$.

We see $X^2 = X$ so X is an idempotent of S.

Also $Y = \{0, 1 + p_1 + p_2 + p_3 + p_4 + p_5\} \in S$ is such that $Y^2 = \{0\}$ is a zero divisors of S.

We see S is a non commutative finite semiring which has both zero divisors and idempotents.

We see S is non commutative for if $A = \{p_1\}$ and $B = \{p_2\}$. Clearly $AB \neq BA$.

Example 3.6: Let S = {Collection of all subsets of the ring Z} be the subset semiring of infinite order of the ring Z. Clearly S is commutative.

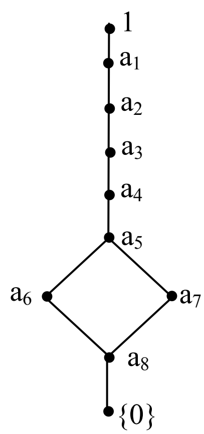
Z has no zero divisors.

$A = \{0, 1\}$ is an idempotent in S as $A^2 = \{0, 1\} = A$.

Example 3.7: Let S = {Collection of all subsets of the semiring $Z^+ \cup \{0\}$} be the commutative subset semiring of infinite order of the semiring $Z^+ \cup \{0\}$.

Example 3.8: Let S = {Collection of all subsets of the semiring which is a distributive lattice L given by the following figure} be the subset semiring of the lattice L.

The lattice L is as follows:



S has idempotents and o (S) = $2^{10} - 1$.

Example 3.9: Let
S = {Collection of all subsets of the semifield $Q^+ \cup \{0\}$} be the subset semiring of the semifield $Q^+ \cup \{0\}$ of infinite order. The only non trivial idempotent is A = {0, 1} $\in$ S.

The subset {1} acts as the multiplicative identity. The subset {0} acts as the additive identity. S has no zero divisors.

Example 3.10: Let S = {Collection of all subsets of the semifield $R^+\cup\{0\}$} be the subset semiring of the semifield $R^+\cup\{0\}$. S has no zero divisors. S is of infinite order and is commutative.

Example 3.11: Let S = {Collection of all subsets of the field C} be the subset semiring of the complex field C. S is an infinite complex subset semiring which is a semifield.

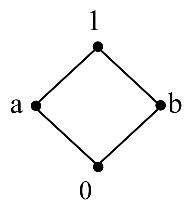
Inview of this we have the following result.

THEOREM 3.2: *Let S = {Collection of all subsets of the semifield or a field} be the subset semiring of the semifield or a field; then S is a semifield.*

Proof is direct and hence left as an exercise to the reader.

Example 3.12: Let
S = {Collection of all subsets of the lattice L given in the following figure} be the subset semiring of L.

S has zero divisors. L is as follows:



S = {{0}, {0}, {b}, {a}, {a, b}, {0, a, b}, {0, a}, {0, b}, {1, a, b}, {1, a}, {1, b}, {0, 1}, {0, a, 1}, {0, b, 1}, {1, a, b, 0}}.

We see {0, a} × {0, b} = {0},
{0, a} {b} = {0} and
{0, b} {a} = {0}.

Thus S has zero divisors and S has idempotents also.

Take {a} {a} = {a},

{1} {1} = 1,

{a, 0} × {a, 0} = {a, 0},

{1, 0} {1, 0} = {1, 0},

{0, b} {0, b} = {0, b},

{1, a} {1, a} = {1, a},

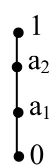
{1, b} {1, b} = {1, b},

and so on.

Example 3.13: Let

S = {Collect ion of all subsets of the lattice L giv en in the following} be a subset semiring of L.

L is as follows:

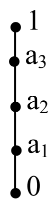


S = {{0}, {1}, {a}, {b}, {1, a}, {1, b}, {0, a}, {0, b}, {1, 0}, {0, a, b}, {0, b, 1}, {0, a, 1}, {1, a, b}, {0, a, b, 1}, {a, b}} is a semiring of order 15.

Clearly S is a commutative semiring. S has no zero divisors but has idempotents. S is a semifield of order 15.

Example 3.14: Let S = {Collection of all subsets of the finite lattice L given in the following} be the subset semiring of S.

L is



Then S = {{0}, $\{a_1\}$, $\{a_2\}$, $\{a_3\}$, {1}, $\{0, a_1\}$, $\{0, a_2\}$, $\{0, a_3\}$, {0, 1}, $\{a_1, a_2\}$, $\{a_1, a_3\}$, $\{a_1, 1\}$, $\{a_2, 1\}$, $\{a_2, a_3\}$, $\{a_3, a_1\}$, $\{0, a_1, a_2\}$, $\{0, a_1, a_3\}$, $\{0, a_2, a_3\}$, $\{0, a_1, 1\}$, $\{0, a_2, 1\}$, $\{0, a_3, 1\}$, $\{a_1, a_2, a_3\}$, $\{1, a_1, a_2\}$, $\{1, a_1, a_3\}$, $\{1, a_2, a_3\}$, $\{1, a, a_2, a_3\}$, $\{0, a_1, a_2, a_3\}$, $\{1, 0, a_1, a_2\}$, $\{0, 1, a, a_3\}$, $\{0, 1, a_3, a_2\}$, $\{0, 1, a_1, a_2, a_3\}$} is a semifield of order $2^5 - 1 = 31$.

Inview of all these examples we have the following result.

THEOREM 3.3: *Let S = {Collection of all subsets of a distributive lattice L} be the subset semiring. If the lattice L is a chain lattice certainly S is a semifield.*

Proof follows from the simple fact that ab = 0 is not possible in L unless a = 0 or b = 0.

Hence the claim.

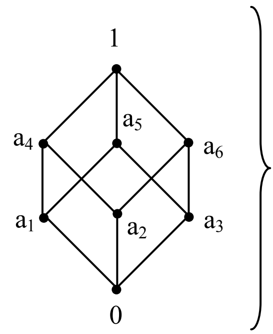
If L is a distributive lattice or a Boolean algebra and is not a chain lattice.

Corollary 3.1: Let S = { Collection of all subsets of a Boolean algebra B of order greater than or equal to four} be the subset semiring. Then S is only a semiring and is not a semifield.

The proof is direct hence left as an exercise to the reader.

Example 3.15: Let S = {Collection of all subsets of the Boolean algebra

B =



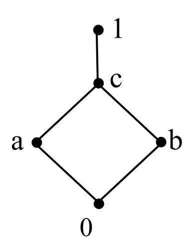
be the subset semigroup of order $2^8 - 1$. S has zero divisors so S is not a semifield.

For take A = {a_1, 0} and B = {0, a_2} in S we see A × B = {0}. Likewise A_1 = {0, a_3} and B_1 = {0, a_1} in S are such that $A_1 \times B_1$ = {0}. So S is not a semifield.

Now we have seen semifield of finite or any desired order cannot be got but they are of order 3, 7, 15, 31, 63, 127, 255 and so on.

We now proceed onto give more examples.

Example 3.16: Let S be the collection of all subsets of L given by



S = {{0}, {1 }, {a}, {b}, {c}, {a,0}, { 0,b}, {0,c}, {0,1}, {0,a , b}, {0,a,c}, {0,a,1}, {0,b,1}, {0,c ,1}, { 0,b,c}, {1,a,b}, {1,a,c}, {1,b,c}, {a,b,c}, {a,1}, {b,1}, {c ,1}, {a,b}, {b,c}, {c,a}, {0,a,b, c}, {0,a,b,1}, {0,a,c,1}, {0,c,b,1 }, {a,b,c,1}, {0,1,a,b,c}} is th e subset semiring of order $2^5 - 1 = 31$.

This has zero divisors for {0, a} {0, b} = {0},
{0, a} {b} = {0},
{0, b} {a} = {0},
and {a}, {b} = {0}.

This has several idempotents
$\{0, a\}^2 = \{0, a\}$, $\{0, a, 1\}^2 = \{0, a, 1\}$,
$\{0, b, 1\}^2 = \{0, b, 1\}$, $\{0, c, a\}^2 = \{0, a, c\}$ and so on.

Example 3.17: Let S be the collection of all subsets of the lattice which is a Boolean algebra of order 16. Then S is a subset semiring of order $2^{16} - 1$.

Clearly S is commutative is not a s emifield as it h as zero divisors.

Now having seen exa mples of subset se miring using lattices, field and rings.

We se e an example of a subset se miring which is not a subset semifield.

Example 3.18: Let S be the collection of all subsets of a ring Z_{12}. S is a su bset semiring of order $2^{12} - 1$. Clearly S has zero divisors so S is not a subset semifield but S is commutative.

For $\{0, 4\} \times \{0, 3\} = \{0\}$,
{0,4,8} {0, 3, 6} = {0},
{6} {4} = {0},
{4} {0,3,6} = {0} and so on.

Now we proceed onto define the notion of subset subsemirings and subset ideals of a subset semiring S.

DEFINITION 3.2: *Let S = {Collection of all subsets of the ring / field / semiring / semifield} be the subset semiring. $T \subseteq S$; if T* *under the operations of S is a subset semiring we define T to be a subset subsemiring of S.*

We will first illustrate this situation by some examples.

Example 3.19: Let

S = {Collection of all subsets of the field Z_5} be the subset semiring of order $2^5 - 1 = 32$.

Consider T = {{0}, {1}, {2}, {3}, {4}} $\subseteq$ S is a subset subsemiring of S.

For observe the tables of T;

+	{0}	{1}	{2}	{3}	{4}
{0}	{0}	{1}	{2}	{3}	{4}
{1}	{1}	{2}	{3}	{4}	{0}
{2}	{2}	{3}	{4}	{0}	{1}
{3}	{3}	{4}	{0}	{1}	{2}
{4}	{4}	{0}	{1}	{2}	{3}

Clearly (S, +) is a sem igroup infact it is also a group under +.

The table (T, ×) is as follows:

×	{0}	{1}	{2}	{3}	{4}
{0}	{0}	{0}	{0}	{0}	{0}
{1}	{0}	{1}	{2}	{3}	{4}
{2}	{0}	{2}	{4}	{1}	{3}
{3}	{0}	{3}	{1}	{4}	{2}
{4}	{0}	{4}	{3}	{2}	{1}

We see T \ {0} under × is a group. Thus T is a field so trivially a semifield hence is also a semiring. Thus T is a subset subsemiring of S.

Now we see the subset semiring has both a subset field as well as a subset semifield.

Example 3.20: Let

S = {Collection of all subsets of the ring Z_6} be the subset semiring of the ring Z_6.

S is only a subset semiring and is not a subset semifield.

Consider T = {{0},{1}, {2}, {3}, {4}, {5}} ⊆ S. T is a subset ring as well as subset subsemiring. T is not subset semifield or subset field.

Consider the subset P = {{0}, {0,2,4}, {0,2}, {0,4}} ⊆ S.

P is a subset subsemiring, not a subset ring or subset field but is also a subset semifield.

The tables of P are

+	{0}	{0,2}	{0,4}	{0,2,4}
{0}	{0}	{0,2}	{0,4}	{0,2,4}
{0,2}	{0,2}	{0,2,4}	{0,4,2}	{0,2,4}
{0,4}	{0,4}	{0,2,4}	{0,4,2}	{0,2,4}
{0,2,4}	{0,2,4}	{0,2,4}	{0,2,4}	{0,2,4}

The table of (P, ×) is as follows:

×	{0}	{0,2}	{0,4}	{0,2,4}
{0}	{0}	{0}	{0}	{0}
{0,2}	{0}	{0,4}	{0,2}	{0,2,4}
{0,4}	{0}	{0,2}	{0,4}	{0,2,4}
{0,2,4}	{0}	{0,2,4}	{0,2,4}	{0,2,4}

{P \ {0}, ×} is not a group . However P is onl y a semifield and not a field or a ring.

Let M = {{0}, {0,3}, {3} } ⊆ S. M is a subset subsemiring of S. We find the tables of M.

+	{0}	{3}	{0,3}
{0}	{0}	{3}	{0,3}
{3}	{3}	{0}	{3,0}
{0,3}	{0,3}	{0,3}	{0,3}

×	{0}	{3}	{0,3}
{0}	{0}	{0}	{0}
{3}	{0}	{3}	{0,3}
{0,3}	{0}	{0,3}	{0,3}

{S, ×} is a subset subsemiring also a subset semifield.

Thus S has subset subsemirings.

Now w e proceed onto characte rize tho se subset se mirings which are S marandache subset se mirings and those that are Smarandache semiring of level II.

We will first give some examples.

Example 3.21: Let S = {Collection of all subsets of the field Z_7} be the subset semiring. S is a Smarandache subset semiring of le vel II a s S contains a subfield.

For A = {{0}, {1}, { 2}, {3}, {4}, {5}, {6}} ⊆ S is a field in S.

Example 3.22: Let S = {Collection of all subsets of the field Z_{19}} be the subset semiring of order $2^{19} - 1$. Clearly S is a S maradache subse t semiring of level II for A = $\{\{g\} \mid g \in Z_{19}\} \subseteq$ S is a fiel d isomorphic to Z_{19}.

Hence the claim.

Inview of this we have the following theorem.

THEOREM 3.4: *Let*
S = {Collection of all subsets of the field Z_p; p a prime} be the subset semiring of order 2^p–1.

S is a Smarandache subset semiring of level II.

The proof is direct and hence left as an exercise to the reader.

Example 3.23: Let S = {Collection of all subset of the ring Z_{15}} be the subset semiring of the ring Z_{15}.

Consider P = {{0}, {3}, {6}, {9}, {12}} $\subseteq$ S.

We prove P is a subset field isomorphic with Z_5 and (P,+) of P is given in the following;

Table of (P, +) is as follows:

+	{0}	{3} {6}	{9}		{12}
{0}	{0} {3}	{6} {9}			{12}
{3}	{3} {6}	{9}		{12}	{0}
{6}	{6} {9}		{12}	{0}	{3}
{9}	{9}	{12}	{0}	{3} {5}	
{12}	{12}	{0}	{3} {5}	{9}	

Clearly (S, +) is group under +.

Consider the table (P, ×)

×	{0}	{3}	{6}	{9}	{12}
{0}	{0}	{0}	{0}	{0}	{0}
{3}	{0}	{9}	{3}	{12}	{6}
{6}	{0}	{3}	{6}	{9}	{12}
{9}	{0}	{12}	{9}	{6}	{3}
{12}	{0}	{6}	{12}	{3}	{9}

Clearly P \ {0} is a group under product thus {P, +, ×} is a field isomorphic to Z_5. Thus S is a Smarandache subset semiring of level II.

Example 3.24: Let
S = {Collection of all subsets of the field Z_{13}} be the subset semiring. S is a Smarandache semiring of level II for P = {{0}, {1}, {2}, …, {12}} ⊆ S is a subset field isomorphic to Z_{13}.

In view of all this we have the following theorem.

THEOREM 3.5: *Let*
S = {Collection of all subsets of the field Z_p} be the subset semiring of Z_p. S is a Smaradache subset semiring of level II.

The proof is simple for every such S has a subset field P ⊆ S which is isomorphic to Z_p.

Hence the claim.

Now we give one more example before we proceed to give a result.

Example 3.25: Let
S = {Collection of all subsets of the ring Z_{30}} be the subset semiring of the ring Z_{30}. S is a Smarandache semiring of level II as S contains subset fields.

For take $P_1 = \{\{0\}, \{10\}, \{20\}\}$, $P_2 = \{\{0\}, \{15\}\}$ and $P_3 = \{\{0\}, \{6\}, \{12\}, \{18\}, \{24\}\}$ subsets of S. Each P_i is a subset field of S. So S is a Smarandache semiring of level II.

Inview of this observation we give the following theorem.

THEOREM 3.6: *Let S = {Collection of all subsets of the ring Z_n, n a composite number of the form $n = p_1, p_2, \ldots, p_t$ each p_i is a distinct prime} be the subset semiring of the ring Z_n. S is a Smarandache subset semiring of level II.*

Proof follows from the fact that S has t distinct subsets say $P_1 = \{\{0\}, \{p_1\}, \{2p_1\}, \ldots\}$

$P_2 = \langle\{p_2\}\rangle$, $P_3 = \langle\{p_3\}\rangle, \ldots, \langle\{p_t\}\rangle = P_t$ are subset field in S.

Hence the claim of the theorem.

Example 3.26: Let
S = {Collection of all subsets of the group ring $Z_{12}S_{20}$} be the subset semiring which is of finite order but non commutative. S is a Smarandache subset semiring of level II.

Example 3.27: Let
S = {Collection of all subsets of the group ring $Z_{31}D_{2,29}$} be the subset semiring which is of finite order but non commutative. S is a Smarandache subset semiring of level II.

Example 3.28: Let S = {Collection of all subsets of the ring Z} be the subset semiring of the ring Z. S is not a Smarandache semiring of level II.

Example 3.29: Let
S = {Collection of all subset semiring of the field Q}. S is a Smarandache semiring of level II for $P = \{\{a\} \mid a \in Q\} \subseteq S$ is a subset field isomorphic to Q.

Example 3.30: Let
S = {Collection of subsets of the field C or R} be the subset semiring; S is a Smarandache subset semiring of level II for S contain $P = \{\{a\} \mid a \in C\}$ or $P_1 = \{\{a\} \mid a \in R\}$ are subset fields of S.

Example 3.31: Let S = {Collection of all subsets of the group ring QG or RG or CG; G any group} be the subset semiring of the group ring. S is Smarandache subset semiring of level II as they contain subset fields isomorphic to Q or R or C.

Inview of this we give conditions for an infinite subset semiring to be Smarandache subset semiring of level II.

THEOREM 3.7: *Let S = {Collection of subsets of the field Q or R or C or the group CG or QG or RG} be the subset semiring of the field Q or R or C or the group ring CG or QG or RG, then S is a Smarandache subset semiring of level II.*

The proof is direct from the fact that S contains subset which are fields isomorphic to Q or R or C, hence the claim.

All these results hold good if in the group ring, the group G is replaced by a semigroup that is the results continue to hold good for semigroup ring also.

Now having studied about subset Smarandache semirings of level II we now proceed on to study Smarandache subset semigroup.

A subset semiring S is said to be a Smarandache semiring if S has a proper subset which is a semifield.

We now proceed onto give examples of this situation.

Example 3.32: Let S = {all subsets of the ring Z} be the subset semiring. S is a Smarandache subset semiring as S contains a set $P = \{\{g\} \mid g \in Z^+ \cup \{0\}\} \subseteq S$ is a subset semifield of S.

Hence the claim.

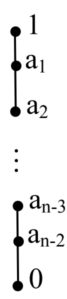
Example 3.33: Let
S = {Collection of all subsets of the semifield $Q^+ \cup \{0\}$} be the subset semiring of $Q^+ \cup \{0\}$, the semifield. S is a subset Smarandache semiring as S contains subset semifields isomorphic to $Z^+ \cup \{0\}$ and $Q^+ \cup \{0\}$.

We make the following observations.

Clearly the subset semirings given in examples 3.32 and 3.33 are not subset Smarandache semiring of level II.

However all subset Smarandache semirings of level II are subset Smarandache semiring as every field is a semifield and a semifield in general is not a field.

Example 3.34: Let
S = {Collection of all subsets of a chain lattice L} be the subset semiring of the lattice $L = C_n =$



S is a Smarandache subset semiring as $P = \{\{m\} \mid m \in L\}$ is a semifield in S.

Example 3.35: Let S be the collection of all subsets of a Boolean algebra of order 64. S is a subset semiring of order 2^{64}. We see S has idempotents and zero divisors.

Example 3.36: Let S be the collection of subsets of a Boolean algebra of order 16 with $\{a_1\}$, $\{a_2\}$, $\{a_3\}$ and $\{a_4\}$ as its atoms is a subset semiring of order 2^{16}. The set
$P = \{\{0\}, \{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_3, a_4\}\} \subseteq S$ is a subset semiring which is a semifield. So S is a Smarandache subset semiring.

Example 3.37: Let S is the collection of all subsets of the semigroup ring $Z_7S(5)$. S is a subset semiring. H has no zero divisors. But is not a subset semifield as S is non commutative.

However S is a Smarandache subset semiring of level II as well as Smarandache subset semiring.

Inview of all these examples we give the following theorem.

THEOREM 3.8: *Let S = {Collection of all subsets of the Boolean algebra with $a_1, a_2, \ldots, a_n$ as atoms}. S is a subset semiring of order 2^{2^n}. Take*
$P = \{\{0\}, \{a_1\}, \{a_1\ a_2\}, \{a_1\ a_2\ a_3\}, \ldots, \{a_1, a_2, \ldots, a_n\} \subseteq S$ is a subset semifield.

Thus S is not a subset semifield but has atleast n subset semifield.

S is a S-subset semiring and S is not a S-subset semiring of level II.

The proof is direct follows from the basic properties of a Boolean algebra. Left as an exercise for the reader.

Now we can define the notion of subset ideals of a semiring in an analogous way. This task is left as an exercise to the reader. Here we give some examples of them.

Example 3.38: Let S = {set of all subsets of the ring Z_{24}} be the subset semiring. Take I = {Collection of all subsets of the set $\{0, 2, 4, 6, 8, 10, \ldots, 22\} \subseteq Z_{24}$}, I is a subset ideal of the semiring S.

Take J = {Collection of all sub sets of the set {0, 3, 6, 9, 12, 15, 18, 21}} $\subseteq$ S, J is a subset ideal of the semiring.

Example 3.39: Let S = {Set of all subsets of field Z_3} be the subset semiring. We see S has no subset ideals.

Example 3.40: Let S = {set of all subsets of the field Z_{19} be the subset semiring of the field Z_{19}}. S has no subset ideals S has subset subsemirings.

Inview of this we have the following result.

THEOREM 3.9: *Let S = {set of all subsets of the field Z_p; p a prime} be subset semiring of the field Z_p. S has no subset ideals but has subset subsemirings.*

The proof is direct and hence left as an exercise to the reader.

Example 3.41: Let S = {Collection of subsets of a ring Z_{36}} be the subset semiring of the ring Z_{36}.

The set P = {Collection of all subsets of the subring T = {0, 2, 3, 4, …, 34} $\subseteq Z_{36}$} $\subseteq$ S is a subset ideal of the semiring S.

J = {Collection of all subsets of the subring R = {0, 3, 6, …, 33} $\subseteq Z_{36}$} $\subseteq$ S is a subset ideal of the semiring S.

M = {{a} | a $\in Z_{36}$} $\subseteq$ S, M is only a subset subsemiring of S and is not a subset ideal of S.

M is also a ring.

Inview of this property we define a concept of Smarandache quasi semiring.

DEFINITION 3.3: *Let S be any semiring. If P $\subseteq$ S is such that P is a ring under the operations of S we define P to be Smarandache quasi semiring.*

We will give examples them.

Example 3.42: Let
S = {collection of all subsets of the ring Z_{20}} be the subset semiring.

We see
P = {{ a} | a $\in Z_{20}$ = {{0} , {1}, {2}, …, {18}, {19}} $\subseteq$ S is a subset ring in S. So S is a subset quasi Smarandache semiring.

Infact S has more number of subset rings, for
M = {{0}, { 5}, {10}, {15}} $\subseteq$ S is again a subset ring and so on.

Example 3.43: Let
S = {Collection of all subsets of the field Z_{23}} be the subset semiring of the field Z_{23}. We see S has subset field P = {{a} | a $\in Z_{23}$} $\subseteq$ S; S is not a quasi Smarandache subset semiring as P is only a field.

Here we use only the fact every field is not a ring so we cannot call it as a S-quasi subset semiring. This is also in keeping with the definition of Smarandache subset semiring.

Example 3.44: Let S = {Collection of a subsets of a chain lattice C_5 = {1, a_1, a_2, a_3, 0}} be a subset semiring. Clearly S is not a quasi Smarandache subset semiring.

Example 3.45: Let
S = {Collection of subsets of the Boolean algebra of order 2^5} be a subset semiring. Clearly S is not a quasi Smarandache subset semiring.

Example 3.46: Let
S = {Collection of all subsets of the field Z_{37}} be the subset semiring. S is not a quasi Smarandache subset semiring.

In view of all these we give the following theorems.

