INTEGRAL EIGEN-PAIR BALANCED CLASSES OF GRAPHS: RATIOS, ASYPTOTES, DENSITY AND AREAS

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Abstract

The association of integers, conjugate pairs and tightness with the eigenvalues of graphs provide the motivation for the following definitions. A class of graphs, with the property, that for each graph (member) of the class, there exists a pair a,b of non-zero, distinct, eigenvalues, whose *sum* (or *product*) yields the **same integer***,* either as a fixed constant, or a function of an inherent aspect of the graph (such as its size), is said to be *sum-eigen*(a+b)*pair balanced* (or *product-eigen*(a.b)*pair balanced,* respectively). For example, complete graphs on n vertices, are *eigen-bi-balanced* with sum-eigen*(n-2)*pair balanced and product-eigen*(1-n)*pair balanced, and since a,b are non-zero their reciprocals (which affect the tightness of a graph) are defined, so that this class has the eigen-balanced *ratio* of 1/a+1/b=(a+b)/(a.b)= (n-2)/(1-n) =f(n) which is *asymptotic* to the constant value of -1. The absolute value of the integral of f(n) multiplied by the average degree yields the *area* (n-1)(n-ln(n-1)) – we show that this is the *maximum* area for most known classes of eigen-bi-balanced graphs. We investigate the effect of this asymptotic ratio on the *energy* of the molecular representation of graphs. Cycles are generally neither sum-eigen-pair, nor product-eigen-pair balanced, while paths are only sum- eigen-pair balanced. In this paper, we introduce a class of graphs, involving q cliques each of size q, and show that this class is eigen-bi-balanced with respect to the sum -1 and product 1-q so that it has ratio 1/(q-1) asymptotic to 0, and has area q(2q+2ln(q-1)), and discuss its eigen-bi-balanced *criticality.*

1. Introduction

All graphs which we shall consider will be finite, simple, loopless and undirected.

1.1 Integers, conjugate pairs and the eigenvalues of a graph

Summing of the eigenvalues of the adjacency matrix graph is connected to the *energy* of physical structures (See [1]). There has been interest in classes of graphs whose *pairs* of eigenvalues satisfy certain conditions. In [13], graphs are considered with reciprocal *pairs* of eigenvalues:

$(\lambda, \frac{1}{\lambda})$ whose *product* is the **integer** 1. *Pairs* of eigenvalues (1,-1), summing to 0, and whose product is -1, are considered in [7]. Ervin Van Dam's paper on regular graphs [5] with 4 eigenvalues considers the *eigenvalue pair* of real *conjugates*:

$\frac{a \pm \sqrt{b}}{2}$ and shows that if a matrix has an eigenvalues

$$\frac{a + \sqrt{b}}{2}$$

Then it has an eigenvalues

$$\frac{a - \sqrt{b}}{2}$$

of the same multiplicity, and visa-versa.

Adding the *pair o*f conjugates $\frac{a+\sqrt{b}}{2};\frac{a-\sqrt{b}}{2}$ we obtain the **integer** “a”. Their product is $\frac{a^2-b}{4}$ which is **integral** provided the numerator is a multiple of 4. The paper shows that there are graph whose matrices have conjugate pairs of eigenvalues whose sum does not necessarily sum to the *same* integer a. In [3] integral trees, where the eigenvalues of trees being **integral** are investigated.

The importance of pairs of numbers, whose sum and product produce the same integral constant exists outside the linear algebra of matrices (see, for example, “Proof of Lyapunov exponent *pairing* for systems of constant kinetic energy” by C P Dettmann and G P Morris [7]). The cryptography article by M Hamada [9] considers conjugate code *pair c*onsisting of linear codes [n,k’] and [n,k’’] satisfying the constant (integral) sum term k’+k’’=n+k, where n is the dimension of the vector space involved and k is the k-digit secret information sent. The paper by A M Kadin [11] “Spatial structures of the Cooper pair” investigates the Cooper *pair* of opposite wavevectors k and –k which *balance* by summing to 0 and whose product is $-k^2$, while in “Note on the rheology of dilute suspension of dipolar spheres with weak Brownian *couples*” by E J Hinch and L G Leal [10] the notion of an isolated particle in the absence of rotary Brownian motion is considered under the condition that the hydrodynamic and external field *couples* exactly *balance* one another.