THEOREM 3.10: *Let S = {Collection of all subsets of the lattice L (distributive) or a Boolean algebra} be a subset semiring. S is not a quasi Smarandache subset semiring.*

The proof is direct hence left as an exercise to the reader.

THEOREM 3.11: *Let*
S = {Collection of all subsets of the field Z_p, p a prime} be the subset semiring. S is not a quasi Smarandache subset semiring only a Smarandache subset semiring.

THEOREM 3.12: *Let S = {Collection of all subsets of the ring Z_n or the group ring Z_nG; n a composite number} be the subset semiring of the ring Z_n or the group ring Z_nG. S is a quasi Smarandache subset semiring.*

This proof is also left as an exercise to the reader.

Now we define extension of a subset semifield in a different way as for the first time we make some special modifications as subset semirings are built using the a set. So our extension is done in the following way.

DEFINITION 3.4: *Let*
S = {Collection of all subsets of the chain lattice C_n ($n < \infty$)} be the subset semiring of the chain lattice. Clearly S is a semifield. We see every proper subsemifield $T \subseteq S$; S is defined as an extension semifield of the subsemifield T.

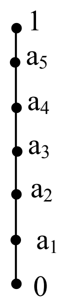
However we are not always guaranteed of the subse mifield or semifield.

If we replace the chain lattice by a field then we can have extension of the semifield.

We will illustrate this situation by examples.

Example 3.47: Let
S = {Collection of all subsets of the chain lattice C_7} be the subset semiring. S is a subset semifield.

P = {{0}, {a_1}, {a_2}, {a_3}, {a_4}, {a_5}, {1}} $\subseteq$ S is a subset subsemifield of S.



Clearly S is a extension of the subset subsemifield P.

Example 3.48: Let
S = {Collection of all subsets of the field Z_{43}} be the subset semiring of the field Z_{43}. S is a subset semifield. P = {{g} | g $\in$ Z_{43}} $\subseteq$ S; P is a subset subsemifield and S is an extension of the subset subsemifield P.

We see we can build almost all properties related with semirings / semifields as in case of subset semirings / subset semifields.

Now we proceed onto introduce the notion of set ideals of a subset semiring.

As in case of ring we define in case of subsemiring for the first time the notion of set ideals.

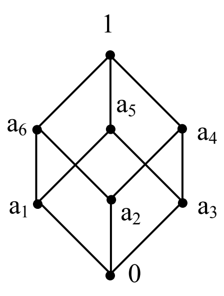
DEFINITION 3.5: *Let S be a semiring and P a proper subset of S. M a proper subset subsemiring of S. P is called a set left subset ideal of S relative to the subsemiring of M if for all m $\in$ M and p $\in$ P, mp and pm $\in$ P.*

Similarly one can define set right ideals of a semiring over a subsemiring.

In case S commutative or P is both set left ideal and set right ideal of the sem iring then we define P to be a set ideal of the semiring relative to the subsemiring M of S.

We will give examples of this new structure.

Example 3.49: Let S be the semiring given by the following lattice.



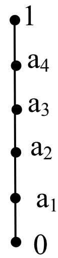
We see $\{0, a_1\}$ = B is a subsemiring.

M = $\{0, a_2, a_3\} \subseteq$ S; M is a set ideal of the subset semiring over B the subset subsemiring.

Example 3.50: Let S = $Z^+ \cup \{0\}$ be the semiring.
P = $\{3Z^+ \cup 5Z^+ \cup \{0\}\} \subseteq$ S be a proper subset of S.

M = $\{2Z^+ \cup \{0\}\} \subseteq$ S, is the subsemiring. P is a set ideal of the semiring over the subsemiring M of S.

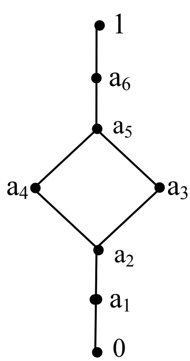
Example 3.51: Let S be the semiring.



B = {0, a_1} is subset subsemiring of S.

P = {a_4, a_3,1,0} $\subseteq$ S; P is a set ideal of the subset semiring over the subring B = {0, a_1}.

Example 3.52: Let S be the semiring.



B = {0, a_2, a_1} be the subset subsemiring of S.
M = {a_4, a_3, a_1, a_2, 0} $\subseteq$ S is a set ideal of S over B.

Now we proceed onto give examples set ideal of the subset semiring over a subset subsemiring.

Example 3.53: Let
S = {Collection of all subsets of the ring Z_6} be the subset semiring of the ring Z_6. B = {{0}, {2}, {4}} $\subseteq$ S be a subset subsemiring of S. P = { {0}, {3}, {0, 3}} $\subseteq$ S is a set ideal subset semiring of the subsemiring.

Let P = {{0}, {3}, {0, 3}} $\subseteq$ S be the subset subsemiring of S.

M = {{0}, {0, 2}, {0, 4}, {0, 2, 4}, {2}, {4}, {2, 4}} $\subseteq$ S is set ideal of the subset semiring of the subsemiring P.

Example 3.54: Let S = {Collection of all subsets of the ring Z_{12}} be the subset semiring.

P = {{0}, {3}, {6}, {9}} be the subset subsemiring.

M = {{0}, {4}, {8}, {0, 4}, {0, 8}, {4, 8}, {0, 4, 8}} $\subseteq$ S, is a set subset ideal of the subset semiring over the subset subsemiring.

Example 3.55: Let S = {Collection of all subsets of the ring Z_{30}} be the subset semiring of the ring Z_{30}. M = {0, 10, 20} $\subseteq$ S be a subset subsemiring of S. P = { 0, 15, 6, 3} $\subseteq$ S is the set subset ideal subset semiring of the subset subsemiring M of S.

Example 3.56: Let S = {Collection of all subsets of the ring R = $Z_{20} \times Z_9$} be the subset semiring of the ring R.

Consider R = {(0, 0), (5, 3), (5, 0), (15, 0), (15, 3), (10, 0), (10, 3)} $\subseteq$ S be the subset subsemiring. Take T = {(0, 0), (4, 0), (8, 0), (12, 0), (16, 0), (0, 3), (10, 3), (10, 0)} $\subseteq$ S is the set subset ideal of the subset semiring of the subset subsemiring P of S.

Example 3.57: Let S = {Collection of all subsets of the ring Z_{40}} be the subset semiring of the ring Z_{40}. Take P = {{0}, {10}, { 20}, {30}} to be the subset subsemiring of the subset semiring S.

M = {{0}, { 8}, {0,8}, {16}, {16,0} , {16,8}, {16, 8, 0}} $\subseteq$ S. M is a set subset ideal of the subset semiring S over the subset subsemiring P of S.

Example 3.58: Let S = {Collection of $Z^+(g) \cup \{0\}$, $g^2 = 0$} be the subset semiring of the semiring $Z^+(g) \cup \{0\}$.

Take P = {{3ng} | n ∈ $Z^+ \cup \{0\}$} to be a subset subsemiring of S.

Clearly M = {{2ng}, {5ng}, {11 ng}} ⊆ S is a set subset ideal of the subset se miring S over the subset subsemiring P of S.

Example 3.59: Let
S = {Collection of all subsets of the ring Z_{16}} be the subset semiring of the ring Z_{16}. P = {{0}, {0, 8}, {8}} ⊆ S is a subset subsemiring of S.

T = {{0}, {2}, {0, 2}, {0, 3}, {0, 3}, { 6, 0}, {6}} ⊆ S is a set subset ideal of the subset se miring S over the subset subsemiring P of S.

We see B_1 = {{0}, {4}, {0, 4}, {0, 10}} ⊆ S is also a s et subset ideal of the subset semiring of S over P.

We can have sever al such set ide al su bset se mirings for a given subset subsemiring P of S.

Example 3.60: Let
S = {Collection of all subsets of the ring Z_{28}} be the subset semiring of the ring Z_{28}.

Take P = {{0}, {0, 14}, {14}} to be a subset subsemiring of S.

M_1 = {{0}, {2}} ⊆ S is a set subset ideal of the subset semiring one P.

M_2 = {{0}, {0, 2}} ⊆ S is again a set subset ideal of the subset semiring over P.

$M_3 = \{\{0\}, \{0, 4\}\} \subseteq S$ is also a set subset ideal of the subset semiring over P.

$M_4 = \{\{0\}, \{4\}\} \subseteq S$ is also a set subset ideal of the subset semiring over P.

$M_5 = \{\{0\}, \{6\}\} \subseteq S$ is also a set subset ideal of the subset semiring over P and so on.

Example 3.61: Let
S = {Collection of all subsets of the ring Z_{42}} be the subset semiring of the ring Z_{42}.

Take $P = \{\{0\}, \{7\}, \{14\}, \{21\}, \{28\}, \{35\}\} \subseteq S$ be the subset subsemiring of S.

Consider
$M_1 = \{\{0\}, \{6\}, \{12\}, \{18\}, \{24\}, \{30\}, \{36\}\} \subseteq S$, M_1 is a set subset ideal of the subset subsemiring over P of S.

$M_2 = \{\{0\}, \{6\}\} \subseteq S$ is also a set subset ideal subset semiring of the subset subsemiring P of S. Clearly M_1 is a ideal set subset ideal subset semiring which contains M_2.

Let $M_3 = \{\{0\}, \{12\}\} \subseteq S$ be a set ideal subset semiring of the subset subsemiring.

We sometime write just set ideals instead of set subset ideals for the reader can understand the situation by the context. Thus we have several such set ideals of the subset semiring.

Example 3.62: Let
S = {Collection of all subsets of the ring $Z_{12} \times Z_8 = R$} be the subset semiring of the ring $R = Z_{12} \times Z_8$.

$P = \{\{(0, 0)\}, \{(4, 0)\}, \{(8, 0)\}, \{(0, 4)\}, \{(4, 4)\}, \{(8, 4)\}\} \subseteq S$ is a subset subsemiring of S.

Choose $T_1 = \{\{(0, 0)\}, \{(0, 2), (6, 0)\} \subseteq S$. T is a set ideal subset semiring of S over P.

$T_2 = \{\{(0, 0)\}, \{(0, 2)\}\} \subseteq S$,
$T_3 = \{\{(0, 0)\}, \{(0, 6)\}\} \subseteq S$,
$T_4 = \{(0, 0), (3, 0)\} \subseteq S$, $T_5 = \{(0,0), (9,0)\} \subseteq S$,

$T_6 = \{\{(0, 0)\}, \{(3, 2)\}\} \subseteq S$, $T_7 = \{\{(0, 0)\}, \{(6, 2)\}\} \subseteq S$ and so on are all set ideal subset semiring S over the subset subsemiring P of S.

For the first time we define the concepts of set ideal topological space of semirings and set ideal topological spaces of the subset semirings.

DEFINITION 3.5: *Let S = {Collection of all subsets of a ring or a semiring or a field or a semifield} (or used in the mutually exclusive sense) be the subset semiring of the ring (or semifield or semiring or field). $P \subseteq S$ be a subset subsemiring of S.*

T = {Collection of all set ideals of S over P}, T is given the topology, for any A, B $\in$ T both $A \cap B$ and $A \cup B \in T$; $\{0\} \in T$ and $S \in T$.

T is defined as the subset semiring ideal topological space over the subset subsemiring.

If we replace the subset semiring by a semiring still the definition holds good.

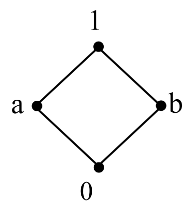
We will illustrate this by an example or two.

Example 3.63: Let
S = {Collection of all subsets of the ring Z_4} be the subset semiring of the ring Z_4.

$P = \{\{0\}, \{2\}\} \subseteq S$ is a subset subsemiring of S.

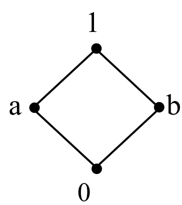
T = {collection of all set ideal of the subset semiring over the subset subsemiring P} = {{0}, {{0}, {2}}, {{0}, {0, 2}}, {{0}, {0, 2}, {2}}, {{0}, {1}, {2}}, {{0}, {1}, {0, 2}, {2}}, {{0}, {0, 1}, {0, 2}} {{0}, {0, 1}, {0, 2}, {1}, {2}}, {{0}, {3}, {2}}, {{0}, {0, 3}, {0, 2}}, {{0}, {0,2}, {0,3}, {2}}, {{0}, {0,2}, {2}, {0, 3}, {3}}, {{0}, {1}, {3}, {2}}, {{0}, {1}, {0, 3}, {2}, {0, 2}, {0, 1}}, {{0}, {0, 3}, {0, 2}, {0, 1}}, {{0}, {0, 1}, {0, 2}, {0, 3}}, S} is a set ideal topological space of the subset semiring over the subset subsemiring {{0}, {2}}.

Example 3.64: Let S = {Collection of all subsets of the semiring



},

be the subset semiring of the semiring



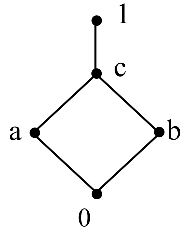
.

Take P = {{0}, {1}} $\subseteq$ S as a subset subsemiring of S.

Let T = {Collection of all set ideals of the subset semiring over the subset subsemiring P} = {{0}, {{0}, {1}}, ({0}, {0, 1}), {{0}, {0, 1}, {1}}, {{0}, {a}}, {{0}, {b}}, {{0}, {0, a}}, {{0}, {0, b}}, {{0}, {a}, {0, a}}, {{0}, {a}, {b}}, {{0}, {a}, {0, b}}, {{0}, {0, a}, {b}}, {{0}, {a}, {1}}, {{0}, {a}, {0, 1}}, {{0}, {a}, {1}, {0, 1}} and so on}.

T is a set ideal subset semiring topological space over the subset subsemiring.

Example 3.65: Let S =



be the semiring.

Let $P_1 = \{0, a\} \subseteq S$ be a subsemiring.

Let T_1 = {Collection of all set ideals of S over the subsemiring P_1} = {{0}, {0, b}, {0, c, a}, {0, 1, a}, {0, a}, {0, a, b}, {0, a, c, b}, {0, a, c, 1}, {0, a, b, 1}, S} be the set semiring ideal topological space or set ideal topological space of the semiring.

Consider $P_2 = \{0, b\} \subseteq S$ is a subsemiring.

T_2 = {{0}, { 0, b}, { 0, a}, {a, b, 0}, { 0, 1, b}, { 0, 1, a, b}, {0, c, b}, {0, c, 1, b}, {0, a, b, c}, S} = {Collection of all set ideals of S over the subsemiring} is a set ideal topological space of S over P_2.

It is clear $T_1 = T_2$.

Thus we can say even if subsemiring are different yet the collection of all set ideals can be identical.

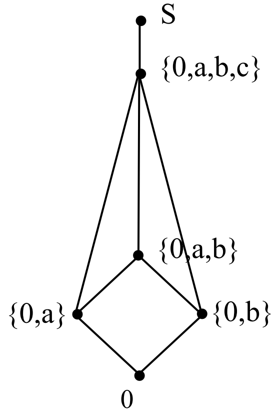
Consider the subsemiring $\{0, 1\} = P_3$ of S.

The collection of all set ideals of S over P_3 be T_3 = {{0}, {0, a}, {0, b}, {0, c}, {0, 1}, {0, a, b}, {0, a, 1}, {0, b, 1}, {0, a, c}, {0, b, c}, {0, 1, c}, {0, a, b, 1}, {0, a, b, c}, {0, a, 1, c}, {0, b, 1, c}, S}, T_3 is a set ideal topological space of the semiring S over the subsemiring T_3. Clearly $T_3 \neq T_1$ or T_2.

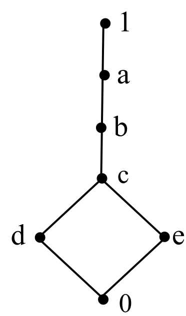
Next consider P_4 = {0, a, b, c} to be a subsemiring of S.

Let T_4 = {Collection of all set ideals of S over the subsemiring P_4} = {{0}, {0, a}, {0, b}, {0, a, b}, {0, a, b, c}, S} be the set ideal topological semiring subspace over P_4.

The lattice associated with T_4 is as follows:



Example 3.66: Let S be the semiring which is as follows:



Consider P = {0, d, e, c} a subsemiring of S. Let T = {Collection of all set ideals of S over the subsemiring P of S} = {{0}, {0, d}, {0, e}, {0, d, c}, {0, e, c}, {0, d, e, c}, {0, d, b}, {0, e, b}, {0, c, b}, {0, d, c, b}, {0, e, b, c}, {0, d, e, b}, {0, d, e, c, b}, {0, d, a}, {0, e, a}, {0, c, a}, … S} be a set ideal topological semiring space over the subsemiring P of S.

Example 3.67: Let $S = Z^+ \cup \{0\}$ be the semiring.

Let $P = \{2n \mid n \in Z^+ \cup \{0\}\}$ be the subsemiring of S.

We see T =

{Collection of all set ideals of S over the subsem iring P} $= \{\{0\}, \{3Z^+ \cup \{0\}, 4Z^+ \cup \{0\}\}, 5Z^+ \cup \{0\}, 72^+ \cup \{0\}, 12Z^+ \cup \{0\}$ and so on}. T is a set ideal topological semiring space over P.

Example 3.68: Let $S = Z^+ [x] \cup \{0\}$ be the semiring.

$P = \{3Z^+ \cup \{0\}\}$ is a subsemiring of S.

T = {Collection of all set ideals of S over the subsemiring P} is the set ideal topolo gical space of the sem iring S over the subsemiring P.

Infact T is an infinite set ideal topolo gical semiring space. Further it is interesting to note that we can have infinite number of infinite set ideal semiring topological spac es as S the semiring has infinite number of subsemiring.

The same type of results h old good in c ase $Z^+[x] \cup \{0\}$ is replaced by $Q^+ \cup \{0\}$ or $Q^+[x] \cup \{0\}$ or $R^+ \cup \{0\}$ or $R^+[x] \cup \{0\}$.

Now we giv e some more set ideal topolo gical semiring spaces of inf inite order which are not sem ifield of the above mentioned type.

Example 3.69: Let
$S = Z^+ \cup \{0\} (g) = \{a + bg \mid a, b \in Z^+ \cup \{0\}, g^2 = 0\}$ be the semiring of dual numbers. Let
$P = \{a + bg \mid a, b \in 3Z^+ \cup \{0\}\} \subseteq S$ be the subsemiring of S.
T = {Collection of all set ideals of the semiring over the subsemiring P} be the set ideal se miring topological space of S over the subsemiring P.

Example 3.70: Let
S = {Collection of all subsets of the field Z_7} be the subset semiring of the field Z_7. P = {{0}, {1}, {2}, …, {6}} be the subset subsemiring of S.

To find the set ideals of the topological space; T of the subset semiring over the subsemiring P.

T = ⟨{{0}, {{0}, {1}, {2}, …, {6}}, {{0}, {0,1}, {0,2}, …, {0, 6}}, {{0}, {0, 1, 2}, {0, 2, 4}, {0, 4, 1}, {0, 3, 6}, {0, 5, 3}, {0, 6, 5}}, {{0}, {0, 1, 3}, {0, 2, 6}, {0, 4, 5}, {0, 3, 2}, {0, 5, 1}, {6, 0, 4}}, {{0}, {0, 1, 6}, {0, 2, 5}, {0, 4, 3}}, {{0}, {0, 1, 2, 3}, {0, 2, 4, 5}, {0, 3, 6, 2}, {0, 4, 1, 5}, {5, 0, 3, 1}, {0, 6, 5, 4}}, {{0}, {0, 1, 2, 4}, {0, 3, 5, 6}}, {{0}, {0, 1, 2, 5}, {0, 2, 4, 3}, {0, 3, 6, 1}, {0, 4, 1, 6}, {0, 5, 3, 4}, {0, 6, 5, 2}}, {{0}, {1, 2, 6, 0}, {2, 4, 5, 0}, {4, 1, 3, 0}, {3, 6, 4, 0}, {5, 3, 2, 0}, {6, 5, 1, 0}}, {{0}, {0, 12, 3, 4}, {0, 2, 4, 6, 1}, {0, 4, 1, 5, 2}, {0, 3, 6, 2, 5}, {0, 2, 4, 6, 3}, {0, 6, 5, 4, 2}} {{0}, {0, 1, 2, 3, 5}, {0, 2, 4, 6, 3}, {0, 4, 1, 5, 6}, {0, 3, 6, 2, 1}, {0, 5, 3, 1, 2}, {0, 6, 5, 4, 2}}, {{0}, {0, 1, 2, 3 , 6}, {0, 2, 4, 6, 5}, {0, 4, 1, 5, 3}}, {{0}, {0, 1, 2, 3, 4, 5}, {0, 2, 4, 6, 1, 3}, {0, 4, 1, 5, 2, 6}, {0, 3, 6, 2, 5, 1}, {0, 5, 3, 1, 6, 4}, {0, 2, 3, 4, 5, 6}}, {{0}, {0, 1, 2, 3, 4, 5, 6}}⟩.

Example 3.71: Let
S = {Collection of all subsets of the field Z_3} be the subset semifield of the field Z_3.

Let P = {{0}, {1}, {2}} be the subset subsemifield of S.

T = {Collection of all set ideals of the subset semiring over subset subsemiring P} = ⟨{{0}, {1}, {2}}, {{0}, {0, 1}, {0, 2}}, {{0}, {0, 1, 2}}, {{0}, {1,2}}}⟩.

T is a set ideal subset topological space of subset semiring over P.

Example 3.72: Let
S = {Collection of all subsets of the rin g $Z_5 \times Z_7$} be the subset semiring of the ring $Z_5 \times Z_7$.

Let P = {(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)}} $\subseteq$ S be the subset subsemiring of the subset semiring.
T = { Collection of all set ideals of S over P} is th e set ideal topological subset semiring space of S over P.

Now w e pro ceed onto give exa mples of S marandache s et ideal of the subset se miring. M, if the subset subse miring P is contained in the set ideal M.

Example 3.73: Let S = {Collection of all subset of the ring Z_6} be the subset semiring of the ring Z_6.

Take P = {{0}, {2}, {4}} $\subseteq$ S a subset semiring of S.

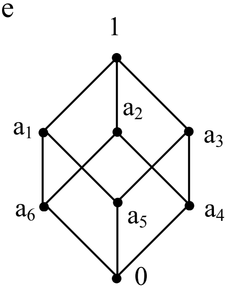
M_1 = {{0}, { 2}, {4}, {0, 3}} is a S marandache set ideal of the subset semiring of S over P.

M_2 = {{0}, { 2}, {4}, { 3}, {3}} $\subseteq$ S is a S marandache set ideal of the subset semiring of S over P.

M_3 = {{0}, {2}, {4}, {0, 3}, {3}} $\subseteq$ S is a S-set ideal of the subset semiring of over P.

M_4 = {{0}, { 2}, {4}, {3}, {1}}, M $_5$ = {{0}, {2}, {4}, {3}, {0, 3}, {1}, {0, 1}} and so on are S-set ideal of the subset semiring over the subset subsemiring.

Example 3.74: Let L be a lattice

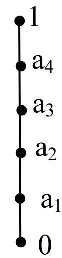


which is a semiring.

Let $P = \{0, a_6\}$ be the subsemiring.

We see $M_1 = \{0, a_6, a_5\}$, $M_2 = \{0, a_6, a_4\}$, $M_3 = \{0, 1, a_6\}$, $M_4 = \{0, a_1, a_6\}$, $M_5 = \{0, a_6, a_3\}$, $M_6 = \{0, a_2, a_6\}$ and so on are all S-set ideals of the subsemiring over P.

Example 3.75: Let S =



be a semifield.

$P = \{0, a_1\} \subseteq S$ is a subsemiring.

$M_1 = \{0, a_1, a_2\}$, $M_2 = \{0, a_1, a_3\}$, $M_3 = \{0, a_3, a_1, a_4\}$, $M_4 = \{0, a_1, 1\}$ are all S-set ideals of the semiring over the subsemiring.

Example 3.76: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in Z^+ (g) \cup \{0\}; 1 \leq i \leq 4 \right\}$$

be the semiring.

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in 3Z^+ (g) \cup \{0\}; 1 \leq i \leq 4 \right\}$$

be the subsemiring of S.

$$M_1 = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \, a_i \in 2Z^+ (g) \cup \{0\}\right\}$$

is only a set ideal of the semiring over the subsemiring P.

Consider

$$M_2 = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \, a_i \in 2Z^+ (g) \cup \{0\}; 1 \leq i \leq 4\right\} \subseteq S$$

is a set ideal over the subsemiring $P_2 = 8Z^+ (g) \cup \{0\}$.

Clearly M_2 is a S-set ideal over the subsemiring P_2 of S.

We have infinitely many S-set ideals for a give subsemiring of S. All these S-set ideals are of infinite order.

Example 3.77: Let $S = Z^+ (g_1, g_2) \cup \{0\}$ where $g_1^2 = g_2^2 = g_1g_2 = g_2g_1 = 0$ be the semiring of dual numbers of order two. Take $P = \{12Z^+ (g_1, g_2) \cup \{0\}\}$ be the subsemiring of S.

$M_1 = \{2Z^+ (g_1, g_2) \cup \{0\}\} \subseteq S$
$M_2 = \{3Z^+ (g_1, g_2) \cup \{0\}\} \subseteq S$ and
$M_3 = \{6Z^+ (g_1, g_2) \cup \{0\}\} \subseteq S$ are all S-set ideals of the subsemiring of S.

Example 3.78: Let

$$S = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \, a_i \in R^+ (g_1, g_2, g_3) \cup \{0\},\ g_1^2 = 0\right.$$

$$\left. g_2^2 = g_2 \text{ and } g_3^2 = 0 \text{ such that } g_jg_i = g_ig_j = 0 \text{ if } i \neq j, \; 1 \leq i, j \leq 3; 1 \leq t \leq 4\right\}$$

be a semiring of infinite order under natural product of matrices. S has infinite number of subsemirings attached with each of these subsemirings; we have infinite number of set ideal topological semiring spaces. All these topological spaces are also of infinite order.

Example 3.79: Let S = {Collection of all subsets of the complex modulo integer ring $C(Z_6)$} be the subset complex modulo integer semiring of finite order.

$P_1 = \{\{0\}, \{3\}\}$, $P_2 = \{\{0\}, \{0, 3\}\}$, $P_3 = \{\{0\}, \{2\}, \{4\}\}$, $P_4 = \{\{0\}, \{0, 2\}\ \{0, 4\}\}$ and so on are subsemirings.

We can associate with them set ideal topological complex modulo integer semiring spaces all of finite order.

Example 3.80: Let S = {Collection of all subsets of the complex modulo integer polynomial ring $C(Z_{11})\,[x]$} be an infinite finite complex modulo integer semiring of infinite order.

This has set ideal topological semiring spaces of infinite order.

Example 3.81: Let S = {Collection of all subsets of the special quasi dual number ring $C(Z_8)\,(g_1, g_2)$ where $g_1^2 = 0$, $g_2^2 = -g_2$, $g_1g_2 = g_2g_1 = 0$}; S has S-set ideal topological subset semiring subspaces also.

Example 3.82: Let $S = \langle Z^+ \cup I\rangle \cup \{0\}$ be the neutrosophic semiring of infinite order.

$Z^+ \cup \{0\} = P$ is subsemiring of S and T = {Collection of all set ideals of the semiring over the subsemiring P} be the set ideal topological semiring space of the subsemiring P.

We can have several such set ideal topological subspaces of infinite order.

This will also be known as the set ideal topological neutrosophic semiring space.

Example 3.83: Let $S = \langle Q^+ \cup I\rangle$ be a neutrosophic semiring. $P = \langle 3Z^+ \cup I\rangle$ is a neutrosophic subsemiring of S.

We can have several set ideal of this neutrosophic semiring over the subsemiring P. This collection T will be a set ideal neutrosophic topological space of the semiring S over the subsemiring P of S.

Example 3.84: Let

$$S = \left\{\left.\begin{pmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \end{pmatrix}\right| a_i \in \langle R^+ \cup I\rangle,\ 1 \le i \le 18\right\}$$

be the neutrosophic semiring of 2×9 matrices under natural product of matrices.

$$P = \left\{\left.\begin{pmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \end{pmatrix}\right| a_i \in \langle Z^+ \cup I\rangle,\ 1 \le i \le 18\right\} \subseteq S$$

be the neutrosophic subsemiring of S.

T = {Collection of all set ideals of S over the neutrosophic subsemiring P}, T is the set ideal neutrosophic semiring topological space of S over the subsemiring P. Clearly T is a S-set ideal topological space.

Now having seen examples of S-set ideal topological spaces of a semiring / subset semiring and set ideal topological spaces of a semiring / subset semiring we now proceed onto describe different types of set ideal topological spaces of a semiring as well as subset semirings.

We take S a semiring or a subset semiring suppose S_1 is a subsemiring / subset subsemiring of S.

T = {Collection of all prime set ideals of S over S_1} we define T to be a prime set ideal semiring topological space over the subset subsemiring S_1 of S.

We will illustrate this by some simple examples.

Example 3.85: Let
S = {Collection of all subsets of the ring Z_{12}}. S_1 = {{0}, {4}, {8}} $\subseteq$ S, P_1 be the subsemiring of S. P_1 = {{0}, {3}, {9}} $\subseteq$ S is a prime set ideal of S over S_1.

P_2 = {{0}, {0, 3}, {0, 9}, {0, 3, 9}} is a prime set ideal of S over S_1.

T = {Collection of all prime set ideals of S over S_1} is the prime set ideal subset semiring topological space of S over S_1.

Interested reader can construct several such prime set ideal topological spaces of semirings / subsemirings of S over S_1.

We can also define the notion of S marandache strongly quasi set ideal topological space of S_1, the semiring / subset semiring over the subsemiring / subset subsemiring of S over S_1.

Example 3.86: Let
S = {Collection of all subsets of the ring Z_{12}} be the subset semiring of the ring Z_{12}.

S_1 = {{0},{0, 6}, {6}} $\subseteq$ S is a subset subsemiring of S.

P_1 = {{0}, {4}, {0, 4}, {6}} $\subseteq$ S is a Smarandache strongly quasi set ideals of S over S_1. We see if T = {Collection of all S-strongly quasi set ideals of the subset semiring over the subsemiring S_1 of S}, then T is a Smarandache strongly quasi set ideal topological space of S over the subset semiring S_1 of S.

If we replace Z_{12} in example 3.86, by $\langle Z_{12} \cup I\rangle$ or $C(Z_{12})$ or $Z_{12}(g_1)$ ($g_1^2 = 0$) or $Z_{12}(g_2)$ ($g_2^2 = g_2$) or $Z_{12}(g_3)$ ($g_3^2 =$

$-g_3$) or by $C(Z_{12})$ (g_1) or $C(Z_{12})$ (g_2) or $C(Z_{12})$ (g_3) we get using the same subset subsemiring get the Smarandache strongly quasi set topological spaces.

This task is left as exercise to the reader.

We propose some problems for the reader.

Problems:

1. Find some interesting features enjoyed by subset semirings.

2. Characterize those subset semirings which are not subset S-semirings.

3. Does there exist a subset semiring which is not a S-subset semiring?

4. Characterize those subset semirings which are S-subset semirings of level II.

5. Give an example of a subset semiring which is not a Smarandache subset semiring of level II.

6. Let S = {Collection of all subsets of the ring Z_{18}} be the subset semiring of the ring Z_{18}.

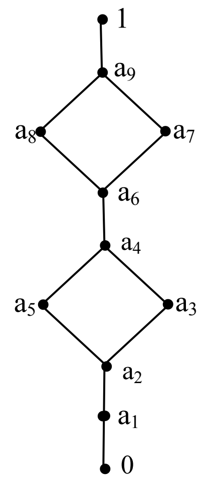
   (i) Find subset subsemirings of S.
   (ii) Is S a Smarandache subset semiring?
   (iii) Is S a Smarandache subset semiring of level II?

7. Let S = {Collection of all subsets of the ring Z_{90}} be the subset semiring.

   (i) Find all subset subsemirings of S.
   (ii) Is S a Smarandache subset semiring of level II?

8. Let S be the collection of all subsets of the lattice L given in the following:

L =



(i) Is S a subset semiring?
(ii) Can S be a subset Smarandache semiring of level II?
(iii) Is S a Smarandache subset semiring?
(iv) Find all subset subsemirings of S.
(v) Can S have zero divisors?
(vi) Can S have idempotents?

9. Study pr oblem S when L is replaced b y the Bo olean algebra of order 32 in problem 8.

10. Let S be the collection of all subsets of the ring $Z_7 \times Z_9$.

(i) Find order of S.
(ii) Prove S is a S-subset semiring.
(iii) Prove S is not a semifield.
(iv) Prove S is a Smarandache subset semiring of level II.
(v) Can S have idempotents?
(vi) Find all S-subset subsemirings of S.
(vii) Does there exist a subset subsemiring which is not a S-subset subsemiring?

11. Show if S is a collection of subsets of a Boolean algebra of order 2^m.

    (i) S is never a S-subset semiring of level II-prove.
    (ii) Is S is a S-subset semiring?
    (iii) S has zero divisors - prove.
    (iv) S has atleast m subset semifields - prove.
    (v) Can S have idempotents?

12. Can any subset semiring built using lattices be a S-subset semiring of level II?

13. Let S be the collection of all subsets of the ring $R = Z_3 \times Z_4 \times Z_{12}$.

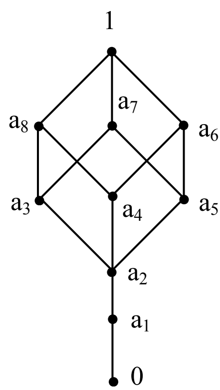
    (i) Prove S is a subset semiring.
    (ii) Prove S is a S-subset semiring of level II.
    (iii) Prove S has zero divisors.
    (iv) Can S have S-subset subsemirings?
    (v) Find idempotents in S.
    (vi) Can S have Smarandache zero divisors?
    (vii) Can S have Smarandache idempotents?

14. Let S = {be the collection of all subsets of the semigroup ring $Z_2S(3)$} be the subset semiring of the semigroup ring $Z_2 S(3)$.

    (i) Find subset subsemirings of S.
    (ii) Can S have idempotents?
    (iii) Can S have zero divisors?
    (iv) Is S a S-subset semiring?
    (v) Can S have S-subset ideals?
    (vi) Is S a S-subset semiring of level II?

15. Let S = {Collection of all subsets of the lattice L

L =



}

be the subset semiring.

(i) Is S a S-subset semiring?
(ii) Can S have zero divisors?
(iii) What is the order of S?
(iv) How many subset ideals can be built using S?

16. Let S = {Collection of all subsets of the ring Z_{24}} be the subset semiring of the ring Z_{24}.

(i) Can S have a subset which is a ring?
(ii) Can S be quasi S-subset semigroup?
(iii) Is S a S-subset semigroup?

(iv) Can S be a S-subset semigroup of level II?
(v) Can S have S-subset ideals?

17. Let
S = {Collection of all subsets of the group ring Z_{24} S_5} be the subset semiring.

Study problems (i) to (v) given in problem 16.

18. Let
S = {Collection of all subsets of the group ring $Z_6D_{2,5}$} be the subset semiring.

(i) Find order of S.
(ii) Can S have idempotents?
(iii) Prove S has zero divisors.
(iv) Can S have S-zero divisors?
(v) Can S have S-subset ideals?
(vi) Can S have S-subset subsem irings which are not S-subset ideals?
(vii) Is S a S-subset semiring?
(viii) Is S a S-subset semiring of level II?
(ix) Is S a quasi S-subset semiring?
(x) Can S have S-idempotent?

19. Let S = {Collection of all subsets of the ring $Z_{12} \times Z_{17}$} be the subset semiring over the ring $Z_{12} \times Z_{17}$.

(i) Prove S is a quasi S-subset semiring.
(ii) Prove S is a S-subset semiring of level two.
(iii) Prove S is a S-subset semiring.
(iv) Find zero divisors and idempotents in S.
(v) Can S have S-zero divisors and S-idempotents?
(vi) Can S have subset subsemir ing which are not subset ideals?

20. Let S = {Collection of all subsets of the ring $M_{2\times 2} = \{(a_{ij}) = m \mid a_{ij} \in Z_{12}; 1 \leq i, j \leq 2\}$} be the subset semiring which is commutative and of finite order.

(i) Find the order of S.
(ii) Can S have S-left subset ideals?
(iii) Can S have right subset ideals which are not left subset ideals?
(iv) Can S have S-subset ideals?
(v) Can S be a S-subset semiring of level II?
(vi) Prove S is a quasi S-subset semiring.

(vii) Can S have left zero divisors which are not right zero divisors?
(viii) Can S have S-idempotents?

21. Let S = {Collection of all subsets of the ring $C(Z_{19})$} be subset semiring.
Study questions (i) to (viii) proposed in problem 20.

22. Let S = {Collection of all subsets of the complex modulo mixed dual number ring $C(Z_{12})$ (g_1, g_2, g_3) where $i_F^2 = 11$, $g_1^2 = 0$, $g_2^2 = g_2$ and $g_3^2 = g_3$ with $g_ig_j = 0$ if $i \neq j$, $1 \leq i, j \leq 3$} be the subset semiring.
Study problems (i) to (viii) proposed in problem 20.

23. Let S = {Collection of all subsets of the ring

$$R = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \middle| a_i \in C(Z_{10})\ (g_1, g_2);\ a_1 + a_2g_1 + a_3g_2; \right.$$

$a_i = x_i + y_i i_F$, $i_F^2 = 9$, $x_i, y_j \in Z_{10}$, $1 \leq i \leq 3$, $g_1^2 = 0$, $g_2^2 = g_2$, $g_1g_2 = 0$} where the product on R is the natural product of the matrices} be the subset semiring.

(i) Study the special features enjoyed by S.
(ii) Study all the questions proposed in problem 20.

24. Let S = {Collection of all subsets of the semiring $Z^+(g_1, g_2) \cup \{0\}$, $g_1^2 = 0$ and $g_2^2 = g_2$, $g_1g_2 = 0$ } be subset semiring. Study properties associated with S.

25. Let S = {Collection of all subsets of ring $M_{3\times 3}$ = {M = (a_{ij}) | $a_{ij} \in C(Z_{10})$ (g_1, g_2) where $1 \leq i, j \leq 3$, $g_1^2 = 0$, $g_2^2 = g_2$, $g_1g_2 = g_2g_1 = 0$}} be the subset semiring.

(i) S is not a subset field.
(ii) Prove S has left subset ideals which are not right subset ideals.

(iii) Can S have S-zero divisors?
(iv) Can S be a S-subset semiring of level II?
(v) Give some special properties enjoyed by S.

26. Obtain some interesting features enjoyed by set ideal topological spaces of the subset semiring S using the ring Z_n (n a composite number) over any subset subsemiring of S.

27. Let S = $\{\{Z^+ \cup \{0\}\}$ be a semiring. Prove S has infinite number of set ideal topological spaces.

28. Let S = $\{\langle 3Z^+ \cup I\rangle \cup \{0\}$ be neutrosophic semiring$\}$.

(i) Prove S has infinite number subsemirings.
(ii) Prove using any subsemiring S_1 of S we can have an infinite set ideal topological semiring space of S over S_1.
(iii) Can S have S-set ideal topological space semiring?

29. Study problem (28) if S is replaced by $\langle Q^+ \cup I\rangle \cup \{0\}$ and $\langle R^+ \cup I\rangle \cup \{0\}$.

30. Let S = {Collection of sub sets of the ring $C(Z_7)$ (g) = R} be the subset semiring of the ring R.

(i) Find the number of subset subsemirings of S.
(ii) Find all subset ideals of the subset semiring.
(iii) How many of these subset ideals of S are Smarandache?
(iv) Does there exist subset subsemirings which are not subset ideals?

31. Obtain some special features enjoyed by Smarandache strong special ideal topological space of the subset semiring of the ring $C(Z_n)$.

(i) n a composite number.
(ii) n a prime number.

32. Study the above problem in case of ring $C(Z_n)$ replaced by $C(Z_n)$ (g) of dual numbers.

33. Analyse the same problem (32) in case of special dual like numbers $C(Z_n)$ (g_1); $g_1^2 = g$

34. Let S = {Collection of all subsets of the ring $C(Z_{16})$} be the subset semiring.

    (i) Find the number of set ideal topological spaces of S over the subset subsemirings of S.
    (ii) How many of these set ideal topological spaces are S-set ideal topological spaces?
    (iii) How many of them are Smarandache quasi strong set ideal subspaces?

35. Let S = {subsets of the ring Z_{24}} be the subset semiring.

    (i) Find the number of elements in S.
    (ii) How many subset subsemirings are in S?
    (iii) How many of them are S-subset subsemirings?
    (iv) How many S-set subset ideal topological space of S over subset subsemirings exist?
    (v) How many of them are S-strong quasi set subset topological spaces?

36. Let S = {subsets of the ring R = $C(Z_6) \times Z_7$} be the subset semiring of the ring R.

    (i) Find the total number of subset subsemirings.
    (ii) Find the total number of S-subset subsemirings of S.
    (iii) Find the total number of subset ideals of the subset semiring of S.
    (iv) How many of them are S-ideals of the subset semiring S?

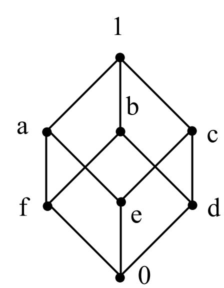
37. Let $S = P = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \langle Z^+ \cup I \rangle \cup \{0\} \right\}$ be a neutrosophic semiring of S.

(i) Find S-set ideals of S over the subsemiring $S_1 = Z^+ \cup \{0\}$.
(ii) Find set ideals of S which are not S-set ideals of S over the subsemiring $S_1 = Z^+ \cup \{0\}$ of S.
(iii) Find the set ideal topological space of the semiring over the subsemiring S.
(iv) Can S be a S-semiring?

38. Let $S = \{(a_1, a_2, a_3, a_4) \mid a_i \in \langle Q^+ \cup I \rangle) (g_1\ g_2);\ g_1^2 = 0,\ g_2^2 = 0, 1 \leq i \leq 4\}$ be a semiring.

(i) Is S a S-semiring?
(ii) Find S-set ideal topological semiring spaces of S.
(iii) Find two set ideal topological semiring spaces of S which are not Smarandache set ideal topological spaces of S.

39. Let P = {all subsets of the semiring

L = 

}

be a subset semiring.

(i) Is P a S-subset semiring?
(ii) How many subset subsemirings of P are there?
(iii) Find all S-subset subsemirings of P.

(iv) Find the total number of set ideal topological spaces of the subset semiring P.

40. Let B = {Collection of all subsets of the ring R = C(Z_9) (g_1, g_2), $g_1^2 = 0$, $g_2^2 = g_2$, $g_1g_2 = g_2g_1 = 0$} be the subset semiring of the ring R. Study question (i) to (iv) of problem 39 for this B.

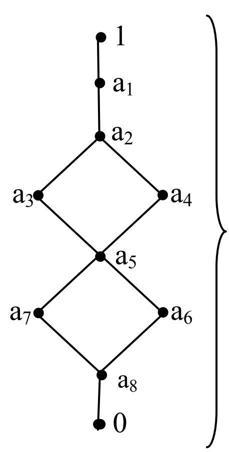
41. Let S = {Collection of all subsets of the ring Z_{48}} be the subset semiring of the ring Z_{48}.

(i) Find all the subset subsemirings of S.
(ii) Find all subset ideals of S.
(iii) Find all set subset ideal topological spaces of S.

42. Find the difference between the subset semiring built using a chain lattice and a Boolean algebra.

43. Find the difference between subset semirings built using $Z^+ \cup \{0\}$ and Z i.e., using a semiring and a ring respectively.

44. Find all the zero divisors of the subset semiring.
S = {Collection of all subsets of the semiring.



(i) Can S have S-zero divisors?
(ii) Can S have S-idempotents?
(iii) Find the nu mber of S-set ideal topo logical subset semiring space of S.

45. Let S = {set of all subsets of the ring

$$R = \left\{M = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| \; a_i \in C(Z_4)(g_1, g_2),\ 1 \leq i \leq 4,\right.$$

$g_1^2 = g_2^2 = g_2g_1 = g_1g_2 = 0\}\}$ be the subset semiring.

(i) Can S have right subset zero divisors which are not left subset zero divisors?
(ii) Can S have S-zero divisors?
(iii) How many S-subset subsemirings does S have?
(iv) Find the number of distinct set ideal su bset semiring topological spaces of S using S-subset subsem irings of S.

46. Let S = {Collection of all subsets of the ring

$$R = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| \; a_i \in C(Z_{12})(g), \text{ where } g^2 = -g,\right.$$

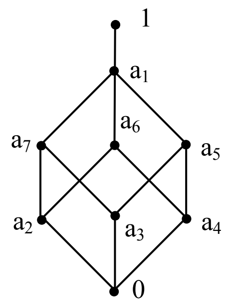
$1 \leq i \leq 10\}$ be the subset semiring of R.

(i) Find the number of elements in S.
(ii) Find the number a subset subsemirings of S.
(iii) Find the nu mber of the set ideal topological spaces of the subset semirings.

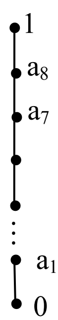
47. Find the difference bet ween the set id eals of a se miring and a semifield.

48. Let S be the semiring.

S =



and S_1 =



be the semifield.

(i) Compare the set ideal topological spaces of S and S_1.

49. Suppose S = {collection of subsets of the ring R = Z_{15}} be the subset semiring.

(i) How many subrings S contains?
(ii) Find set ideals over these subrings of S.

(iii) Will the collection of set ideals over subrings be a set ideal topological space?

50. Give example of semirings which has no S-subrings?

51. Does there exist a S-subset semiring which has no proper subset P which is a ring?

52. Can the semiring S = {Collection of all subsets of the ring

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| \; a_i \in C(Z_{10}) (g_1, g_2, g_3) \text{ where } g_1^2 = 0 \right.$$

$g_2^2 = g_2$ and $g_3^2 = g_3$; $g_1g_2 = g_2g_1 = 0$ $g_1g_3 = g_3g_1 = 0$, $g_2g_3 = g_3 = g_3g_2$; $1 \leq i \leq 6$} be the ring under natural product of matrices} be a subset semiring?

(i) Find the number of elements in S.
(ii) Find the number of subset subsemirings.
(iii) How many of these subset subsemirings are S-subset subsemirings?
(iv) Find the number of set ideal topological subset semiring spaces over every subset subsemiring.

53. Enumerate some interesting properties enjoyed by set ideal subset semiring topological spaces.

54. Find some nice applications of subset semirings.

55. Can we say with every subset semiring have a set ideal topological space associated with a lattice?

56. When will the lattice in problem 55 be a Boolean algebra?

57. Find the set ideal topological subset semiring space of the subset semiring S = {Collection of all subsets of Z_{126}} over the subset subsemiring
P = {{0}, {18}, {36}, {54}, {72}, {90}, {108}} ⊆ S.

    (i) How many elements are in that set ideal topological space?
    (ii) How many subset subsemirings are there in S?
    (iii) Find the number of S-subset subsemirings of S.

58. Let S be the subset semiring of the ring Z_{12}. Let F be a subset field in S.

    (i) Find all the set ideals of the subset semiring over the field F.
    (ii) Let T = {Collection of all set ideals of S over F} be a set ideal topological space over F. Find o(T).
    (iii) Let R ⊆ S be a subset ring which is not a subset field. M = { Collection of all set ideals of S over R} be set ideal topological space over R. Find o(M).
    (iv) Compare M and T.

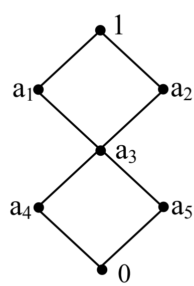
59. Let S = {Collection of all subsets of the field Z_{11}} be the subset ideal of S.

    (i) Let T = {collection of all set ideal of S over the subset field P = {{0}, {1}, {2}, { 3}, {4}, {5}, {6}, {7}, {8}, {9}, {10}}} be set ideal topological space of S over P. Find o(T). Find the lattice associated with T.
    (ii) If P is replaced by
    M = {{0}, {0, 1}, {0,2}, {0, 3}, …, {0, 10}} in T.
    Study T.