In [2] the importance of the quadratic part of a characteristic equation which has the form:

$$x^2-\tau.x+\delta$$

This quadratic gives rise to the two eigenvalues:

$$a,b=\frac{\tau\pm\sqrt{\tau^2-4\delta}}{2}$$

With the *sum* and *product* being $\tau, \ \delta$ respectively – this is refer to as the eigenpair, but we shall focus on the *pair of eigenvalues* a,b (or *eigen-pair*). The above provides the motivation for the following definitions.

Generally, there exists two eigenvalues (associated with the adjacency martix of a graph) whose sum or product is integral, or a function of its number of vertices v. Is it possible to get the **same** integer, when adding or multiplying two distinct, non-zero, eigenvalues, which is either a fixed constant, or a function of an inherent property of the graph, for **all** graphs belonging to a certain *class* of graphs?

For example, complete graphs K_{nv} on n vertices have pair eigenvalue sum balancing to f(n)=n-2, and product g(n)=1-n for each n≥ 2, and the complete bi-partite graphs $K_{n/2,n/2}$ on n vertices have pair sums (of non-zero eigenvalues) balancing to 0, and product balancing to $-n^2/4$.

1.1.1 Definition of a function of a member of a class of graphs

We define a *function of a member* belonging to a class of graphs, as a real function f(p) of an inherent property p of the member in the class, such as the number of vertices or the clique number of a graph etc.

In this paper we combine the two ideas of a pair of eigenvalues and their balanced integral sum and product with respect to a class of graphs, to introduce a definition which is a form of *integral-eigenvalue balance* associated with classes of graphs. We investigate *classes of graphs* on v vertices with pairs (a,b) of distinct non-zero eigenvalues such that a+b=s or a.b=t where s,t are the *same integer* (respectively) for each graph in the class **or** the same *function* of each graph in the class.

Main definitions

1.2 Integral sum eigen-pair balanced classes of graphs

Classes $\Im$ of graphs on v elements are said to be *sum*(s)*eigen-pair (integral) balanced* if there exists a pair of a,b of *distinct non-zero* eigenvalues (eigenvalues considered once so we ignore multiplicities) of the matrices associated with each class of the structures such that a+b= s = **same integer** as a fixed constant for each member in the class, or s is the **same integer** as a **function of each member** in the class. The sum balance is *exact,* if s is the same integer as a fixed constant for each member in the class, or *non-exact* .

We provide a few examples:

1. The class of complete graphs K_n is non-exact sum*(n-2)*eigen-pair balanced for $n \geq 3$, since the eigenvalues of the associated matrices are -1 and n-1.
2. The class of complete bipartite graphs $K_{p,p}$ on n=2p vertices has as its associated eigenvalues p,-p and 0 so that they are exact sum*(0)*eigen-pair balanced.
3. The class of complete bipartite graphs $K_{p,k}$ on p+k vertices, p and k different, have eigenvalues $-\sqrt{pk}\ ;\sqrt{pk}\ ;(0)^{m+n-2}$ so that they are exact sum*(0)*eigen-pair balanced (this includes the star graphs with radius 1).
4. Wheels with p spokes are sum*(2)*eigen-pair balanced (see theorem 10 in [12]). They have conjugate pairs:
   $$\frac{2 \pm \sqrt{4+4p}}{2}$$
5. The path P_n on $n \geq 2$ vertices and n-1 edges has eigenvalues (see[3])
   $$2\cos(\frac{\pi j}{n+1}); \quad j = 1,2,\ldots,n$$
   Note that:
   $$\cos(\frac{n\pi}{n+1}) = \cos(\pi - \frac{\pi}{n+1}) = \cos\pi\cos(\frac{\pi}{n+1}) + \sin\pi\sin(\frac{\pi}{n+1}) = -\cos(\frac{\pi}{n+1})$$
   So that the non-zero pair :
   $$2\cos(\frac{n\pi}{n+1});\ 2\cos(\frac{\pi}{n+1})$$