60. Let $S = \{ \langle Z^+ \cup I \rangle (g_1, g_2) \cup \{0\} \mid g_1^2 = g_2^2 = 0, g_1g_2 = g_2g_1 = 0\}$ be a semiring.

(i) Is S a S-semiring?
(ii) For $P = Z^+ \cup \{0\} \subseteq S$, the subsemiring find the set ideal topological space T_1 of S over P.
(iii) If $R = \langle Z^+ \cup I \rangle \cup \{0\}$ be the subsemiring of S. Find the set ideal topological space T_2 of S over R.
(iv) Compare T_1 and T_2.
(v) If $M = \{(Z^+ \cup \{0\}) (g_1)\}$ be the subsemiring of S. Find T_3 the set ideal topological space of S over M.
(vi) Compare T_2 and T_3.
(vii) Let $N = \{\langle Z^+ \cup I \rangle (g_1) \cup \{0\}\}$ be the subsemiring of S. Find T_4 the set ideal topological space of S over N.
(viii) Which is the largest set ideal topological space T_1 or T_2 or T_3 or T_4?

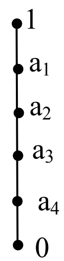
61. Let S =



be the semiring.

(i) Find the set ideals of S over the subsemirings $P_1 = \{0, a_4\}$, $P_2 = \{0, a_5\}$ and $P_3 = \{0, a_4, a_5, a_3\}$.

62. Let S =

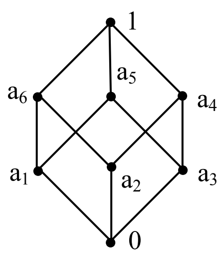


be a semiring. $P_1 = \{0, a_4\}$, $P_2 = \{0, a_3, a_4\}$ and $P_3 = \{0, a_4, a_3, a_2\}$ be subsemirings.

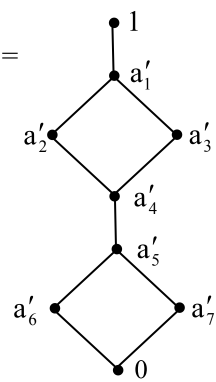
Find all set ideals of S ov er the subsemirings P_1, P_2 and P_3.

63. Let $S = L_1 \times L_2$ where

L_1 =



and L_2 =

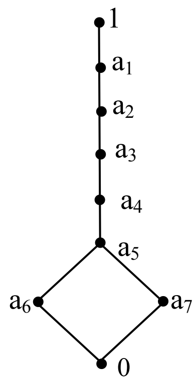


be a semiring.

(i) Let $R = \{0, a_1\} \times \{0, a'_6\}$ be the subsem iring. Fin d all the set ideals of S over the subring R.

(ii) If $M = L_1 \times \{0\}$ be the subring. Find the set ideal topological space of S over M.

64. Let S =



be a semiring.

(i) Find all the set ideals of S over every subsemiring.
(ii) Suppose P = {all subsets of the sem iring S} be the subset sem iring of S. Find all set ideals of P over every subsemiring.

65. Let P = {Collection of all subsets of $C(Z_{20})$} be the subset semiring.

(i) How many subset subsemirings are in P?
(ii) Find how many of the subset subsem irings are Smarandache.
(iii) How many distinct set ideal topol ogical subset semiring spaces of P exist?

66. Let M = {Collection of all subsets of $C(Z_{20})$ (g) $g^2 = 0$} be the subset sem iring. Stud y questions (i), (ii), (iii) of problem (65) in case of M.

67. Let S = {The Boolean algebra of o rder 16} be the semiring.

(i) Find all subsemirings.
(ii) Find all S-subsemirings.
(iii) Find all set ideal topological spaces of S over the subsemirings.
(iv) Can S have S zero divisors?

68. Distinguish between set ideal topological spaces of a semigroup and a semiring.

69. Compare the set ideal topological spaces of a semiring and a ring.

70. Let $S = Z^+ \cup \{0\}$ be the semiring.

(i) Find all set ideal topological semiring spaces of S.
(ii) Find all set ideal topological semigroup spaces of the semigroup $T = Z^+ \cup \{0\}$.

71. Let $S = Z_5$ be the ring find the set ideal topological space T_1 of S.

If $S_1 = (Z_5, \times)$ be the semigroup find the set ideal topological space T_2 of the semigroup S. Compare T_2 and T_1.

72. Let M = {subsets of the semigroup $\{Z_{12}, \times\}$} be a subset semigroup. N = {subsets of the ring Z_{12}} be a subset semiring. Compare M and N.

73. Let P = {subsets of the semigroup $\{Z_{19}, \times\}$} be the subset semigroup and R = {subsets of the ring $\{Z_{19}, \times\}$} be the subset semiring. Compare P and R.

Chapter Four

SUBSET SEMIVECTOR SPACES

In this chapter we for the first ti me define the new notion different types of subset semivector spaces over fields, rings and semifields.

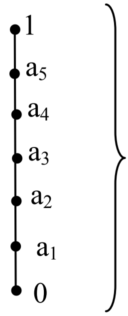
This study is not only innovative but will be useful in due course of time in applications. Several interesting results about them are derived and developed in this book.

These subset semivector spaces are different from set vector spaces for these deal with subsets of a semigroup or a group.

DEFINITION 4.1: *Let S = {Collection of all subsets of a semigroup P} and M a semifield. If S is a semivector space over M then we define S to be a subset semivector space over the semifield M.*

We will illustrate this situation by some examples.

Example 4.1: Let S = {Collection of all subsets of the chain lattice L =



be the subset semivector space over the semifield L.

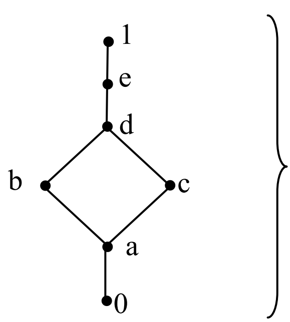
Elements of S are of the form $\{0, a_1, a_3\}$, $\{0\}$, $\{a_1\}$, $\{a_2, a_3\}$ and $\{a_1, a_3\} \in S$.

We can multiply the elements of S by elements from $\{0, a_1, a_2, \ldots, a_5, 1\}$. $a_2 \{a_2, a_3\} = \{a_2\}$, $a_1 \{0, a_1, a_3\} = \{0, a_1\}$.

This is the way product is made. Clearly S is a semivector space over the semifield L.

Example 4.2: Let
S = {set of all subsets of the semiring $Z^+ \cup \{0\}$} be the subset semivector space of S over the semifield $F = Z^+ \cup \{0\}$. Clearly S is of infinite order.

Example 4.3: Let S = {Set of all subsets of the lattice L =



be subset semivector space over the semifield L. Clearly the number of elements in S is finite.

Example 4.4: Let S = {set of all subsets of the semiring $Z^+(g_1, g_2) \cup \{0\}$, $g_1^2 = g_2^2 = 0$, $g_1g_2 = g_2g_1 = 0\}$ be the subset semivector space over the semifield $F = Z^+ \cup \{0\}$.

Let A = $\{9 + 8g_1, 11 + 5g_1\}$ and B = $\{10g_1, 12g_2, 5+g_1\}$

A+B = $\{9+18g_1, 9+8g_1+12g_2 + 14+9g_1, 11+15g_1, 11+5g_1 + 12g_2 + 16 + 6g_1\} \in S$.

Let $8 \in F$ $8(A) = \{72 + 64g_1, 88+40g_1\}$ and so on.

Example 4.5: Let S = {Collection of all subsets of the semiring

$$P = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in Z^+(g) \cup \{0\}, g^2 = 0\right\}\}$$

be the subset semivector space over the semifield $F = Z^+ \cup \{0\}$.

$$\text{For } A = \left\{\begin{bmatrix} 3 & 2 \\ g & 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 5 & g \end{bmatrix}, \begin{bmatrix} 8 & 1 \\ 0 & 2 \end{bmatrix}\right\} \text{ and}$$

$$B = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 9 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} g & 2g \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}\right\} \in S \text{ we have}$$

$$A + B = \left\{\begin{bmatrix} 3 & 2 \\ g & 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 5 & g \end{bmatrix}, \begin{bmatrix} 8 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 12 & 4 \\ g & 4 \end{bmatrix}, \begin{bmatrix} 13 & 2 \\ 5 & 4+g \end{bmatrix},\right.$$

$$\begin{bmatrix} 17 & 3 \\ 0 & 6 \end{bmatrix}, \begin{bmatrix} 3+g & 2+2g \\ g & 0 \end{bmatrix}, \begin{bmatrix} 4+g & 2g \\ 5 & g \end{bmatrix}, \begin{bmatrix} 8+g & 1+2g \\ 0 & 2 \end{bmatrix},$$

$$\left.\begin{bmatrix} 3 & 2 \\ g & 5 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 5 & 5+g \end{bmatrix}, \begin{bmatrix} 8 & 1 \\ 0 & 7 \end{bmatrix}\right\} \in S.$$

Suppose $12 \in F$ then

$$12A = \left\{\begin{bmatrix} 36 & 24 \\ 12g & 0 \end{bmatrix}, \begin{bmatrix} 48 & 0 \\ 60 & 12g \end{bmatrix}, \begin{bmatrix} 96 & 12 \\ 0 & 24 \end{bmatrix}\right\} \in S.$$

$$12B = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 108 & 24 \\ 0 & 48 \end{bmatrix}, \begin{bmatrix} 12g & 24g \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 60 \end{bmatrix}\right\} \in S.$$

Thus S is a subset semivector space over F.

Example 4.6: Let

S = {set of all subsets of the semiring $\langle Q^+ \cup I\rangle \cup \{0\}$} be the subset semivector space over the semifield $Q^+ \cup \{0\}$ (also over the semifield $Z^+ \cup \{0\}$).

Example 4.7: Let S = {set of all subsets of the semiring

$$\left\{\begin{pmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \end{pmatrix} \middle|\ a_i \in Q^+ \cup \{0\},\ 1 \le i \le 14\right\}$$

be a subset semivector space over the semifield $Z^+ \cup \{0\}$.

$$\text{Let } A = \left\{\begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 5 & 1 \\ 0 & 1 & 1 & 2 & 0 & 0 & 3 \end{pmatrix}\right\}$$

and

$$B = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 8 & 4 & 0 & 3 & 6 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 3 & 0 \\ 4 & 2 & 0 & 5 & 6 & 0 & 0 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 1 & 2 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 7 & 6 \end{pmatrix}\right\} \in S.$$

We see $A + B = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 8 & 4 & 0 & 3 & 6 & 1 \end{pmatrix}, \right.$

$$\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 3 & 0 \\ 4 & 2 & 0 & 5 & 6 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 7 & 6 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 2 & 3 & 5 & 5 & 11 & 8 \\ 0 & 9 & 5 & 2 & 3 & 6 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 3 & 0 & 8 & 1 \\ 4 & 3 & 1 & 7 & 6 & 0 & 5 \end{pmatrix},$$

$$\left. \begin{pmatrix} 2 & 2 & 0 & 4 & 4 & 5 & 1 \\ 0 & 1 & 6 & 2 & 0 & 7 & 9 \end{pmatrix} \right\} \in S.$$

For if $8 / 3 \in Q^+ \cup \{0\} = F$ then

$$8/3A = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 8/3 & 0 & 0 & 8/3 & 0 & 40/3 & 8/3 \\ 0 & 8/3 & 8/3 & 16/3 & 0 & 0 & 8 \end{pmatrix} \right\} \in S.$$

$$8/3\ B = \left\{ \begin{pmatrix} 8/3 & 16/3 & 8 & 32/3 & 40/3 & 16 & 56/3 \\ 0 & 64/3 & 32/3 & 0 & 8 & 16 & 8/3 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 0 & 8/3 & 0 & 16/3 & 0 & 8 & 0 \\ 32/3 & 16/3 & 0 & 40/3 & 16 & 0 & 0 \end{pmatrix},$$

$$\left. \begin{pmatrix} 8/3 & 16/3 & 0 & 8 & 32/3 & 0 & 0 \\ 0 & 0 & 40/3 & 0 & 0 & 56/3 & 16 \end{pmatrix} \right\} \in S.$$

This is the way operations are perfor med on the subset semivector space S over the semifield $F = Q^+ \cup \{0\}$.

Example 4.8: Let S = {Collection of all subsets of the polynomials of degree less than or equal to 5 in $Z^+[x] \cup \{0\}$} be the subset semivector space over the semifield $F = Z^+ \cup \{0\}$.

Let $A_1 = \{\{x, x^2\}\}$, $A_2 = \{0, 1+x^3, x^4\}$ $A_3 = \{7x^2 + 6x + 3, 8x^5 + 3, 0\}$, $B_1 = \{\{0\}\}$, $B_2 = \{x+1, x^3+3\}$ and $B_3 = \{x^2 + x + 3, x^4 + 3x^2 + x + 8\}$ be in S.

We can find $A_i + B_j$, $A_i + A_j$ and $B_i + B_j$, $1 \leq i, j \leq 3$, which are as follows:

$A_1 + A_2 = \{\{x, x^2, 1+x+x^3, x+x^4, x^2+x^3, x^2+x^4\} \in S$.

$A_3 + B_2 = \{7x^2 + 7x + 4, x^3 + 7x^2 + 6x + 6, 8x^5 + x + 4, x^3 + 8x^5 + 6, x+1, x^3+3\} \in S$ and

$B_1 + B_3 = \{x^2 + x + 3, x^4 + 3x^2 + x + 8\} \in S$.

We see S is a subset semivector space over the semifield $Z^+ \cup \{0\} = F$.

Clearly $A_i \times B_j$ or $A_i \times A_j$ or $B_i \times B_j$ are not defined in S, $1 \leq i, j \leq 3$.

For take $A_2 \times A_3 = \{0 (7x^2 + 6x + 3), 0(8x^5 + 3), 0 \times 0, (1+x^3) \times (7x^2 + 6x + 3), (1+x^3) (8x^5 + 3), (1+x^3) \times 0, x^4 (7x^2 + 6x + 3), x^4 (8x^5 + 3), x^4, 0\} \notin S$.

We see as in case of usual semilinear algebras define in case of subset semilinear algebras, they are basically subset semivector spaces which are closed under some product and the product is an associative operation. Clearly all operations on subset semivector spaces do not lead to subset semilinear algebras.

In view of this we see the subset semivector space in example 4.8 is not a subset linear semialgebra or subset semilinear algebra over a semifield.

We will provide a few examples of subset semilinear algebras.

Example 4.9: Let S = {Collection of all subsets of the semiring $R = \{(Z^+ \cup \{0\})(g_1, g_2, g_3) = a_1 + a_2g_1 + a_3g_2 + a_4g_3$ with $a_i \in Z^+ \cup \{0\}$, $1 \leq i \leq 4$; $g_1^2 = 0$, $g_2^2 = 0$, $g_3^2 = g_3$, $g_ig_j = g_jg_i = 0$, $i \neq j$, $1 \leq i, j \leq 3\}\}$ be the subset semivector space over the semifield $F = Z^+ \cup \{0\}$.

Let $A = \{0, 5, 7g_1, 8+4g_2\}$ and
$B = \{8 + 2g_1 + g_2, 3g_1 + 4g_2 + 5g_3\} \in S$.

We see $A + B = \{0, 8+2g_1 + g_2, 3g_1 + 4g_2 + 5g_3, 13 + 2g_1 + g_2, 5 + 3g_1 + 4g_2 + 5g_3, 8 + 9g_1 + g_2, 10g_1 + 4g_2 + 5g_3, 16 + 2g_1 + 5g_2, 8 + 3g_1 + 7g_2 + 5g_3\} \in S$.

Consider $A \times B = \{0, 40 + 10g_1 + 5g_2, 15g_1 + 20g_2 + 15g_3, 56g_1 + 64 + 16g_1 + 36g_2, 24g_1 + 32g_2 + 40g_3\} \in S$. We see S is a subset semilinear algebra over the semifield $Z^+ \cup \{0\}$.

Example 4.10: Let S = {Collection of all subsets of the semiring

$$R = \left\{\left.\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\right| a_i \in Q^+ \cup \{0\}; 1 \leq i \leq 4\}\right\}$$

be the subset semivector space over $Z^+ \cup \{0\}$. Clearly S is a subset semilinear algebra over $Z^+ \cup \{0\}$.

$$\text{Let } A = \left\{\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}\right\} \text{ and}$$

$$B = \left\{\begin{pmatrix} 6 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 7 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\right\} \in S.$$

We know

$$A + B = \left\{\begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 9 & 5 \end{pmatrix}, \begin{pmatrix} 6 & 2 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 5 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 12 & 2 \end{pmatrix}\right\} \in S.$$

Now

$$A \times B = \left\{\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 6 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 7 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 6 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 7 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix} 0 & 0 \\ 12 & 2 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 21 & 6 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 9 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 30 & 5 \end{pmatrix}, \begin{pmatrix} 7 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 5 & 0 \end{pmatrix}\right\} \in S.$$

Thus S is a subset semilinear algebra over the semifield $F = Z^{+} \cup \{0\}$.

Example 4.11: Let S = {Collection of subsets of $Z^{+}[x] \cup \{0\}$} be a subset semivector space over the semifield $F = Z^{+} \cup \{0\}$.

Let $P = \{1 + 5x + 3x_2, 8x_3 + 9\}$ and $Q = \{9x_6 + 2x + 9, 12x_2 + 1, 5x_7\} \in S$.

We see $P + Q = \{10 + 7x + 3x_2 + 9x_6, 8x_3 + 10 + 12x_2, 5x_7 + 8x_3 + 9, 15x_2 + 5x + 2, 5x_7 + 3x_2 + 5x + 1\} \in S$.

Consider $P \times Q = \{(1 + 5x + 3x_2)(9x_6 + 2x + 9), (3x_2 + 5x + 1) \times (12x_2 + 1), (1 + 5x + 3x_2) \times 5x_7, (8x_3 + 9) \times 5x_7, (8x_3 + 9), (9x_6 + 2x + 9), (8x_3 + 9) \times (12x_2 + 1)\} \in S$.

Thus S is a subset sem ilinear algebra of over the semifield $F = Z^+ \cup \{0\}$.

It is interesting to note the following result.

THEOREM 4.1: *Let S be a subset semivector space over the semifield F. Then S in general is not a subset semilinear algebra over the semifield F.*

However if S is a subset semilinear algebra over the semifield F then S is a subset semivector space over the semifield F.

Proof is left as an exercise to the reader.

We can as in case of semivector spaces define in case of subset semivector spaces also the notion of subset semivector subspaces, semitransformation, semi basis and so on.

The definitions are infact a matter of routine.

However we will give some examples.

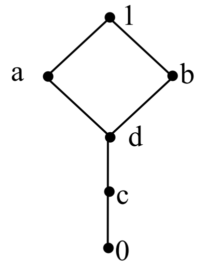
Example 4.12: Let

S = {Collection of all subsets of the sem iring $Q^+[x] \cup \{0\}$} be the subset semivector space over $F = Z^+ \cup \{0\}$, the semifield.

T = {Collection of all subsets of the sem iring $Z^+[x] \cup \{0\}$} $\subseteq$ S is a sub set se mivector subspace of S over the se mifield $F = Z^+ \cup \{0\}$.

Let W = {Collection of all subsets with cardinality are i.e., $\{x\}$, $\{x^2\}$, $\{10x\}$, $\{15x^2\}$ so on} $\subseteq$ S; W is again the subset semivector subspace of S over the semifield $F = Z^+ \cup \{0\}$.

Example 4.13: Let S = {Collection of all subsets of the semiring



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be a subset semivector space over the semifield;
F = {0, c, d, a, 1}. S is clearly a subset semilinear algebra over F.

Take A = {0, 1, a, b} and B = {d, c, b, a} $\in$ S. We see A + B = {d, c, a, b, 1} and AB = {0, d, c, b, a} $\in$ S. Thus S is a subset semilinear algebra over F.

Take P = {all subsets from the set {0, d, c}} $\subseteq$ S; P is a subset subsemivector space over F or subset semivector subspace of S over the semifield F.

Example 4.14: Let S = {Collection of all subsets of the semiring $Z^+(g_1, g_2) \cup \{0\}$, $g_1^2 = g_2^2 = g_1g_2 = g_2g_1 = 0$} be a subset semivector space over the semifield $F = Z^+ \cup \{0\}$.

P = {Collection of all subsets of the semiring $Z^+(g_1) \cup \{0\}$} be the subset semivector subspace of S over the semifield $F = Z^+ \cup \{0\}$.

We can have several such subset semivector subspaces of S over F.

Infact these are also subset semilinear subalgebras of S as S is a subset semilinear algebra over the semifield F.

Example 4.15: Let S = {Collection of all subsets of the semiring

$$R = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_i \in Z^+ (g) \cup \{0\},\ 1 \leq i \leq 3\} \right.$$

be the subset semivector space over the semifield $F = Z^+ \cup \{0\}$.

Take V = {Collection of all subsets from

$$P = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_i \in Z^+ (g) \cup \{0\} \subseteq R\}, \right.$$

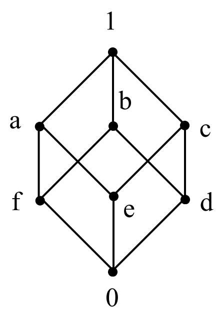
V is a subset semivector subspace of S over the semifield $F = Z^+ \cup \{0\}$.

Example 4.16: Let

S = {Collection of all subsets of the semiring $Z^+ [x] \cup \{0\}$} be the subset semivector space over the semifield $F = Z^+ \cup \{0\}$.

Let P = {Collection of all singleton subsets $\{ax^i\}$ where $a \in Z^+ \cup \{0\}$; $0 \leq i \leq \infty$} $\subseteq$ S; be the subset semivector subspace of S over the semifield $F = Z^+ \cup \{0\}$.

Example 4.17: Let S = {Collection of all subsets of the semiring B =



be the subset semivector space over the semifield F = {0, F, a, 1} $\subseteq$ B.

Consider the set P = {Collection of all subsets of the subsemiring T = {0, b, f, d} $\subseteq$ B}, P is a subset semivector subspace of S over F = {0, 1, a, f} $\subseteq$ B.

Example 4.18: Let S = {Collection of all subsets of the semiring $\langle Q^+ \cup I\rangle (g_1, g_2, g_3) \cup \{0\}$ where $g_ig_j = 0$; $1 \leq i, j \leq 3$} be the subset semivector space over the semifield $F = Z^+ \cup \{0\}$.

P_1 = {Collection of all subsets of $\langle Q^+ \cup I\rangle \cup \{0\}$} $\subseteq$ S is a subset semivector subspace of S over F the semifield.

P_2 = {Collection of all subsets of the semiring $Q^+ (g_1) \cup \{0\}$} $\subseteq$ S is a subset semivector subspace of S over F.

P_3 = {Collection of all subsets of the semiring $(\langle Q^+ \cup I\rangle \cup \{0\}) (g_1)$} $\subseteq$ S is a subset semivector subspace of S over $F = Z^+ \cup \{0\}$.

Now we can define the notion of transformation of subset semivector spaces, it is a matter of routine so left as an exercise to the reader.

First we make it clear that transformation of two subset semivector spaces is possible if and only if they are defined over the same semifield.

We will illustrate this situation by an example or two.

Example 4.19: Let S = {Collection of all subsets of the semiring $Z^+(g) \cup \{0\}$, $g^2 = 0$} be a subset semivector space over the semifield $Z^+ \cup \{0\}$.

Let S_1 = {Collection of all subsets of $Z^+ \cup \{0\}$} be a subset semivector space over the semifield $Z^+ \cup \{0\}$.

Define $T : S \to S_1$ as

$T(A) = A$ if $A \subseteq Z^+ \cup \{0\}$;

$T(A_1) = \{B_1\}$ if $A_1 = \{a + bg\}$ then
$T(A_1) = \{a\}$ for all $A, A_1 \in S$.

That is $A = \{0, 9, 25, 32, 47, 59\} \in S$ then
$T(A) = A \in S_1$.
$A_1 = \{3 + 5g, 7g, 8+4g, 6, 80, 14g + 7\} \in S$.
$T(A_1) = \{3 + 8, 0, 6, 80, 7\} \in S_1$.

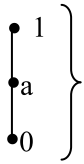
This is the way the semilinear transformation T from S to S_1 is defined.

Interested reader can study the notion of semilinear transformation of subset semivector spaces defined over a semifield S.

We can also have the notion of basis and study of basis for the subset semivector spaces is interesting.

We will first illustrate this situation by an example or two.

Example 4.20: Let V = {Collection of all subsets of the semiring



be the subset semivector space over the semifield F = {0, a, 1}.

V = {{0}, {1}, {a}, {0, 1}, {0, a}, {a, 1}, {0, a, 1}}.

The basis for V is {{1}}.

For a{1} = {a}, $\quad$ {1} ∪ {a} = {1, a},
0{1} = {0}, $\quad$ {0} ∪ {a} = {0, a},
{1} ∪ {0} = {1, 0} and
{1, 0} ∪ {a} = {0, 1, a}.

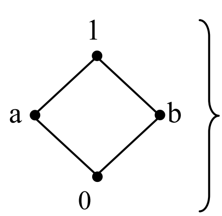
Hence the claim.

Inview of this we have the following theorem.

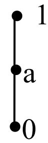
THEOREM 4.2: *Let S = {Collection of all subsets of a chain lattice C_n} be the subset semivector space over the semifield, the chain lattice C_n. {1} is a semibasis of S and C_n. The basis of S is unique and has no more basis.*

Proof is direct hence left as an exercise to the reader.

Example 4.21: Let S = {Collection of subsets of the Boolean algebra



be the subset se mivector space over the se mifield F = {0, a, 1;



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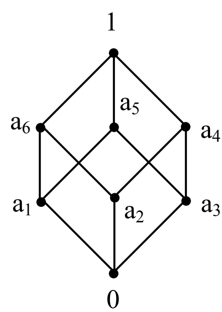
The basis B for S over F is as follows:

B = {{1}, { b}}. a{1} = {a}, 1.{b} = {b}, o{1} = {0}. {a} ∪ {1} = {1, a}, { 1, a} ∪ {b} {1, b, a} {1, a, b} ∪ {0} = {1, a, b, 0} and so on.

This basis is also unique.

Example 4.22: Let S = {Collection of all subs ets of the Boolean algebra of order 8



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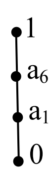
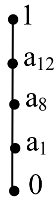
be the subset semivector space over the semifield F



The basis B for S over F is {{1}, {a_2}, {a_3}}. This basis is also unique.

Example 4.23: Let S = {Collection of all subsets of the semiring which is a Boolean algebra of order 16} be the subset semivector space over the semifield

F =



The basis B of S over F is {{1}, $\{a_2\}$, $\{a_3\}$, $\{a_4\}$} where a_1, a_2, a_3, a_4 are the atoms of the Boolean algebra of order 2^4.

Inview of all these we have the interesting theorem.

THEOREM 4.3: *Let S ={Collection of all subsets of a Boolean algebra B of order 2^n over a chain F of length (n + 1) of the Boolean algebra B} be the subset semivector space over the semifield F. If $\{a_1, \ldots, a_n\}$ are the atoms of B and if $a_i \in F$ then the basis of S is $\{\{1\}, \{a_1\}, \ldots, \{a_n\}\}$.*

Proof is direct hence left as an exercise to the reader.

Example 4.24: Let
S = {Collection of all subsets of the semiring $Z^+ \cup \{0\}$} be the subset semivector space over the semifield F = $Z^+ \cup \{0\}$.

Can S have a finite basis over F? Find a basis of S over F.

Example 4.25: Let S = {Collection of all subsets of the semiring $Q^+ (g) \cup \{0\}$, $g^2 = 0$} be the subset semivector space over the semifield F = $Z^+ \cup \{0\}$.

S has an infinite basis over F.

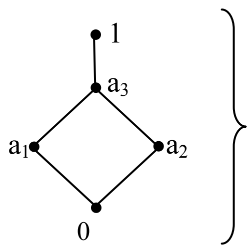
Now having seen examples of subset semivector spaces and basis associated with them now we just indicate we can also define the concept of line ar operato r. This is also a matter of routine and is left as an exercise to the reader.

Now we pr oceed of to define spe cial ty pe I subset semivector spaces.

DEFINITION 4.3: *Let S = {Collection of all subsets of a semiring R which is not a semifield}; we define S to the subset semivector space over the semiring R to be type I subset semivector space.*

We will illustrate this situation by some examples.

Example 4.26: Let S b e the collection of all subsets of a semiring R =

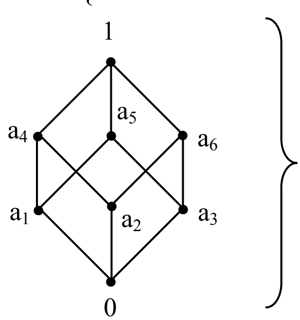


be the subset semivector space of type I over the semiring R.

S has subset se mivector subspaces. I nfact S is a subset semilinear algebra of type I over R.

B = {{1}} is a basis of S over R.

Example 4.27: Let S = {Collection of all subs ets of the semiring B =



be the subset semivector space over the semiring B.

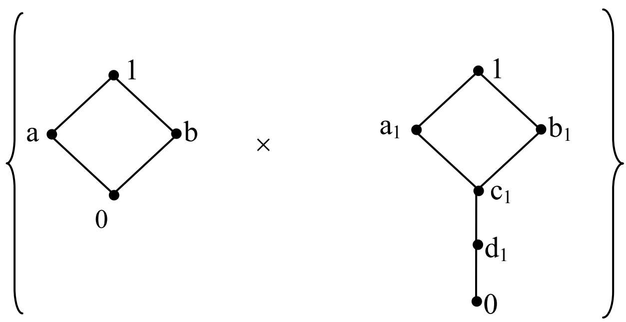
Infact S is also a subset semilinear algebra over B.

We see the basis of S over B is {1} and so dimension of S over B is {1}.

Example 4.28: Let S = {Collection of all subsets of the semiring R $= (Z^+ \times \cup \{0\}) \times (Z^+ \cup \{0\})$} be the subset semivector space over the semiring R.

Example 4.29: Let S = {Collection of all subsets of the semiring

R =



$R = L_1 \times L_2$} be the subset semivector space of type I over the semiring $L_1 \times L_2 = R$.

Clearly S contains elements of the form $A = \{(0, 0)\}$, $B = \{(0, a_1), (0, d_1), (a, b_1), (1, 1)\}$ and so on. If $(a, a_1) \in L$.

$(a, a_1) B = \{(0, a_1), (0, d_1), (a, a_1), (a, c_1)\}$.

If $C = \{(a, a_1), (b, b_1)\}$

$A + C = \{(a, a_1), (a, a_1) (a, 1) (1, 1), (b, 1), (b, b_1), (1, b_1)\} \in$ S.

This is the way operations are performed on S.

The concept of transformation, basis and finding subset semivector subspaces of type I are all a matter of routine and is left as an exercise to the reader.

It is only important to note in case one wants to define semi transformation of subset semivector spaces of type I it is essential that both the spaces must be defined on the same semiring.

Now we proceed onto define strong type II semivector spaces.

DEFINITION 4.4: *S = {Collection of all subsets of the ring R} is defined as the special subset semivector space of type II over the ring R.*

Example 4.30: Let
S = {Collection of all subsets of the ring Z_6} be the special subset semivector space of type II over Z_6.

Example 4.31: Let
S = {Collection of all the subsets of the ring $C(Z_{12})$} be the special subset semivector space of type II over the ring $C(Z_{12})$.

Example 4.32: Let
S = {Collection of all subsets of the ring R = $C(Z_6)(g)$; $g^2 = 0$} be the special subset semivector space of type II over the ring $C(Z_6)(g)$.

Example 4.33: Let
S = {Collection of all subsets of the ring R = Z_{24}} be the special subset semivector space over R of type II.

If A = {(0, 2, 17, 9, 4)} and B = {(9, 2, 20, 5, 7, 3)} are in S.

Then A + B = {9, 2, 20, 5, 7, 3, 11, 4, 22, 21, 15, 0, 18, 14, 12} $\in$ S.

Now AB = {0, 18, 4, 16, 10, 14, 6, 3, 20, 23, 13, 12, 21, 15, 8} $\in$ S.

Now if 2 $\in$ R then 2A = {0, 4, 18, 8, 14} $\in$ S.

2B = {18, 4, 10, 14, 6, 16} $\in$ S.

This is the way operations are perform ed on special subset semivector s paces of ty pe II, infact S is a spe cial subset semilinear algebra of type II.

Example 4.34: Let S = {Collection of all subsets of the ring

$$R = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in C(Z_4),\ 1 \leq i \leq 5 \right\}$$

under the natural product $\times_n$ of matrices} be the special subset semivector space of type II over the ring R.

$$\text{Let } A = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1+i_F \\ 2+3i_F \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3i_F \\ 2i_F+1 \\ 0 \end{bmatrix} \right\} \text{ and}$$

$$B = \left\{ \begin{bmatrix} 0 \\ 3+i_F \\ 0 \\ 2+i_F \\ 0 \end{bmatrix}, \begin{bmatrix} i_F \\ 2i_F \\ 0 \\ 3i_F \\ 2+i_F \end{bmatrix}, \begin{bmatrix} 3 \\ 3i_F \\ 2 \\ 2i_F \\ 1 \end{bmatrix} \right\} \in S.$$

Now

A + B =

$$\left\{\begin{bmatrix}2\\3+i_F\\1+i_F\\0\\0\end{bmatrix},\begin{bmatrix}2+i_F\\2i_F\\1+i_F\\2+2i_F\\2+i_F\end{bmatrix},\begin{bmatrix}1\\3i_F\\3+3i_F\\2+i_F\\1\end{bmatrix},\begin{bmatrix}1\\1+i_F\\3i_F\\3+3i_F\\0\end{bmatrix},\begin{bmatrix}1+i_F\\2+2i_F\\3i_F\\1+i_F\\2+i_F\end{bmatrix},\begin{bmatrix}0\\2+3i_F\\2+3i_F\\1\\1\end{bmatrix}\right\} \in S.$$

$$A\times B = \left\{\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix},\begin{bmatrix}2i_F\\0\\0\\1\\0\end{bmatrix},\begin{bmatrix}2\\0\\2+2i_F\\2\\0\end{bmatrix},\begin{bmatrix}0\\2+2i_F\\0\\i_F\\0\end{bmatrix},\begin{bmatrix}i_F\\0\\0\\3i_F+2\\0\end{bmatrix},\begin{bmatrix}3\\2i_F\\2i_F\\2i_F\\0\end{bmatrix}\right\} \in S.$$

Let $1 + 2i_F \in R$.

we find $(1 + 2i_F)\ A = \left\{\begin{bmatrix}2\\0\\3i_F+3\\3i_F\\0\end{bmatrix},\begin{bmatrix}1+2i_F\\2\\3i_F+2\\1\\0\end{bmatrix}\right\}$ is in S.

$(1 + 2i_F)\ B = \left\{\begin{bmatrix}0\\3i_F+1\\0\\i_F\\0\end{bmatrix},\begin{bmatrix}i_F+2\\2i_F\\0\\3i_F+2\\i_F\end{bmatrix}\right\}$ is in S.

This is the way operations are performed on S as a special subset semivector space of type II.

Infact S is a special subset semilinear algebra of type II over R.

Example 4.35: Let S = {Collection of all subsets of the ring

$$R = \left\{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| \; a_i \in Z_6\,(g);\, g^2 = 0,\, 1 \leq i \leq 4\right\}\}$$

be the special subset semilinear algebra of type II which is non commutative.

Further even the special subset semivector space of type II is non commutative for ra ≠ ar for all r ∈ R, a ∈ S. We can only say ar and ra ∈ S.

Thus for the first time we encounter with this type of special non commutative structure.

Infact AB ≠ BA for A, B ∈ S.

We will first show these facts.

$$\text{Let } A = \left\{\begin{pmatrix} 3 & 3 \\ 2i_F & 2+i_F \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 3i_F \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}\right\}$$

$$\text{and } B = \left\{\begin{pmatrix} 3 & 3i_F \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 4i_F & 4 \\ 0 & 2 \end{pmatrix}\right\} \in S.$$

We see

$$A + B = \left\{\begin{pmatrix} 0 & 3+3i_F \\ 2i_F & 4+i_F \end{pmatrix}, \begin{pmatrix} 5 & 4+3i_F \\ 3 & 2+3i_F \end{pmatrix},\right.$$

$$\begin{pmatrix} 3 & 3+3i_F \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 3+4i_F & 1 \\ 2i_F & 4+i_F \end{pmatrix}, \begin{pmatrix} 2+4i_F & 2 \\ 3 & 2+3i_F \end{pmatrix}, \begin{pmatrix} 4i_F & 1 \\ 4 & 2 \end{pmatrix} \Bigg\}$$

$\in$ S.

$$A + A = \left\{ \begin{pmatrix} 0 & 0 \\ 4i_F & 2+2i_F \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \right\} \text{ and}$$

$$B + B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 2i_F & 2 \\ 0 & 4 \end{pmatrix} \right\} \in S$$

Consider

$$AB = \left\{ \begin{pmatrix} 3 & 3i_F \\ 0 & 4+2i_F \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2i_F & 4 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 3 & 3i_F \end{pmatrix}, \begin{pmatrix} 2i_F & 4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

$\in$ S.

Now we find

$$BA = \Bigg\{ \begin{pmatrix} 3 & 0 \\ 4i_F & 2+2i_F \end{pmatrix}, \begin{pmatrix} 3i_F & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2i_F & 2+4i_F \\ 4i_F & 4+2i_F \end{pmatrix},$$

$$\begin{pmatrix} 2i_F & 4i_F \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix} \Bigg\} \in S.$$

Clearly $AB \neq BA$.

Now we take $x = \begin{pmatrix} 3 & 2 \\ i_F & 4i_F \end{pmatrix} \in R$ and find xA and Ax.

$$xA = \left\{ \begin{pmatrix} 3+4i_F & 1+2i_F \\ 4+3i_F & 2+5i_F \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2i_F & 4i_F \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 4i_F & 3i_F \end{pmatrix} \right\} \in S.$$

Consider

$$Ax = \left\{ \begin{pmatrix} 0 & 0 \\ 2i_F + 4 & 4 + 4i_F \end{pmatrix}, \begin{pmatrix} 1 + 3i_F & 0 \\ 2i_F + 5 & 2 \end{pmatrix}, \begin{pmatrix} 3i_F & 0 \\ 0 & 2 \end{pmatrix} \right\} \in S.$$

Clearly $Ax \neq xA$.

Thus this is a unique and an interesting feature of special subset semivector space of type II over the ring, that is if the ring R is non commutative so is S as a special subset semivector space of type II.

Also as special subset semilinear algebra of type II is doubly non commutative if the underlying ring R over which it is defined is non commutative.

This feature is very different from usual vector spaces V as vector spaces are defined over abelian group under '+' and it is always assumed for every $a \in F$, $v \in V$; $av = va$ and we only write av.

Example 4.36: Let
S = {Collection of all subsets of the ring $Z_8(g) = g^2 = 0$} be the special subset semivector space of type II over the ring Z_8. Let A = {0, 4g, 2+4g, 2g} and B = {0, 4g, 4, 4+4g} $\in$ S we see AB = (0).

Thus we see in case of special subset semilinear algebra of type II over the ring Z_8 has zero divisors.

Example 4.37: Let S = {Collection of all subsets of the ring

$$R = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \right| a_i \in Z_{10};\ 1 \leq i \leq 8 \right\}\}$$

be the special subset semivector space over the ring R of type II. R is a ring under natural product $\times_n$ of matrices.

$$\text{Let } A = \left\{ \begin{bmatrix} 0 & 5 \\ 1 & 0 \\ 2 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 8 & 1 \\ 1 & 1 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ 2 & 0 \\ 1 & 0 \\ 3 & 4 \end{bmatrix} \right\} \text{ and}$$

$$B = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 2 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 3 \\ 5 & 0 \end{bmatrix} \right\} \text{ be two elements of S.}$$

$$\text{We see } A + B = \left\{ \begin{bmatrix} 0 & 6 \\ 2 & 0 \\ 4 & 4 \\ 3 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 1 & 1 \\ 5 & 5 \\ 8 & 4 \end{bmatrix}, \begin{bmatrix} 7 & 1 \\ 9 & 0 \\ 3 & 3 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 8 & 0 \\ 8 & 2 \\ 4 & 4 \\ 5 & 7 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 5 & 3 \\ 3 & 0 \\ 3 & 2 \\ 3 & 9 \end{bmatrix}, \begin{bmatrix} 6 & 0 \\ 2 & 1 \\ 4 & 3 \\ 8 & 4 \end{bmatrix} \right\} \in S.$$

$$A \times B = \left\{ \begin{bmatrix} 0 & 5 \\ 1 & 0 \\ 4 & 4 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 6 & 6 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 8 & 0 \\ 2 & 2 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 1 \\ 3 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 3 & 0 \\ 5 & 0 \end{bmatrix} \right\} \in S.$$

We can easily verify AB = BA.

$$\text{Also if } x = \begin{bmatrix} 3 & 1 \\ 5 & 2 \\ 4 & 5 \\ 0 & 2 \end{bmatrix} \in R \text{ then}$$

$$xA = \left\{ \begin{bmatrix} 0 & 5 \\ 5 & 0 \\ 8 & 0 \\ 0 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 5 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ 0 & 0 \\ 4 & 0 \\ 0 & 8 \end{bmatrix} \right\} \in S.$$

Clearly xA = Ax.

Infact S is a special subset semilinear algebra of type II over the ring R.

We can have special subset semivector subspaces / semilinear subalgebras of type II of S over R.

This is a matter of routine.

Further we can also have the new notion of quasi subring special subset semivector spaces over subrings of R. To this end we give an example or two.

Example 4.38: Let
S = {Collection of all subsets of the ring $R = Z_{12}$} be the special subset semivector space of type II over the ring R.

Let
T = {Collection of all subsets of the ring $\{0, 2, 4, 6, 8, 10\} \subseteq R$} be a quasi subring special subset semivector space over the subring $\{0, 3, 6, 9\} \subseteq R = Z_{12}$.

Example 4.39: Let
S = {Collection of all sub sets of the ring R = {Z_6 [x]}} be a special subset semivector space of type II over the ring R.

Now we have P = {Collection of subsets of S such that every element in every subset is of even degree or the constant term} $\subseteq$ S is a quasi set special subset semivector subspace of S of type II over the subring Z_6.

Example 4.40: Let
S = {Collection of all subsets of the ring R = ($Z_6 \times Z_{12}$)} be the special subset semivector space of type II over the ring R.

M = {Collection of all sub sets of the ring R_1 ({0, 3} $\times Z_{12}$)}$\subseteq$S; is a special subset semivector subspace of type II over the subring {0} $\times$ {0, 6}.

Example 4.41: Let
S = {Collection of all subsets of the ring Z[x]} be the special subset semivector space of type II over the ring Z[x].

Consider P = {Collection of all subsets of S in which every polynomial is of even degree or constant polynomial} $\subseteq$ S, P is a quasi subring strong subset semivector space over the subring Z of S.

Now having seen the properties of quasi subring special subset semivector subspace now we proceed onto define special strong subset semivector space / semilinear algebra of type III over a field.

We give examples of them.

Example 4.42: Let
S = {Collection of all subsets of the field Z_{11}} be the special strong semivector space of type III over the field Z_{11}.

Example 4.43: Let
S = {Collection of all subsets of the field Z_2} = {{0}, {0, 1}, {1}} be the special strong subset semivector space / semilinear algebra of type III over Z_2.

Example 4.44: Let
S = {Collection of all subsets of the field Z_7} be the special strong subset semivector space of type III over the field Z_7.

Example 4.45: Let
S = {Collection of all subsets of the field Z_{13}} be the special strong subset semivector space of type III over the field Z_{13}.

Now we can go from one type to another type and so on.

Example 4.46: Let
S = {Collection of all subsets of the ring Z_{12}} be the special subset semivector space of type II over the ring Z_{12}. We see S has a special strong subset semivector subspace of type III over the semifield $\{0, 4, 8\} \subseteq Z_{12}$.

We call such special subset semivector spaces as Smarandache special subset semivector spaces of type III.

Example 4.47: Let
S = {Collection of all subsets of the ring $R = Z_5 \times Z_{12}$} be the special subset semivector space of type II over the ring R.

S is a S-special strong subset semivector subspace of type III over the field $F = \{Z_5 \times \{0\}\} \subseteq R$.

Example 4.48: Let S = {Collection of all subsets of the ring

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in Z_{19} \right\}$$

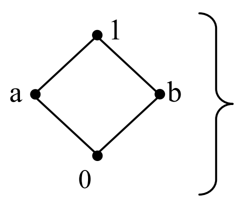
be the special subset semivector space of type II over the ring R. S is a S-strong special subset semivector space of type III as S contains a strong special subset semivector subspace over the field

$$F = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| \; a \in Z_{19} \right\}.$$

Example 4.49: Let S = {Collection of subsets of the ring Z_{48}} be the special subset semivector space of type II over the ring Z_{48}.

S is a S-special subset semivector subspace of type III over the field $\{0, 16, 32\} \subseteq Z_{48}$.

Example 4.50: Let S = {Collection of all subsets of the semiring



be the subset semivector space over the semifield {0, a}.

Clearly this S cannot have subset semivector sub spaces which can be special subset semivector spaces of type II or strong special subset semivector spaces of type III.

Thus it is not possible to relate them. This is true in case of subset semivector spaces of type I also.

Example 4.51: Let
S = {Collection of all subsets of the semifield $Q^+ \cup \{0\}$} be a subset semivector space over semifield $Q^+ \cup \{0\}$.

This cannot be shifted to any of the three types of subsemivector spaces.

We can have subspaces of them.

Further concept of linear transformation of these subset semivector spaces over rings or fields or semifields or semirings; or is used in the mutually exclusive sense.

Example 4.52: Let S = {Collection of all subsets of the semigroup semiring F = $Z^+S(3) \cup \{0\}$} be the subset semivector space over the semifield $Z^+ \cup \{0\}$. We see S is also a subset semilinear algebra which is clearly non commutative.

For take A = $\{5 + 8p_1, 3+9p_4, p_1 + p_2 + p_3 + p_4\}$ and B = $\{p_1, p_2, p_3\} \in$ S.

Clearly AB ≠ BA. For AB = $\{5p_1 + 8, 3p_1 + 9p_3, 1 + p_5 + p_4 + p_3, 5p_2 + p_5, 3p_2 + 9p_1, 1 + p_4 + p_5 + p_1, 5p_3 + p_4, 3p_3 + 9p_2, 1 + p_4 + p_5 + p_2\}$.

It can be easily verified AB ≠ BA.

It is pertinent to keep on record that if we use semigroup semirings or group semirings using non commutative semigroups and groups we get non commutative subset semilinear algebras.

We will give one or two examples of them before we proceed onto define the notion of set subset semivector subspaces and topologies on them.

Example 4.53: Let S = {Collection of all subsets of the group semiring $Z^+S_{12} \cup \{0\}$} be the subset semivector space over the field $Z^+ \cup \{0\}$. Clearly S is a non commutative subset semilinear algebra over $Z^+ \cup \{0\}$.

Example 4.54: Let S = {Collection of all subsets of the group semiring R = $Z^+ D_{2.10} \cup \{0\}$} be a subset semivector space over the semiring R of type I.

Clearly xs ≠ sx for s ∈ S and x ∈ R. Infact S is a doubly non commutative subset semilinear algebra over the semiring R.

Example 4.55: Let
S = {Collection of all subsets of the field Z_{43}} be the special strong subset semivector space over the field Z_{43} of type III.

S has P = {{ a} | a ∈ Z_{43}} be a subset vector space over the field Z_{43} and we call such strong special semivector spaces of type III which has subset vector spaces as a super strong Smarandache semivector spaces of type III.

As in case of usual vector spaces we can in case of semivector spaces and subset semivector spaces of all types define topology. We call the topologies over subspaces of a semivector space as topological semivector spaces.

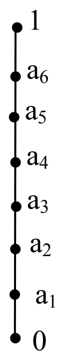
We shall first describe them with examples.