has the sum 0 so that paths are exact sum*(0)*eigen-pair balanced

6. The cycle C_n on n veritices and edges has eigenvalues (see [3]): $2\cos(\frac{2\pi j}{n}); \quad j = 0,1,2,...,n-1$
   The 3-cylce (complete graph on 3 vertices) has eigenvalues (Ignoring multiplicities):-1 and 2. This cycle is sum*(1)*eigen-pair balanced.
   The 4-cycle has eigenvalues:
   2; 0; -2, which is sum*(0)*eigen-pair balanced
   The 5-cycle has eigenvalues:
   $(2)^1;\ (\frac{-1+\sqrt{5}}{2})^2;\ (\frac{-1-\sqrt{5}}{2})^2$
   Adding the 2 distinct irrational eigenvalues we get the integer -1 so this graph is sum*(-1)*eigen-pair balanced (of the exact kind).
   The 6-cycle has eigenvalues $(2)^1;(1)^2;(-1)^2;(-2)^1$ so that adding 2 and -1 makes it sum*(1)*eigen-pair balanced which is the same as the 3-cycle and 4-cycle.
   However the 7-cycle does not contain two distinct eigenvalues whose sum is 1: $(2)^1;(1{,}247)^2;(-0{,}445)^2;(-1{,}82)^2$.
   Thus the class of cycles is not sum*(1)*eigen-pair balanced.
   Even cycles are sum*(0)*eigen-pair balanced, since if n=2k

   $$2\cos(\frac{\pi j}{k}); \quad j = 0,1,2,...,2k-1$$

   then j= 0 and j=k yields eigenvalues of 2 and -2 respectively.
7. Strongly regular graph of degree k have exactly three distinct eigenvalues: k, whose multiplicity is 1 (See [8]).

   $$\frac{1}{2}\left[(\lambda-\mu)+\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right]$$

whose multiplicity is

$$\frac{1}{2}\left[(v-1)-\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right]$$

and

$$\frac{1}{2}\left[(\lambda-\mu)-\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right]$$

whose multiplicity is

$$\frac{1}{2}\left[(v-1)+\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right]$$

Strongly regular graphs for which:
$2k+(v-1)(\lambda-\mu)=0$
are called conference graphs because of their connection with symmetric conference matrices.

Their parameters reduce to
$srg\left(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4}\right)$.
Strongly regular graphs for which
$2k+(v-1)(\lambda-\mu)\neq 0$
have integer eigenvalues with unequal multiplicities.
The complement of an srg(v,k,λ,μ) is also strongly regular. It is an srg(v, v−k−1, v−2−2k+μ, v−2k+λ).
Note that if we ignore the largest eigenvalue of strongly regular graphs, adding the remaining 2 yields the integer:
λ-μ
so that they are sum*(λ-μ)*eigen-pair balanced (non-exact)

8. The eigenvalues of DDG are provided in [8] – they have 5 distinct eignvalues. Two of the eigenvalues are:
$\pm\sqrt{k-\lambda_1}$

So that divisible design graphs are sum*(0)*eigen-pair balanced (exact). Examples of bipartite graphs with four distinct eigenvalues are the incidence graphs of symmetric 2-(*v*, *k*, l) designs. It is proven in [4] by Cvetkovic´, Doob and Sachs [6, p. 166] that these are the only examples, i.e. a connected bipartite regular graph with four distinct eigenvalues must be the incidence graph of a symmetric 2-(*v*, *k*, l) design. Moreover its spectrum is

$(k)^1;(\sqrt{k-\lambda})^{v-1};(-\sqrt{k-\lambda})^{v-1};(-k)^1$

Note that these graphs are sum*(0)*eigen-pair balanced (exact).

9. The p-regular hypercube on 2^p vertices and $p2^{p-1}$edges has eigenvalues (see [3]):

   $(p-2k)^{\binom{p}{k}};k=0,1,2,\ldots,p$

   These graphs are also Cayley graphs generated by Z_2^n

   Using the largest eigenvalues p and p-2k, for p,2k different, k not 0 , this class of graphs wlll be sum*(2p-2k)*eigen-pair balanced.