Example 4.56: Let S = {Collection of all row vectors (a_1, a_2, a_3) | $a_i \in Z^+ \cup \{0\}$, $1 \leq i \leq 3$} be a semivector space over the semifield $Z^+ \cup \{0\}$.

If T = {Collection of all semivector subspaces of S}; T is a topological space with usual '∩' and '∪'; that is for A, B ∈ T; A ∩ B ∈ T and A ∪ B is the smallest semivector subspace containing A and B of S over $Z^+ \cup \{0\}$; this topological space T is defined as the topological semivector subspace of a semivector space.

Example 4.57: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_4 \\ a_2 & a_5 \\ a_3 & a_6 \end{bmatrix} \middle| \; a_i \in L \text{ where L is a chain lattice} \right.$$



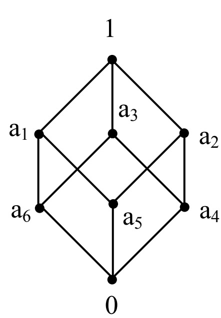
$1 \leq i \leq 6\}$ be the semivector space over the semifield L.

Clearly if T = {Collection of all semivector subspaces of S over the semifield L}, then T is a topological semivector subspace of S over L.

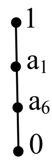
Example 4.58: Let $S = Z^+ [x] \cup \{0\}$ be a semivector space over the semifield $F = Z^+ \cup \{0\}$.

T = {Collection of all semivector subspaces of S over F}. T is a topological semivector subspace of dimension infinity.

Example 4.59: Let $S = \left\{ \begin{pmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{pmatrix} \middle| \; d_i \in L = \right.$



; $1 \leq i \leq 6\}$ be a semivector space over the semifield

F = 

T = {Collection of all semivector subspaces of S over F}; T is a topological semivector subspace of S.

It is pertinent to keep on record that for a given semivector space S over a semifield F we can have one and only one topological semivector subspace of S over F.

To overcome this we define the notion of set semivector subspaces of a semivector space defined over a semifield.

DEFINITION 4.5: *Let S be a semivector space over a semifield F. Let P $\subseteq$ S be a proper subset of S and K $\subseteq$ F be a subset of F. If for all p $\in$ P and k $\in$ K, pk and kp $\in$ P then we define P to be a quasi set semivector subspace of V defined over the subset K of F.*

We will first illustrate this situation by some examples.

Example 4.60: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \, a_i \in Z^+ \cup \{0\};\ 1 \leq i \leq 4 \right\}$$

be a semivector space over the semifield F = $Z^+ \cup \{0\}$.

Let K = $\{3Z^+ \cup 5Z^+ \cup \{0\} \subseteq$ F be a subset of F.

Consider

$$P = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \, a, b \in Z^+ \cup \{0\} \right\} \subseteq S.$$

P is a quasi set semivector subspace of S over the subset K.

Example 4.61: Let
$S = \{(3Z^+ \cup \{0\} \times 5Z^+ \cup \{0\} \times 7Z^+ \cup \{0\} \times 11Z^+ \cup \{0\})\}$ be a semivector space over the semifield $F = Z^+ \cup \{0\}$.

Consider $P = \{(3Z^+ \cup \{0\} \times \{0\} \times \{0\} \times \{0\}), (\{0\} \times \{0\} \times \{7Z^+ \cup \{0\}\} \times \{11Z^+ \cup \{0\}\}) \subseteq S$ and $K = \{5Z^+ \cup 7Z^+ \cup 6Z^+ \cup \{0\}\} \subseteq F$.

Clearly P is a quasi set semivector subspace of S over the subset K of F.

Example 4.62: Let

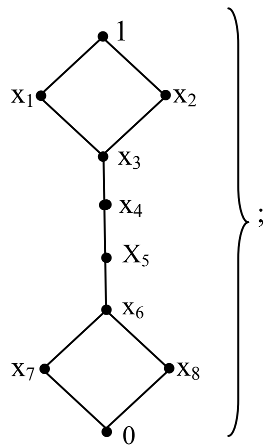
$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \, a_i \in Z^+ \cup \{0\}; 1 \leq i \leq 8 \right\}$$

be a semivector space over the semifield $F = Z^+ \cup \{0\}$. Take

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ 0 & 0 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ a_1 & a_2 \\ a_3 & a_4 \\ 0 & 0 \end{bmatrix} \middle| \, a_i \in Z^+ \cup \{0\}; 1 \leq i \leq 4 \right\} \subseteq S$$

P is a quasi set semivector subspace of S over the subset $K = \{3Z^+ \cup 7Z^+ \cup \{0\}\} \subseteq F$.

Example 4.63: Let S = $(a_1, a_2, a_3, a_4, a_5) \mid a_i \in L =$



S is a semivector space over the semifield $F = \{1, x_1, x_3, x_4, x_5, x_6, x_7, 0\}$.

Take $P = \{(a_1, 0, a_2, 0, 0), (0, 0, a_5, a_3, a_4) \mid a_i \in L; 1 \leq i \leq 5\} \subseteq S$ and $K = \{1, x, x_2, 0\} \subseteq F$.

Clearly P is a quasi set semivector subspace of S over the subset $K \subseteq F$.

Example 4.64: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \right.$$ be a semivector space over the semifield

$F = Q^+ \cup \{0\}$ (that is $a_i \in Q^+ \cup \{0\}$); $1 \leq i \leq 8\}$.

Take P = $\left\{\begin{bmatrix} a_1 & 0 \\ 0 & a_3 \\ a_2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ a_1 & 0 \\ 0 & a_2 \\ 0 & a_3 \end{bmatrix} \middle|\; a_i \in Q^+ \cup \{0\};\ 1 \leq i \leq 3\right\} \subseteq S$

and $K = \{3Z^+ \cup \{0\} \cup 16Z^+\} \subseteq F$.

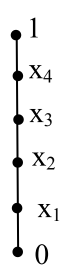
Both P and K are just subsets.

We see P is a quasi set semivector subspace of S over the set K.

We can have many such quasi set semivector spaces.

For the same subset we can have several subsets in S which are quasi set semivector subspaces of S over the same set K.

Example 4.65: Let $S = \{(a_1, a_2) \mid a_i \in L =$



$; 1 \leq i \leq 2\}$ be a semivector space over the semifield L.

Let $P_1 = \{(0, a_1) \mid a_i \in L\} \subseteq S$ is a quasi set semivector subspace of S over the set $K = \{0, x_3, x_1\} \subseteq L$. $P_2 = \{(0, 0)\}$ is a trivial quasi set semivector subspace of S over the set $K = \{0, x_3, x_1\} \subseteq L$.

$P_3 = \{(1, 0), (x_1, 0), (x_3, 0)\} \subseteq S$ $P_4 = \{(0, 1), (0, x_1), (0, x_3)\} \subseteq S$
$P_5 = \{(x_1, 0)\}$, $P_6 = \{(0, x_1)\} \subseteq S_1$, $P_7 = \{(0, x_1), (0, x_3)\}$,

$P_8 = \{(x_1, 0), (0, x_1)\}$, $P_9 = \{(x_3, 0), (x_1, 0)\}$,

$P_{10} = \{(0, x_1), (0, x_3), (x, 10)\}$, $P_{11} = \{(0, x_1)\ (x_3, 0), (x_1, 0)\}$ and so on are all quasi set semivector subspace of S over the set K.

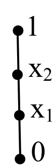
We can get several such quasi set semivector sub spaces depending on the subsets of the semifield.

Now we can using the collection of all quasi set semivector subspaces T of S over a subset K of the semifield F; that is T = {Collection of all quasi set semivector subspaces of S over the set K $\subseteq$ F}; can be given a topology and this topological space will be defined as the quasi set topological semivector subspace of S over K.

For a given subset in the semifield we can have a quasi set topological semivector subspace of S; thus we can have several such quasi set topological semivector subspaces of S over the appropriate subsets K of F.

Thus by this method of constructing quasi set semivector subspaces we can get several quasi set topological semivector subspaces as against only one topological semivector subspace.

Example 4.66: Let $S = \{(a_1, a_2, a_3) \mid a_i \in L =$



$1 \leq i \leq 3\}$ be the semivector space over the semifield L.

Take K = {0, 1} a subset of L. $P_1 = \{(0, 0, 0)\}$, $P_2 = \{(0, 0, 0), (x_1, 0, 0)\}$ $P_3 = \{(0, 0, 0), (0, x_1, 0)\}$, $P_4 = \{(0, 0, 0), (0, 0, x_1)\}$, $P_5 = \{(0, 0, 0), (x_1, x_1, 0)\}$, $P_6 = \{(0, 0, 0), (x_1, 0, x_1)\}$, $P_7 = \{(0, 0, 0), (0, x_1, x_1)\}$, $P_8 = \{(0, 0, 0), (x_1, x_1, x_2)\}$, $P_9 = \{(0, 0, 0), (x_1, x_2, 0)\}$, $P_{10} = \{(0, 0, 0), (x_1, 0, x_2)\}$, $P_{11} = \{(0, 0, 0), (0,$

$x_1, x_2)\}$, $P_{12} = \{(0, 0, 0), (x_2, x_1, 0)\}$, $P_{13} = \{(0, 0, 0), (x_2, 0, x_1)\}$, $P_{14} = \{(0, 0, 0), (0, x_2, x_1)\}$, $P_{15} = \{(0, 0, 0), (x_1, x_1, x_2)\}$, $P_{16} = \{(0, 0, 0), (x_1, x_2, x_1)\}$, $P_{17} = \{(0, 0, 0), (x_2, x_1, x_1)\}$, $P_{18} = \{(x_2, x_2, x_1), (0, 0, 0)\}$, $P_{19} = \{(x_2, x_1, x_2), (0, 0, 0)\}$, $P_{20} = \{(x_1, x_2, x_2), (0, 0, 0)\}$, $P_{21} = \{(x_2, 0, 0), (0, 0, 0)\}$, $P_{22} = \{(0, x_2, 0), (0, 0, 0)\}$, $P_{23} = \{(0, 0, x_2), (0, 0, 0)\}$, $P_{24} = \{(0, 0, 0), (x_2, x_2, 0)\}$, $P_{25} = \{(0, 0, 0), (x_2, 0, x_2)\}$, $P_{26} = \{(0, 0, 0), (0, x_2, x_2)\}$, $P_{27} = \{(0, 0, 0), (x_2, x_2, x_2)\}$, $P_{28} = \{(0, 0, 0), (1, 0, 0)\}$ and so on.

Using these $P_1, P_2, \ldots, P_{26} \ldots$ as atoms we can generate a Boolean algebra of finite order where $\{(0, 0, 0)\}$ is the least element and S is the largest element. These elements form a quasi set topological semivector subspace of S over the set $K = \{0, 1\}$.

Take $K_1 = \{0, x_1\} \subseteq L$ is a subset of L. The quasi set semivector subspaces of S over K_1 are as follows:

$P_1 = \{(0, 0, 0)\}$, $P_2 = \{(0, 0, 0), (0, 0, x_1)\}$ $P_3 = \{(0, 0, 0), (0, x_1, 0)\}$, $P_4 = \{(0, 0, 0), (x_1, 0, 0)\}$ $P_5 = \{(0, 0, 0), (x_1, x_1, x_1)\}$, $P_6 = \{(0, 0, 0), (1, 0, 0), (x_1, 0, 0)\}$, $P_7 = \{(0, 0, 0), (0, 1, 0), (0, x_1, 0)\}$, $P_8 = \{(0, 0, 0), (0, 0, 1), (0, 0, x_1)\}$, $P_9 = \{(0, 0, 0), (x_1, x_1, 0)\}$, $P_{10} = \{(0, 0, 0), (0, x_1, x_1)\}$, $P_{11} = \{(0, 0, 0), (x_1, 0, x_1)\}$, $P_{12} = \{(0, 0, 0), (1, 1, 0), (x_1, x_1, 0)\}$, $P_{13} = \{(0, 0, 0), (0, 1, 1)$ $(0, x_1, x_1)\}$, $P_{14} = \{(0, 0, 0), (1, 0, 1)$ $(x_1, 0, x_1)\}$, $P_{15} = \{(0, 0, 0), (1, 1, 1)$ $(x_1, x_1, x_1)\}$, $P_{16} = \{(0, 0, 0), (0, x_1, 0)$ $(x_1, x_1, 0)\}$, $P_{18} = \{(0, 0, 0), (0, x_1, 1)$ $(0, x_1, x_1)\}$ and so on.

This collection of a quasi set topological semivector subspace over the set $K_1 = \{0, x_1\} \subseteq L$ is distinctly different from the quasi set topological semivector subspace over the set $K = \{0, 1\} \subseteq L$.

This is the way quasi set topological semivector subspaces are defined over subsets of L.

We see with each of these quasi set topological semivector subspaces we can define an associated lattice of the topological space which may be finite or infinite.

Now having seen examples of quasi set semivector subspaces of a semivector space we now proceed onto give examples of quasi set subset semivector subspaces of a subset semivector space.

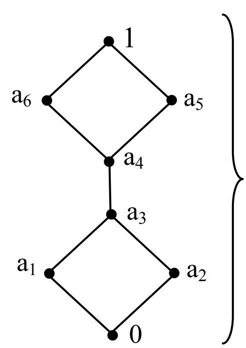
Example 4.67: Let S be the collection of all subsets of the semiring $Z^+[x] \cup \{0\}$ over the semifield $F = Z^+ \cup \{0\}$.

We see if we take $K_1 = \{0, 1\}$ as a subset in F. T = {Collection of all quasi set subset semivector subspaces of S over the set K_1}, T is the quasi set subset topological semivector subspace of S over K_1.

Example 4.68: Let S = {Collection of all subsets of the semiring $Z^+ \cup \{0\} \times Z^+ \cup \{0\} \times Z^+ \cup \{0\}$} be the subset semivector space over the semifield $Z^+ \cup \{0\}$.

Let $K = \{0, 1\} \subseteq Z^+ \cup \{0\}$; T = {Collection of all quasi set subset semivector subspaces of S over the set K} is a quasi set topological semivector subspace of S over K.

Example 4.69: Let S = {Collection of all subsets of the semiring



be the subset semivector space over the semifield $F = \{0, a_1, a_3, a_4, a_6, 1\}$. Take $K = \{0, 1\} \subseteq F$; T = {Collection of all set quasi subset semivector subspaces of S over K} be the set quasi subset topological semivector subspace of S over K.

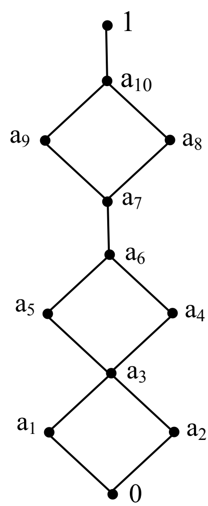
Take $K_1 = \{0, a_1\} \subseteq F$; T_1 = {Collection of all set quasi subset semivector subspaces of S over K_1} is the quasi set subset topological semivector subspace of S over K_1.

We can find several subsets of F and find the related quasi set subset topological semivector subspaces of S over the subsets of F.

Now we have seen examples of quasi set subset topological semivector subspaces over subsets of a semifield.

We now onto study the same concept in case of (subset) semivector space over the semiring by examples.

Example 4.70: Let $S = \{(x_1, x_2, x_3) \mid x_i \in L =$



$1 \leq i \leq 3\}$ be the semivector space of type I over the semiring L.

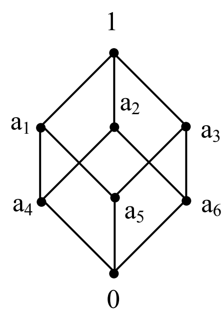
Take $K_1 = \{a_1, a_2\} \subseteq L$.

Let T_1 = {Collection of all quasi set semivector subspaces of S over K_1} be the quasi set topological semivector subspace of type I of S over K_1.

Take $K_2 = \{0, 1, a_6\} \subseteq L$. Let T_2 = {Collection of all quasi set semivector subspaces of S over the set K_2} be the quasi set topological semivector subspace of type I of S over $K_2 \subseteq L$.

We can have several such quasi set topological semivector subspaces of type I over subsets of L over which the semivector space of type I is defined.

Example 4.71: Let S = {L[x] | L =



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be the polynomial semiring. S is the semivector space of type I over the semiring L. Take $K_1 = \{0, a_4, a_6\} \subseteq L$. Let T_1 = {Collection of all quasi set semivector subspace of type I over the set K_1} to be the quasi set topological semivector subspace of S over the set K_1.

Take $K_2 = \{a_1, a_2, a_3\} \subseteq L$, T_2 = {Collection of all quasi set semivector subspaces of S over K_2} be the quasi set topological semivector subspaces of type I of S over K_2.

We can have several such quasi set topological semivector subspaces of type I.

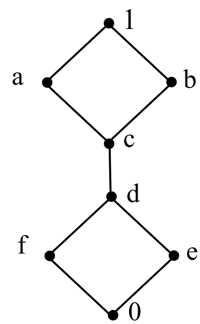
Example 4.72: Let S = $\{Q^+ (g_1, g_2, g_3) \cup \{0\} \mid g_1^2 = 0,\ g_2^2 = 0$ and $g_3^2 = g_3,\ g_1g_2 = g_2g_1 = g_1g_3 = g_3g_1 = g_2g_3 = g_3g_2 = 0\}$ be a semivector space over a semiring $Z^+ (g_1) \cup \{0\}$ of type I.

Take $K_1 = \{0, 1, g_1\} \subseteq Z^+ (g_1) \cup \{0\}$. Let T_1 = {Collection of all quasi set semivector subspaces of S over the set K_1} be the quasi set semivector subspace of S of type I over the set K_1. By taking different subsets in $Z^+(g_1) \cup \{0\}$, we get the corresponding quasi set topological semivector subspace of S over K_1.

Example 4.73: Let S = $\{L (g_1, g_2) \mid g_1^2 = 0,\ g_2^2 = 0,\ g_1g_2 = g_2g_1 = 0$ and L is the Boolean algebra B of order 2^4 with a_1, a_2, a_3 and a_4 as atoms$\}$ be the semivector space over the semiring B of type I.

Let $K_1 = \{0, a_1, a_2, 1\} \subseteq L$. T_1 = {Collection of all quasi set semivector subspaces of S over the subset K_1} be the quasi set topological vector subspace of type I of S over the subset K_1.

Example 4.74: Let S = {Collection of all subsets of the semiring



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be the subset semivector space over the semiring L of type I.

Let K = $\{0, f, a, 1\} \subseteq L$.

If

T = {Collection of all quasi set semivector subspaces of S over K}; then T is defined as the quasi set topological semivector subspace of S over K of type I.

Now having seen examples of quasi set topological semivector subspaces over semiring of type I we now proceed onto describe with examples quasi set special topological semivector subspaces of type II defined over a ring R.

Example 4.75: Let

S = {Collection of all subsets of a ring R = Z_{12}} be the special semivector space of type II over the ring Z_{12}.

Let P = {0, 3, 2} $\subseteq Z_{12}$ be a subset of Z_{12} = R.

T = {Collection of all quasi set subset semivector subspaces of S over P}

= ⟨{{0}, {0, 1, 3, 2, 4, 6, 8, 9}, {0, 5, 3, 10, 8, 6, 4, 9}, {0, 7, 2, 9, 4, 6, 8, 3}, {0, 11, 10, 8, 4, 9, 6}}⟩, generates a quasi set subset topological semivector subspace of type II over a ring R = Z_{12}.

Now take K_1 = {0, 1} $\subseteq$ R. T = {Collection of all quasi set semivector subspaces of type II over the set K_1} = {{0}, {0, 2}, {0, 1}, {0, 3}, {0, 4}, {0, 5}, {0, 6}, …, {0, 11}, {0, 2, 3}, {0, 2, 4}, …, {0, 10, 11}, {0, 2, 3, 4}, …, {{0, 9, 10, 11}, … S} is the quasi set topological semivector subspace of type II over the set {0, 1} $\subseteq$ R.

Example 4.76: Let S = {Collection of all subsets of the ring Z} be the special semivector space of type II over the ring Z.

Take K = {0, 1, –1} $\subseteq$ Z. T = {Collection of all quasi set subset semivector subspaces of S over K} is the quasi set topological semivector subspace of S over the set K of type II.

T_1 = {{0}, {0, –1, 1}, {0, 2, –2}, {0, 3, –3}, …, {0, n, –n}, …, {0, 2, –1, 1, –2}, …, {0, n, –n, 1, –1}, …, {0, 2, –2, 3, –3}, …, {0, n, –n, m, –m}, …, {0, n, –n, m, –m, r, –r}, …, S}.

Suppose K_1 = {0, 1}, we get another quasi set subset topological semivector subspace of type II (say T_2) different from using K. $T_1 \neq T_2$.

However if K_3 = {0, –1} and if T_3 is the space associated with it we see $T_1 = T_3$ however $T_3 \neq T_2$.

Thus we see at times even if the subset over which the topologies are defined are distinct still the topological spaces are the same.

Now having seen subset quasi set topological semivector subspaces of type II we now proceed onto describe with examples the notion of quasi set topological semivector subspaces of type III over subsets of a field.

Example 4.77: Let S = {Collection of subsets of a field Z_7} be the special strong subset semivector space over the field Z_7 of type III.

Let K = {0, 1} $\subseteq Z_7$ be a subset of the field.
T = {Collection of all quasi set subset semivector subspaces of S over the set K of type III}. {{0}, {0, 1}, {0, 2}, …, {0, 6}, {0, 1, 2}, { 0, 1, 3}, …, {0 , 6, 5}, {0, 1, 2, 3}, {0, 1 , 2, 4}, …, {0, 4, 5, 6}, {0, 1, 2, 3, 4}, …, {0, 3, 4, 5, 6}, {0, 1, 2, 3, 4, 5}, …, {0, 2, 3, 4, 5, 6}, {0, 1, 2, …, 6}} is the quasi set subset topological semivector subspace of S over the set K of type III.

If we change K say by K_1 = {0, 2} and the associated space be T_1 then T_1 is different from T.

Example 4.78: Let
S = {Collection of all subsets of the ring Z_3S_3} be the special strong quasi set subset semivector space over the field Z_3.

We can have only three strong special quasi set subset topological semivector subspaces over the sets {0, 1}, {0, 2} or {2, 1} of Z_3.

Now having seen examples of special strong quasi set topological semivector subspaces of type III over subsets of a field of finite characteristic, we now proceed onto give examples of such spaces over infinite fields and finite complex modulo integer fields.

Example 4.79: Let
S = {Collection of all subsets of the ring $C(Z_{13})$} be the special strong subset semivector space over the field Z_{13} of type III.

Let $K_1 = \{0, 1\} \subseteq C(Z_{13})$. T_1 = {Collection of all quasi set subset semivector subspaces of S over the set K_1 of type III} is the quasi set subset topological semivector subspace of S of type III over K_1.

Consider $K_2 = \{0, 1, 2\} \subseteq C(Z_{13})$. T_2 = {Collection of all quasi set subset semivector subspaces of S over the set K_2 of type III} is the quasi set subset topological semivector subspace of S over K_2 of type III and so on.

Example 4.80: Let S = {Collection of all subset of the complex modulo integer $R = \{(a_1, a_2, a_3, a_4) \mid a_i \in C(Z_{19}), 1 \leq i \leq 4\}$} be the strong special subset semivector space of S over the field Z_{19} of type III.

Take $K_1 = \{0, 1\} \subseteq Z_{19}$, T_1 = {Collection of all quasi set subset semivector subspaces of type III over the set K_1 of type III} is the strong special quasi set subset topological semivector subspace of type III over K_1.

Let $K_2 = \{0, 1, 7\} \subseteq Z_{19}$ be a subset of Z_{19}. T_2 = {Collection of all special strong quasi set subset semivector subspaces of S over the set K_2 of type III} is the strong special subset quasi set topological semivector subspace of type III over the subset $K_2 \subseteq Z_{19}$.

Clearly $T_1 \neq T_2$.

Example 4.81: Let S = {Collection of all subsets of Q} be the special strong semivector space of type III over the field Q.

If A = {0, 7, –8, 5} and B = {6, –2, –10, 3/8, 4} are in S. Then A + B = {6, –2, –10, 3/8, 4, 13, 5, –3, 59/8, 11, –2, –10, –18, –7 5/8, 11, 3, –5, 43/8, 9} and AB = {0, 42, – 14, –70, 21/8, 28, –48, 16, 80, –3, –32, 30, –10, –50, 15/8, 20} $\in$ S. This is the way operations on S are performed.

If 7 $\in$ Q. 7A = {0, 49, –56, 35} $\in$ S. Now we take K_1 = {0, 1} $\subseteq$ Q to be a subset let T_1 = {Collection of all strong special subset quasi set semivector subspaces of S over K_1} be the strong special subset quasi set topological semivector subspace of S over $K_1 \subseteq$ Q. Take K_2 = {0, 1, –1} we get again a topological space T_2.

We see of course all the while we had taken for all these topological spaces only the operations as $\cup$ and $\cap$.

Now we can also change these operations in case of all these three types of spaces as well as the semivector spaces.

We will give some more examples of semivector spaces defined over infinite fields.

Example 4.82: Let
S = {Collection of all subsets from the complex field C} be the strong special semivector space defined over the field C of type III.

K_1 = {0, 1} gives a special strong subset quasi set topological semivector subspace of type III over K_1 say T_1.

K_2 = {1, 0, –1} gives a space T_2, K_3 = {0, i} gives a space T_3 and K_4 = {0, I, 1, –1} gives a space T_4 all of them are of infinite dimension.

But if we take $K_5 = \{0, 1, 4\}$ clearly this makes every element in the topological space T_5 to be of infinite cardinality. All finite sets get filtered and do not find a place in this space T_5, but find a place in T_1, T_2, T_3 and T_4.

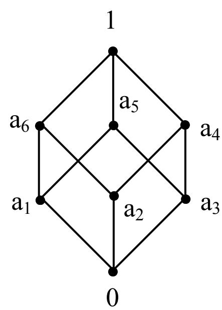
We now proceed onto define another type of operation. Suppose T = {Collection of all quasi set subset semivector subspaces defined over a set in a semifield}.

Suppose A, B $\in$ T in all the earlier topologies we took $A \cup B$ and $A \cap B \in T$.

Now we are going to define a new operation in T and T with new operation will be known as the new topological space denoted by T_N and for A, B $\in T_N$ we define A + B and AB if both A + B and AB $\in T_N = T$ then alone we define T_N to be a new quasi set topological semivector subspace and that both A + B and AB must continue to be quasi set subset semivector subspaces over the subset using which T_N is defined.

We will illustrate this by some simple examples.

Example 4.83: Let S = {Collection of all subsets of the semiring



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be the subset semivector space over the semifield $F = \{0, a_1, a_6, 1\}$. Take $K_1 = \{0, 1\} \subseteq F$.

T_1 = {Collection of all quasi set subset semivector subspaces of S} over the set K_1. Let A = {0, a_1, a_2, a_4} and B = {1, a_6, a_5, a_4} $\in T_1$.

A $\cup$ B = {0, a_1, a_2, a_6, a_4, a_5} and A $\cap$ B = {a_4}.

However A + B = {0, a_6, a_5, a_4, 1}.

Clearly A $\cup$ B $\neq$ A + B.

Further AB = {0, a_1, a_2, a_4}.

Also AB $\neq$ A $\cap$ B.

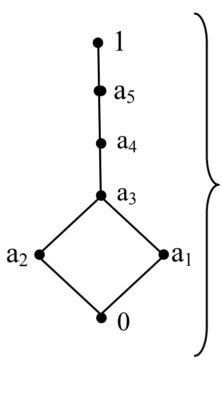
We see AB and A + B are quasi set subset semivector subspaces of S over the set K_1.

So for A, B $\in T_N$, A+B, AB $\in T_N$.

In this case we have T_N to be the new quasi set topological semivector subspace of S over the set K = {0, 1}.

It is left as an open problem that when will T_N exists given T; a quasi set subset topological semivector subspace over K.

Example 4.84: Let S = {Collection of all subsets of the semiring



be the subset semivector space over the semifield F = {0, a_1, a_3, a_5, 1}.

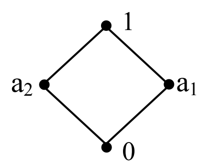
Take $K_1 = \{0, 1\} \subseteq F$.

Let T = {Collection of all quasi set semivector subspaces of S over the set K_1} be the quasi set subset topological semivector subspace of S over K_1.

$T_N = T$ is also a new quasi set subset topological semivector subspace of S over K_1.

Inview of this we are always guaranteed of such new structures that is we can give two topologies on T one is a structure dependent topology where as the other is independent of the structure. The new topology is structure dependent.

Example 4.85: Let S = {Collection of all subsets of the semiring



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be the subset semivector space over the semifield F = {0, a, 1}.

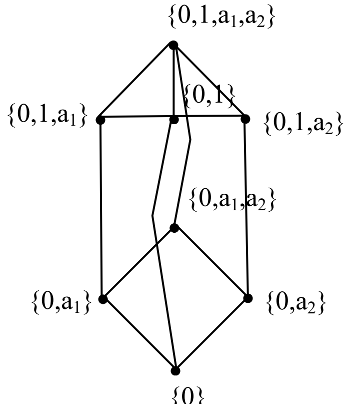
Take K = {0, 1} and T = {Collection of all quasi set semivector subspaces of S over the set K} is the quasi set topological semivector subspaces of S over K.

$T_N = \{\{0\}, \{0, 1\}, \{0, a_1\}, \{0, a_2\}, \{0, 1, a_1\}, \{0, 1, a_2\}, \{0, a_1, a_2\}, \{1, a_1, a_2, 0\}$.

Now $T_N = T$.

But $\{0\} \cup \{0, 1\} = \{0, 1\}, \{0\} \cap \{0, 1\} = \{0\}$.
$\{0, a_1\} \cup \{0, a_2\} = \{0, a_1, a_2, 1\}$, $\{0, a_1\} \times \{0, a_2\} = \{0\}$.

The lattice associated with T is as follows:



We cannot give lattice structure to this but we give the table of $\cup_N$ for T_N.

$\cup_N$	$\{0\}$	$\{0,1\}$	$\{0,a_1\}$	$\{0,a_2\}$	$\{0,a_1,a_2\}$	$\{0,1,a_1\}$	$\{0,a_2,1\}$	$\{0,a_1,a_2,1\}$
$\{0\}$	$\{0\}$	$\{0,1\}$	$\{0,a_1\}$	$\{0,a_2\}$	$\{0,a_1,a_2\}$	$\{0,1,a_1\}$	$\{0,a_2,1\}$	$\{0,a_1,a_2,1\}$
$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1,a_1\}$	$\{0,a_2,1\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1\}$	$\{0,a_2,1\}$	$\{0,1,a_1,a_2\}$
$\{0,a_1\}$	$\{0,a_1\}$	$\{0,a_1,1\}$	$\{0,a_1\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1\}$	$\{0,a_1,a_2,1\}$	$\{0,1,a_1,a_2\}$
$\{0,a_2\}$	$\{0,a_2\}$	$\{0,1,a_2\}$	$\{0,a_2,a_1,1\}$	$\{0,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_2\}$	$\{0,1,a_1,a_2\}$
$\{0,a_1,a_2\}$	$\{0,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,a_1,a_2,1\}$	$\{0,1,a_1,a_2\}$	$\{0,a_1,a_2,1\}$	$\{0,a_2,a_1,1\}$	$\{0,1,a_1,a_2\}$
$\{0,1,a_1\}$	$\{0,1,a_1\}$	$\{0,1,a_1\}$	$\{0,1,a_1\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$
$\{0,1,a_2\}$	$\{0,1,a_2\}$	$\{0,a_2,1\}$	$\{a_1,1,0,a_2\}$	$\{0,1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_2\}$	$\{0,1,a_1,a_2\}$
$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{1,0,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$

The table $\cap_N$ on T_N which is element wise is as follows:

$\cap_N$	{0}	{0,1}	{0,a_1}	{0,a_2}	{0,a_1,a_2}	{0,1,a_1}	{0,a_2,1}	{0,1,a_1,a_2}
{0}	{0}	{0}	{0}	{0}	{0}	{0}	{0}	{0}
{0,1}	{0}	{0,1}	{0,a_1}	{0,a_2}	{0,a_1,a_2}	{0,1,a_1}	{0,1,a_2}	{0,1,a_1,a_2}
{0,a_1}	{0}	{0,a_1}	{0,a_1}	{0}	{0,a_1}	{0,a_1}	{0,a_1}	{0,a_1}
{0,a_2}	{0}	{0,a_2}	{0}	{0,a_2}	{0,a_2}	{0,a_2}	{0,a_2}	{0,a_2}
{0,a_1,a_2}	{0}	{0,a_1,a_2}	{0,a_1}	{0,a_2}	{0,a_1,a_2}	{0,a_1,a_2}	{0,a_1,a_2}	{0,a_1,a_2}
{0,1,a_1}	{0}	{0,1,a_1}	{0,a_1}	{0,a_2}	{0,a_1,a_2}	{0,1,a_1}	{0,1,a_1,a_2}	{0,1,a_1,a_2}
{0,1,a_2}	{0}	{0,1,a_2}	{0,a_1}	{0,a_2}	{0,a_1,a_2}	{0,1,a_1,a_2}	{0,1,a_2}	{0,1,a_1,a_2}
{0,1,a_1,a_2}	{0}	{0,1,a_1,a_2}	{0,a_1}	{0,a_2}	{0,a_1,a_2}	{0,1,a_1,a_2}	{0,a_1,a_2,1}	{0,a_1,a_2,1}

We now give the tables of T under set union and set intersection.

The operation $\cup$ union of set in T.

$\cup$	{0}	{0,1}	{0,a_1}	{0,a_2}
{0}	{0}	{0,1}	{0,a_1}	{0,a_2}
{0,1}	{0,1}	{0,1}	{0,1,a_1}	{0,1,a_2}
{0,a_1}	{0,a_1}	{0,a_1,1}	{0,a_1}	{0,a_1,a_2}
{0,a_2}	{0,a_2}	{0,1,a_2}	{0,a_1,a_2}	{0,a_2}
{0,a_1,a_2}	{0,a_1,a_2}	{1,a_1,0,a_2}	{0,a_1,a_2}	{0,a_1,a_2}
{0,1,a_1}	{0,1,a_1}	{0,a_1,1}	{0,a_1,1}	{0,1,a_1,a_2}
{0,1,a_2}	{0,1,a_2}	{0,1,a_2}	{0,1,a_1,a_2}	{0,1,a_2}
{0,1,a_1,a_2}	{0,a_1,a_2,1}	{0,1,a_1,a_2}	{0,1,a_1,a_2}	{0,1,a_1,a_2}

$\{0,a_1,a_2\}$	$\{0,1,a_1\}$	$\{0,1,a_2\}$	$\{0,1,a_1,a_2\}$
$\{0,a_1,a_2\}$	$\{0,1,a_1\}$	$\{0,1,a_2\}$	$\{0,a_1,a_2,1\}$
$\{0,1,a_1,a_2\}$	$\{0,1,a_1\}$	$\{0,1,a_2\}$	$\{0,1,a_1,a_2\}$
$\{0,a_1,a_2\}$	$\{0,a_1,1\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$
$\{0,a_1,a_2\}$	$\{0,a_1,a_2,1\}$	$\{0,1,a_2\}$	$\{0,1,a_2,a_1\}$
$\{0,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_2,a_1\}$
$\{0,1,a_1,a_2\}$	$\{0,1,a_1\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$
$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,1,a_2\}$	$\{0,1,a_1,a_2\}$
$\{0,1,a_1,a_2\}$	$\{0,1,a_1,a_2\}$	$\{0,a_1,a_2,1\}$	$\{0,1,a_1,a_2\}$

The operation of '$\cap$' of sets in T is as follows:

$\cap$	$\{0\}$	$\{0,1\}$	$\{0,a_1\}$	$\{0,a_2\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\{0,1\}$	$\{0\}$	$\{0,1\}$	$\{0\}$	$\{0\}$
$\{0,a_1\}$	$\{0\}$	$\{0\}$	$\{0,a_1\}$	$\{0\}$
$\{0,a_2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0,a_2\}$
$\{0,a_1,a_2\}$	$\{0\}$	$\{0\}$	$\{0,a_1\}$	$\{0,a_2\}$
$\{0,1,a_1\}$	$\{0\}$	$\{0,1\}$	$\{0,a_1\}$	$\{0\}$
$\{0,1,a_2\}$	$\{0\}$	$\{0,1\}$	$\{0\}$	$\{0,a_2\}$
$\{0,1,a_1,a_2\}$	$\{0\}$	$\{0,1\}$	$\{0,a_1\}$	$\{0,a_2\}$

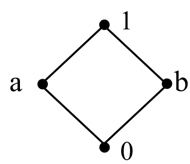
$\{0,a_1,a_2\}$	$\{0,1,a_1\}$	$\{0,a_2,1\}$	$\{0,1,a_1,a_2\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\{0\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
$\{0,a_1\}$	$\{0,a_1\}$	$\{0\}$	$\{0,a_1\}$
$\{0,a_2\}$	$\{0\}$	$\{0,a_2\}$	$\{0,a_2\}$
$\{0,a_1,a_2\}$	$\{0,a_1\}$	$\{0,a_2\}$	$\{0,a_1,a_2\}$
$\{0,a_1\}$	$\{0,1,a_1\}$	$\{0,1\}$	$\{0,1,a_1\}$
$\{0,a_2\}$	$\{0,1\}$	$\{0,1,a_2\}$	$\{0,1,a_2\}$
$\{0,a_1,a_2\}$	$\{0,1,a_1\}$	$\{0,1,a_2\}$	$\{0,1,a_1,a_2\}$

From these four tables we make the fol lowing observations. All the tables are distinct. Further $A \cup_N A \neq A$ in general and $A \cap_N A \neq A$ in general.

Thus if we work with the algebraic structure having two operations then certainly we can have two topologies defined on the same set T of subsets, provided A $\cup_N$ B and A $\cap_N$ B are in T.

Now having seen the new topological space whenever it exists for a given quasi set topological space we proceed onto describe with examples the same concept in case of all the three types of subset topological spaces.

Example 4.86: Let S = {Collection of all subsets of the semiring R



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be the semivector space over the semiring R of type I.
T = {Collection of all quasi set subset semivector subspaces of S over the set K = {0, a, b} $\subseteq$ R} = {{0}, {0, a}, {0, b}, {0, 1, a, b}, {0, a, b}} is a quasi set topological semivector subspace of S over K = {0, a, b}.

The four tables of T and T_N are as follows:

Table under usual union.

$\cup$	{0}	{0,a}
{0}	{0}	{0,a}
{0,a}	{0,a}	{0,a}
{0,b}	{0,b}	{0,a,b}
{0,a,b}	{0,a,b}	{0,a,b}
{0,1,a,b}	{0,1,a,b}	{0,1,a,b}

{0,b}	{0,a,b}	{0,1,a,b}
{0,b}	{0,a,b}	{0,1,a,b}
{0,a,b}	{0,a,b}	{0,1,a,b}
{0,b}	{0,a,b}	{0,1,a,b}
{0,a,b}	{0,a,b}	{0,a,b,1}
{0,a,b,1}	{0,a,b,1}	{0,a,b,1}

The table under usual $\cap$ is as follows:

$\cap$	{0}	{0,a}	{0,b}	{0,a,b}	{0,1,a,b}
{0}	{0}	{0}	{0}	{0}	{0}
{0,a}	{0}	{0,a}	{0}	{0,a}	{0,a}
{0,b}	{0}	{0}	{0,b}	{0,b}	{0,b}
{0,a,b}	{0}	{0,a}	{0,b}	{0,a,b}	{0,a,b}
{0,a,b,1}	{0}	{0,a}	{0,b}	{0,b,a}	{0,a,b,1}

The table for $\cup_N$ is as follows:

$\cup_N$	{0}	{0,a}	{0,b}	{0,a,b}	{0,1,a,b}
{0}	{0}	{0,a}	{0,b}	{0,a,b}	{0,1,a,b}
{0,a}	{0,a}	{0,a}	{0,a,b,1}	{0,a,1,b}	{0,a,b,1}
{0,b}	{0,b}	{1,a,b,0}	{0,b}	{0,a,b,1}	{0,a,b,1}
{0,a,b}	{0,a,b}	{0,a,b,1}	{0,a,b,1}	{0,a,b,1}	{0,a,b,1}
{0,1,a,b}	{0,1,a,b}	{0,a,b,1}	{0,a,b,1}	{0,a,b,1}	{0,b,a,1}

The table for $\cap_N$ is as follows:

$\cap_N$	{0}	{0,a}	{0,b}	{0,a,b}	{0,a,b,1}
{0}	{0}	{0}	{0}	{0}	{0}
{0,a}	{0}	{0,a}	{0}	{0,a}	{0,a}
{0,b}	{0}	{0}	{0,b}	{0,b}	{0,b}
{0,a,b}	{0}	{0,a}	{0,b}	{0,a,b}	{0,a,b}
{0,a,b,1}	{0}	{0,a}	{0,b}	{0,a,b}	{0,a,b,1}

We see $\cap_N = \cap$ however as $\cup \neq \cup_N$ we get a new topological subset semivector space of S.

Now we proceed onto define new topology on quasi set subset semivector subspaces over rings of type II.

Example 4.87: Let
S = {Collection of all subsets of the ring Z_4} be the special subset semivector space over the ring Z_4. $K_1 = \{0, 1\}$ be a subset of the ring Z_4.

T_1 = {Collection of all quasi set subset semivector subspaces of S over the set $K_1 = \{0, 1\} \subseteq Z_4\}$ = {{0}, {0,1}, {0,2}, {0,3}, {0,1,2}, {0,1,3}, {0,2,3}, {0,1,2,3}} is a special quasi set topological subset semivector subspace of S over the set K_1.

The operation $\cap$ on T is given in the following table.

$\cap$	{0}	{0,1}	{0,2}	{0,3}	{0,1,2}	{0,1,3}	{0,2,3}	{0,1,2,3}
{0}	{0}	{0}	{0}	{0}	{0}	{0}	{0}	{0}
{0,1}	{0}	{0,1}	{0}	{0}	{0,1}	{0,1}	{0}	{0,1}
{0,2}	{0}	{0}	{0,2}	{0}	{0,2}	{0}	{0,2}	{0,2}
{0,3}	{0}	{0}	{0}	{0,3}	{0}	{0,3}	{0,3}	{0,3}
{0,1,2}	{0}	{0,1}	{0,2}	{0}	{0,2}	{0,1}	{0,2}	{0,1,2}
{0,1,3}	{0}	{0,1}	{0}	{0,3}	{0,1}	{0,1,3}	{0,3}	{0,1,3}
{0,2,3}	{0}	{0}	{0,2}	{0,3}	{0,2}	{0,3}	{0,2,3}	{0,2,3}
{0,1,2,3}	{0}	{0,1}	{0,2}	{0,3}	{0,1,2}	{0,1,3}	{0,2,3}	{0,1,2,3}

The operation ∪ on T is given by the following table.

∪	{0}	{0,1}	{0,2}	{0,3}	{0,1,2}	{0,1,3}	{0,2,3}	{0,1,2,3}
{0}	{0}	{0,1}	{0,2}	{0,3}	{0,1,2}	{0,1,3}	{0,2,3}	{0,1,2,3}
{0,1}	{0,1}	{0,1}	{0,1,2}	{0,1,3}	{0,2,1}	{0,1,3}	{0,1,2,3}	{0,1,2,3}
{0,2}	{0,2}	{0,1,2}	{0,2}	{0,2,3}	{0,1,2}	{0,1,3,2}	{0,2,3}	{0,1,2,3}
{0,3}	{0,3}	{0,1,3}	{0,2,3}	{0,3}	{0,1,2,3}	{0,1,3}	{0,2,3}	{0,1,2,3}
{0,1,2}	{0,1,2}	{0,1,2}	{0,1,2}	{0,1,2,3}	{0,1,2}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}
{0,1,3}	{0,1,3}	{0,1,3}	{1,0,2,3}	{0,1,3}	{0,1,2,3}	{0,1,3}	{0,2,1,3}	{0,1,2,3}
{0,2}	{0,2}	{0,2,1}	{0,2,3}	{0,2,3}	{0,2,1}	{0,1,2,3}	{0,2,3}	{0,1,2,3}
{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{1,0,2,3}

{T, ∪, ∩} is a special quasi set topological subset semivector subspace of S over the set {0, 1}.

The table of T_N with $\cup_N$ is as follows:

$\cup_N$	{0}	{0,1}	{0,2}	{0,3}
{0}	{0}	{0,1}	{0,2}	{0,3}
{0,1}	{0,1}	{0,1,2}	{0,1,2,3}	{0,1,3}
{0,2}	{0,2}	{0,1,3,2}	{0,2}	{0,1,2,3}
{0,3}	{0,3}	{0,1,3}	{0,1,2,3}	{0,3,2}
{0,2,1}	{0,2,1}	{0,2,3,1}	{0,1,2,3}	{0,1,2,3}
{0,1,3}	{0,1,3}	{0,2,1,3}	{0,1,2,3}	{0,1,2,3}
{0,2,3}	{0,2,3}	{0,2,1,3}	{0,1,2,3}	{0,1,2,3}
{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}

{0,2,1}	{0,1,3}	{0,2,3}	{0,1,2,3}
{0,2,1}	{0,1,3}	{0,2,3}	{0,1,2,3}
{0,2,3,1}	{0,2,1,3}	{0,1,2,3}	{0,1,2,3}
{0,2,3,1}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}
{0,2,1,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}
{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}
{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}
{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}
{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}

Now the table $\cap_N$ is as follows:

$\cap_N$	{0}	{0,1}	{0,2}	{0,3}
{0}	{0}	{0}	{0}	{0}
{0,1}	{0}	{0,1}	{0,2}	{0,3}
{0,2}	{0}	{0,2}	{0}	{0,2}
{0,3}	{0}	{0,3}	{0,2}	{0,1}
{0,1,2}	{0}	{0,1,2}	{0,2}	{0,3,2}
{0,1,3}	{0}	{0,1,3}	{0,2}	{0,1,3}
{0,2,3}	{0}	{0,2,3}	{0,2}	{0,2,1}
{0,1,2,3}	{0}	{1,2,3,0}	{0,2}	{0,1,2,3}

{0,1,2}	{0,1,3}	{0,2,3}	{0,1,2,3}
{0}	{0}	{0}	{0}
{0,1,2}	{0,1,3}	{0,2,3}	{0,1,2,3}
{0,2}	{0,2}	{0,2}	{0,2}
{0,3,2}	{0,3,1}	{0,2,1}	{0,3,2,1}
{0,1,2}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}
{0,1,2,3}	{0,1,3}	{0,1,2,3}	{0,1,2,3}
{0,1,2,3}	{0,1,2,3}	{0,2,1}	{0,2,1,3}
{0,1,2,3}	{0,1,2,3}	{0,1,2,3}	{0,1,2,3}

Clearly $\cap \neq \cap_N$ and $\cup \neq \cup_N$ so T and T_N are two distinct topological spaces defined on the same set of type II.

Now we can also define on
S = {Collection of all subsets of a non commutative ring R}, the special subset semivector space over the ring R.

Clearly T is commutative under ∪ and ∩ so T the collection of all quasi set topological subset semivector space of R of type II is commutative though R is non commutative, but $\{T_N, \cup_N, \cap_N\}$ is a non commutative new topological space as $A \cup_N B \neq B \cup_N A$ and $A \cap_N B \neq B \cap_N A$ in general for A, B ∈ S.

We will just indicate this by a simple example.

Example 4.88: Let
S = {Collection of all subsets of the ring $R = Z_2 S_3$} be the special subset semivector space of R of type II.

Now K = {0, 1} ⊆ R be a subset of R.

We see T = {Collection of all subset quasi set semivector subspaces of S over the set K} is a special quasi set topological semivector subspace of S over the set K ⊆ R.

Now give operations $\cup_N$ and $\cap_N$ on T so that T_N is the new quasi set subset topological semivector subspace of S over K.

Let $A = \{p_1\}$ and $B = \{p_2\} \in T_N$
$A \cap_N B = \{p_5\}$
$B \cap_N A = \{p_4\}$, so $A \cap_N B \neq B \cap_N A$.
Let $A = \{1, p_1\}$ and $B = \{p_2, p_3\} \in T_N$
$A \cup_N B = \{1 + p_2, 1 + p_3, p_1 + p_2, p_1 + p_3\}$
$B \cup_N A = \{p_2 + 1, p_3 + 1, p_2 + p_1, p_3 + p_1\}$
that is $A \cup_N B = B \cup_N A$.

Now having see the non commutative nature of T_N we proceed onto describe the new topology on type III subset semivector spaces by some examples.

Example 4.89: Let
S = {Collection of all subsets of the field F = Z_5} be the special strong subset semivector space of type III over F.

Take K = {0, 1} $\subseteq$ F and T = {Collection of all special strong quasi set semivector subspaces of type III over the set K $\subseteq$ F} = {{0}, {0,1}, {0,2}, {0,3}, {0,4}, {0,1,2}, {0,1,3}, {0,1,4}, {0,2,3}, {0,2,4}, {0,3,4}, {0,1,2,3}, {01,2, 4}, {0,2,3,4}, {0,3,4,1}, {01,2,3,4}} is the special strong quasi set topological semivector subspace of S of type III over K $\subseteq$ F.

Now T_N be the new special strong quasi set topological semivector subspaces of T with $\cup_N$ and $\cap_N$ as operations on T.

We see if A = {0, 3} and B = {0,1,2,3} are in T_N, then
$A \cup B$ = {0,1,2,3}; $A \cup_N B$ = {0,1,2,3,4}

Clearly $A \cup B \neq A \cup_N B$.
We see $A \cap B$ = {0, 3}.
$A \cap_N B$ = {0,3,1,4}.
Clearly $A \cap B \neq A \cap_N B$.

Thus (T_N, $\cup_N$, $\cap_N$) is a new quasi set topological semivector subspace of S over K.

Example 4.90: Let
S = {Collection of all subsets of the field Z_{19}} be the strong special quasi set vector space of type III over Z_{19}.

Take K = {0,1,18} $\subseteq Z_{19}$. T be the collection of all strong special quasi set semivector subspaces of S over K. (T, $\cap$, $\cup$) is a special strong topological quasi set semivector subspace of S over K of type III.

(T_N, $\cup_N$, $\cap_N$) is a special strong quasi set new topological semivector subspace of S over K of type III.

Now having seen examples of type III topological spaces T and new topological spaces T_N of type III we now proceed onto discuss further properties.

Finally we keep on record for every set we can have two topological quasi set semivector or subspaces for all the three types.

We see the special features enjoyed by these new topological semivector subspaces is that in T_N.

$$A \cup_N A \neq A$$
$$A \cap_N A \neq A$$
$$A \cup_N B \neq B \cup_N A \text{ and}$$
$$A \cap_N B \neq B \cap_N A \text{ for all } A, B \in S.$$

It is interesting and innovating means of coupling set theory and the algebraic structure enjoyed by it.

We suggest the following problems.

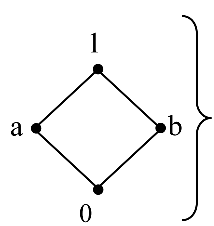
Problems:

1. Find some interesting properties enjoyed by subset semivector spaces over a semifield.

2. Find some special features related with subset semilinear algebras over a semifield.

3. Give an example of a finite subset semivector space which is not a subset semilinear algebra.

4. Let S = {Collection of all subsets of the semiring



be the subset semivector space over the semifield $\{0, a\} = F$.

(i) Does S contain subset semivector subspaces?

(ii) Find a basis of S over $F = \{0, a\}$.

(iii) Can S have more than one basis?

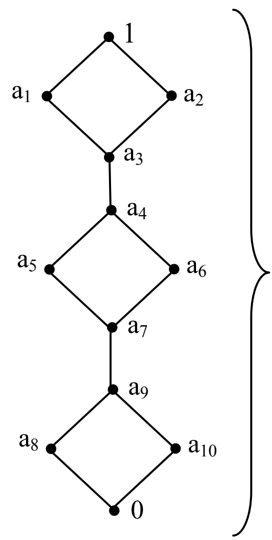
(iv) If S is defined over the semifield $F_1 = \{0, a, 1\}$. Find the differences between the two spaces.

(v) Is S a subset semilinear algebra over F?

5. Obtain a six dimensional subset semivector space over a semifield F.

6. What are the benefits of studying subset semivector spaces?

7. Let S = {Collection of all subsets of the semiring



be the subset semivector space over the semifield $F_1 = \{0, a_8\}$.

(i) Find a basis of S over F_1.

(ii) Can S be made into a subset semilinear algebra over the semifield $F_1 = \{0, a_8\}$?

(iii) Does S contain subset semivector subspaces over F_1?

(iv) If F_1 is replaced by $F_2 = \{0, a_8, a_9\}$ study questions (i) to (iii).

(v) Find the difference between the subset semivector spaces over F_1 and F_2.

(vi) Study the questions (i) to (iii) if F_1 is replaced by the semifield $F_3 = \{1, a_1, a_3, a_4, a_5, a_7, a_9, a_8, 0\}$.

8. Let S = { set of all subsets of the semiring $(Q^+ \cup \{0\})$ (g) | $g^2 = 0\}$ be the subset semivector space over the semifield $F = Z^+ \cup \{0\}$.

(i) Find a basis of S over F.

(ii) Is S an infinite dimensional subset semivector space over F?

(iii) Is S a subset subsemilinear algebra over F?

(iv) Prove S has infinitely many subset semivector subspaces and subset semilinear algebras over F.

9. Let
S_1 = {Collection of all subsets of a semifield $Z^+ \cup \{0\}$ (g)}
and
S_2 = {Collection of all subsets of a semifield $Q^+ \cup \{0\}$} be

two subset semivector spaces defined over the semifield $F = Z^+ \cup \{0\}$.

(i) Find a basis of S_1 and S over F.

(ii) What is the dimension of S_1 and S_2 over F?

(iii) Find a transformation $T : S_1 \to S_2$ over F?

(iv) Can S_1 and S_2 be subset semilinear algebras over F?

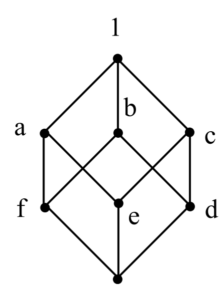
10. Obtain some interesting properties of subset semivector spaces defined over a semifield F.

11. What is the algebraic structure enjoyed by the collection of all linear transformations, $A = \{T : S_1 \to S_2 \mid S_1$ and S_2 two subset semivector spaces over a semifield $F\}$?

12. If S_1 and S_2 are of finite cardinality will A in problem 11 be of finite cardinality?

13. If S_1 and S_2 are of finite dimension, will A in problem 11 be of finite dimension?

14. If $T : S \to S$ is a linear operator of a subset semivector space V = {Collection of all subsets of the semiring



over the semifield $F = \{0, f\}$.

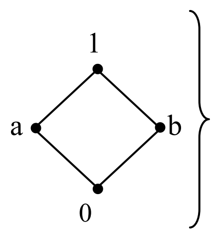
15. If B = {Collection of all linear operators on S} given in problem 14; what is the algebraic structure enjoyed by B?

16. Find some special features enjoyed by;

    (i) Collection of linear operators of a subset semivector space of finite cardinality.

    (ii) Collection of all linear operators of a subset semivector space of finite dimension.

17. Let S = {Collection of all subsets, of the semiring



    be the subset semivector space over the semifield F = {0, a}.

    (i) Find a basis of S over F.

    (ii) Find A = { $T : S \to S$} . Collection of all linear operators on S.
    What is the algebraic structure enjoyed by it?

    (iii) If F = {0, a} is replaced by F_1 = {0, 1, a}. Study problems (i) and (ii).

18. Let S={Collection of all subsets of the semifield $R^+ \cup \{0\}$}.

    (i) Find dimension of S over F.

(ii) Find a basis of S over F.

(iii) If F is replaced by $Q^+ \cup \{0\}$; study questions (i) and (ii).

(iv) If F is replaced by $Z^+ \cup \{0\}$ study questions (i) and (ii).

(v) Find the algebraic structure enjoyed by the collection of all linear operators on S.

19. Let S = {Collection of all subsets of the semiring $(Z^+ \cup \{0\})$ (g_1, g_2, g_3, g_4) | $g_1^2 = 0$, $g_2^2 = g_2$, $g_3^2 = 0$, $g_4^2 = g_4$, $g_i g_j = g_j g_i = 0$ if $i \neq j$; $1 \leq i, j \leq 4$} be the subset semivector space over the semifield $F = Z^+ \cup \{0\}$.

(i) Find a basis of S over F.

(ii) What is the dimension of S over F?

(iii) Find for A = {T : S $\rightarrow$ S}; the algebraic structure enjoyed by A.

20. Let S = {Collection of all subsets of the semiring

$$R = \left\{\left(\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{matrix}\right) \middle| \; a_i \in Z^+ \cup \{0\};\ 1 \leq i \leq 10\right\}$$

under the natural product $\times_n$ of matrices} be a subset semivector space over the semifield $Z^+ \cup \{0\} = F$.

(i) Find a basis of S over F.

(ii) Is S a finite dimensional subset semivector space over F?

(iii) What is the dimension of S over F?

21. Let S = {Collection of all subsets of the semiring

$$R = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \right| a_i \in Z^+(g) \cup \{0\};\ 1 \leq i \leq 5 \text{ under the natural product } \times_n \right\}$$

} be the subset semivector space over the semifield $F = Z^+ \cup \{0\}$.

(i) Find a basis of S over F.

(ii) Prove S is a subset semilinear algebra over F.

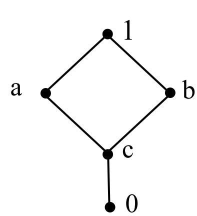
(iii) Find dimension of S over F.

(iv) Find $\{T : S \to S\} = A$, collection of all linear operators on V.

22. Obtain some special and interesting properties enjoyed by the subset semivector spaces of type I.

23. Compare subset semivector spaces with subset semivector spaces of type I.

24. Let S = {Collection of all subsets of the semiring

R = 

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be the subset semivector R of type I.

(i) Find a basis of S over R.

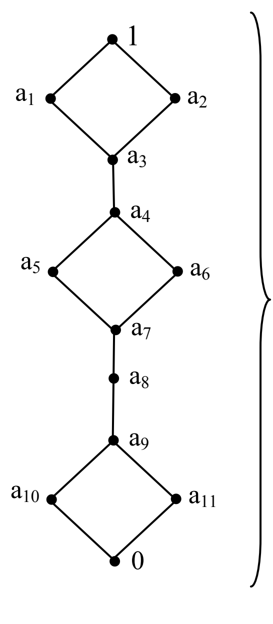
(ii) Prove S contains subset semivector subspaces defined over the semifield $F_1 = \{0, c\}$ or $F_2 = \{0, a, c, 1\}$ or $\{0, a, c\} = F_3$.

(iii) What is dimension of S over R as a subset semivector space of type I?

(iv) Compare dimension of S over R, F_1, F_2 and F_3. When is the dimension of S the largest?

(v) Find basis of S over F_1, F_2 and F_3.

25. Let S = {Collection of all subsets of the semiring R =



be the subset semivector space over the semiring R of type I.

(i) Find dimension of S over R.

(ii) If $S = S_1$ is taken as a subset semivector space over the semifield $\{0, a_{10}\} = F_1$. Compare S_1 with S as a type I space.

(iii) If F_1 is replaced by $F_2 = \{1, a_1, a_3, a_4, a_5, a_7, a_8, a_9, a_{11}, 0\}$ compare S and S_1.

(iv) When is the dimension largest?

(v) Find a basis of S over R, F_1 and F_2.

26. Let S = {Collection of all subsets of the semiring $R = Z^+ (g_1, g_2) \cup \{0\}$, $g_1^2 = g_2^2 = \{0\}$} be the subset semivector space of type I over the semiring $F = 3Z^+ (g_1) \cup \{0\}$.

(i) Find a basis of S over F.

(ii) What is dimension of S over F?

(iii) If A = {Collection of all linear operators from S to S} find the algebraic structure enjoyed by A.

(iv) If F is replaced by $F_1 = Z^+ \cup \{0\}$; study problems (i), (ii) and (iii).

27. Give some special features enjoyed by the special strong subset semivector spaces of type III defined over a field.

Prove every special strong subset semivector space of type III has a proper subset vector space over the field.

28. Let S = {Collection of all subsets of the ring

$$R = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in Z_{12};\ 1 \leq i \leq 6 \right\} \text{ under natural product } \times_n\}$$

be the special subset semivector space over the ring Z_{12}.

(i) Find all linear operators on S.

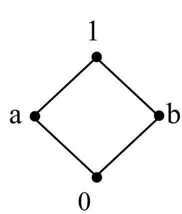
(ii) What is the cardinality of S?

(iii) Find the dimension of S over R.

(iv) Can S contain a proper subs et which is a subset vector space over a field contained in Z_{12}?

(v) How many such subset semivector spaces of S exist?

29. Let S = {Collection of all subsets of the semiring

$L_1 =$ 

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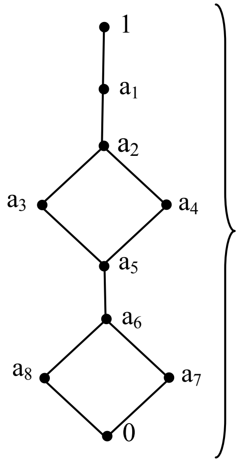
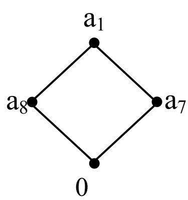
be a subset semivector space of type I over L_1.

P = {Collection of all subsets of the semiring

R = 

be a subset semivector space over the semiring

L_2= 

(i) Can we define semilinear transformation from S to P?

(ii) Compare the two spaces S and P.

(iii) Find a basis of S and a basis of P over the respective semirings.

(iv) Are the basis unique?

(v) Find the number of elements in S and in P.

(vi) Which space is of higher dimension S or P?

30. Let
S = {Collection of all subsets of the semifield $R^+ \cup \{0\}$} be the subset semivector space over the semifield $Z^+ \cup \{0\}$.

(i) What is the dimension of S over $Z^+ \cup \{0\}$?

(ii) If $Z^+ \cup \{0\}$ is replaced by $Q^+ \cup \{0\}$; what is the dimension of S?

(iii) If $Z^+ \cup \{0\}$ is replaced by $R^+ \cup \{0\}$ what is the dimension of S?

(iv) Prove S has infinite number of subset semivector subspaces.

(v) Is S a subset semilinear algebra?

(vi) Find atleast three distinct linear operators on S.

31. Let S = {Collection of all subset of the semiring

$$R = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| \, a_i \in Z^+ \cup \{0\};\ 1 \leq i \leq 4 \right\} \text{ under usual}$$

product} be the subset semivector space over the semfield $Z^+ \cup \{0\}$.

(i) Prove S is of infinite cardinality.

(ii) Is S a non commutative semilinear algebra?

(iii) Can S have a finite basis?

(iv) Can S have more than one basis?

(v) What is the dimension of S over $Z^+ \cup \{0\}$?

32. Let S = {Collection of all subsets of the semiring

$$R = \left\{ \left. \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \right| a_i \in Z^+ \cup \{0\}; 1 \leq i \leq 9 \right\}$$

under usual matrix produ ct} be the subset sem ivector space of type I over the semiring R.

(i) Prove in general xA = Ax for A $\in$ S and x $\in$ R.

(ii) What is the dimension of S over R?

(iii) Find a basis of S over R.

(iv) Can S have several basis?

(v) Find two distinct linear operators on S.

33. Study the above problem if S is a subset semivector space over the semifield $Z^+ \cup \{0\}$.

34. Show S is a non com mutative subset sem ilinear algebra over $Z^+ \cup \{0\}$.

Let S = {Collection of all subsets of the ring R = Z_{45}} be the special subset sem ivector of t ype II over the ring R = Z_{45}.

(i) Is S of finite cardinality?

(ii) Find a basis of S over R.

(iii) Can S have more than one basis?

35. Enumerate the differences between the three types of subset semivector spaces.

36. List out the special features associated with the strong special subset semivector spaces of type IV defined over a field and prove it always contains a proper subset which is a subset vector space over the field.

37. Is it possible to define subset vector space over a field independently other than the one using singleton subsets?

38. Let S = {Collection of all subsets of the ring

$$R = \left\{ \left. \begin{bmatrix} a_1 & a_8 \\ a_2 & a_9 \\ \vdots & \vdots \\ a_7 & a_{14} \end{bmatrix} \right| a_i \in Z_{13};\ 1 \le i \le 14 \right\}$$

under natural product} be the special strong subset semivector space over the field Z_{13} of type III.

(i) How many subset vector subspaces, V of S over Z_{13} exist?

(ii) Find a basis of S over Z_{13}.

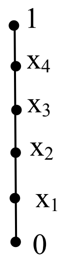
(iii) What is the dimension of S over Z_{13}?

39. Give some interesting properties about quasi set semivector subspaces of a semivector space S over a subset $K \subseteq F$ of the semifield F over which S is defined.

40. Give some examples of quasi set semivector subspaces.

41. Let $S = \{(a_1, a_2, a_3, a_4) \mid a_i \in$

L =



$; 1 \leq i \leq 4\}$

be a semivector space over the semifield L.

(i) How many quasi set semivector subspaces of S exist for the set $K = \{0, 1, x_2, x_3\} \subseteq L$?

(ii) How many quasi set semivector subspaces of S over the set $\{0,1\} \subseteq L$ exist?

(iii) Compare the quasi set semivector subspaces in (i) and (ii).

42. Let S = {Collection of all subsets of the ring Z_{45}} be the special subset semivector space of type II over the ring Z_{45}.

(i) Can S have special strong subset vector subspace over a field?

(ii) Find two linear distinct operators on S.

(iii) Find special subset semivector subspaces of S over Z_{45}.

(iv) Find a basis of S over Z_{45}.

(v) Can S have more than one basis?

(vi) What is the dimension of S over Z_{45}?

43. Give some special features of the quasi set semivector topological subspaces of type I.

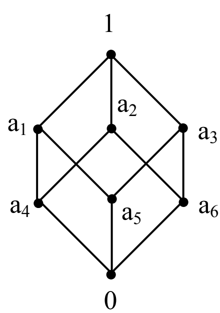
44. Give an example of a quasi set topological semivector subspace of type I using the semifield $Q^+ \cup \{0\}$.

45. Let
$S = \{$Collection of all subsets of the semifield $Z^+ \cup \{0\}\}$ be the subset semivector space over the semifield $Z^+ \cup \{0\}$.

(i) Using $K_1 = \{0, 1\}$, $K_2 = \{0, 3\}$ and $K_3 = \{0,1,7\}$ find the corresponding T_1, T_2 and T_3, the quasi set topological semivector subspaces of S over K_1, K_2 and K_3 respectively.

(ii) Find the quasi set new topological semivector subspaces of S over K_1, K_2 and K_3 respectively using the operations $\cup_N$ and $\cap_N$.
Compare the topologies on these sets.

46. Let S = {Collection of all subsets of the semiring



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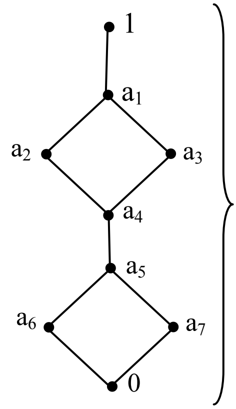
be the semivector space over the semifield
$F = \{0, a_6, a_1, 1\}$, $F \subseteq R$.

(i) Find using the subset $K = \{0, a_2, a_4, a_5\} \subseteq R$, the quasi set topological subset semivector subspace of S.

(ii) Find for the same set new quasi set topological subset semivector subspace of S over K.

47. Let S = {Collection of all subsets of the semiring

P =



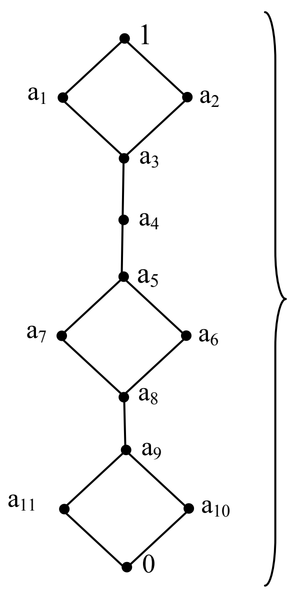
be the subset semivector space of S over P of type I.

(i) Take K = $\{0, a_6, a_7, a_8\}$ to be a subset of S. Find the quasi set topological semivector subspaces of S over the set K.

(ii) Study (i) using the operation $\cup_N$ and $\cap_N$ and compare them.

48. Let S = {Collection of all subsets from the semiring

R =



be the semivector space of type I over R.

Take $K_1 = \{0, a_{11}, a_7\}$ and $K_2 = \{0,1, a_2, a_5\}$, $K_3 = \{0,1, a_2, a_5, a_{11}, a_7\}$ and find the quasi set topological semivector subspaces T^1, T^2 and T^3 over K_1, K_2 and K_3 respectively. Find the new quasi set topological semivector subspaces T_N^1, T_N^2 and T_N^3 of S over K_1, K_2 and K_3 respectively.

Compare them.

49. Let S = {Collection of all subsets of the ring Z_{30}} be the special subset semivector space over the ring Z_{30} of type II.

    (i) Let $K_1 = \{0, 3, 7, 11\} \subseteq Z_{30}$. Find T and T_N. For $K_2 = \{0, 1\}$ find T and T_N.

    (T is the usual quasi set topological subset semivector subspace of S over the respective sets and T_N is T but operation $\cup_N$ and $\cap_N$ is used. This notation will be followed in rest of the problems; that is in the following problems).

50. Let S = {Collection of all subsets of the ring R = {M = (a_{ij}) | M is a 3 × 3 matrix with entries from Z_{42}; $1 \leq i, j \leq 3$}} be a special subset semivector space over the ring R of type II.

    (i) Take $K_1 = \left\{\begin{pmatrix} 5 & 2 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 0 & 8 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\right\} \subseteq R.$

    Find T and $(T_N, \cup_N, \cap_N)$ over K_1.

    (ii) Take $K_2 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 7 \\ -1 & 2 \end{pmatrix}\right\} \subseteq R.$

    Find T and $(T_N, \cup_N, \cap_N)$.

(iii) Replace R by the ring $R_1 = Z_{42}$, take subsets $P_1 = \{0, 1, 4\}$ and $P_2 = \{0, 11, 7\}$ in R_1 and find T^1 and T^2 and T_N^1 and T_N^2.

51. Let S = {Collection of all subsets of the field Z_{43}} be the strong special subspace semivector space over the field Z_{43} of type III.
(i) Take subset $\{0,1\} = K_1$ and find T and T_N.
(ii) Take subset $\{0, 42\} = K_2$ and find T and T_N.

52. Obtain some special properties associated with T_N the new topological subset semivector subspace using $\cup_N$ and $\cap_N$.

53. Let S = {Collection of all subsets of the field F = Q} be the strong special subset semivector space of type III over the field F = Q.

(i) Take $K_1 = \{0, 1\}$, $K_2 = \{0, 1, -1\}$ and $K_3 = \{0, 1, 2\}$ and T^1, T^2 and T^3 to be the quasi set subset special strong topological semivector subspaces of type III.

(ii) Find for these K_i, $1 \leq i \leq 3$, T_N^1, T_N^2 and T_N^3 and compare T_i with T_N^i, $1 \leq i \leq 3$.

(iii) If the same S is taken as a usual semivector space over the semifield $K = Q^+ \cup \{0\}$. Find T_1 and T_N^1 for the set $K_1 = \{0, 1, 11\} \subseteq K' = Z^+ \cup \{0\}$.

54. Let S = {Collection of all subsets of the field Z_{47}} be the strong special semivector space over the field Z_{47} of type III.

(i) Find for the subset $K = \{0, 1, 2\} \subseteq Z_{47}$, T and T_N.
(ii) Find for the subset $K_1 = \{0, 1\} \subseteq Z_{47}$ T_N^1 and T^1.
(iii) Find for the subset $K_2 = \{0, 1, 46\}$ find T_N^2 and T_N. Compare all the 3 sets of spaces.

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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