10. The class of *q-cliqued* graphs on q.q+1 vertices:

    We show in section 3 that the class of q-cliqued graphs (which are q-regular) are **exact** sum*(-1)*eigen-pair balanced and that q is embedded in its conjugate eigenvalues:

    $$\frac{-1\pm\sqrt{1+4(q-1)}}{2}.$$

We also show that the product of these conjugate eigenvalues is an *integral function of q = f(q)* = –(q-1) where q-1 is: the degree of the vertices in a complete graph of size q. The value q is also a function of the size of the graph. This motivates the next definition:

1.3 Integral product eigen-pair balanced classes of graphs.

Classes $\Im$ of graphs on v elements are said to be *product*(t)*eigen-pair (integral) balanced* if there exists a pair of a,b of *distinct non-zero* eigenvalues (counting eigenvalues only once and ignoring multiplicities) of the matrices associated with each class of the structures such that a.b= t = **same integer** as a fixed constant for *each* member in the class, or t is the **same integer** as a **function of each member** in the class. The product balance is *exact,* if t is the same integer as a fixed constant for each member in the class, otherwise *non-exact*.

For example:

1. Complete graphs on n vertices are non-exact product*(1-n)*eigen-pair balanced.
2. Complete bipartite graphs $K_{p,p}$ on 2p vertices are non-exact product*($-p^2$)*eigen-pair balanced.
3. Complete bipartite graphs on $K_{p,k}$, $p \neq k$ on p+k vertices are non-exact product*(-pk)*eigen-pair balanced.
4. Paths on n vertices have eigenvalues:

$$2\cos(\frac{\pi j}{n+1}); \quad j = 1,2,\ldots,n$$

And

$$2\cos(\frac{\pi n}{n+1}) = -2\cos(\frac{\pi}{n+1})$$

So that their product is :

$$-2(1+\cos(\frac{2\pi}{n+1}))$$

which is a function of n but is not generally integral.

5. Cycles on n vertices have associated eigenvalues:

$$2\cos(\frac{2\pi j}{n}); \quad j = 0,1,2,\ldots,n-1$$

Are they product*(t)*eigen-pair balanced?

C_3 on 3 vertices, has eigenvalues 2 and -1 so that the graph has eigen-pair product -2.

C_4 on 4 vertices, has eigenvalues 2, 0 and -2 so that the eigen-pair product is -4.

C_5 on 5 vertices, has eigenvalues 2, $\frac{-1+\sqrt{5}}{2}$ and $\frac{-1-\sqrt{5}}{2}$ with eigen-pair product -1.

C_6 on 6 vertices, has eigenvalues 2, 1, -1 and -2.

Possible products are -1, -2, 2 and -4.

The 7-cycle has eigenvalues: $(2)^1;(1{,}247)^2;(-0{,}445)^2;(-1{,}82)^2$

No product yields an integer?

Even cycles are product-eigen-(4)-pair balanced, since if n=2k then: $2\cos(\frac{2\pi j}{2k});\quad j=0,1,2,\ldots,2k-1$

So that for j=0 we get 2 and j=k we get -2 with product -4.

6. Wheels on p spokes are product *(-p)*eigen-pair balanced (see [12]).
7. If we multiply the two conjugate pairs of strongly regular graphs we obtain the integer:
   μ-k
   so that strongly regular graphs are product*(μ-k)*eigen-pair balanced (non-exact).
8. DDG are product*(λ_1 -k)*balanced (non-exact).
9. Incidence graphs of symmetric

   2-(v, k, l) designs are product *t*eigen-pair balanced for:

   $t=-k^2 \ and \ \lambda-k$

   of the non-exact kind.

10. The class of p-regular hypercubes on 2^p vertices and $p2^{p-1}$ edges has eigenvalues:

    $(p-2k)^{\binom{p}{k}}; k=0,1,2,\ldots,p$

    These graphs are also Cayley graphs generated by Z_2^n

    Using the larges eigenvalues p and p-2k, for p,2k different, k not 0 , this class of graphs wlll be product*(p^2-2pk)*eigen-pair balanced.

11. We show in section 3 below that the q-cliqued graphs are non-exact product*(1-q)*eigen-pair balanced with eigen-pair;

    $$\frac{-1\pm\sqrt{1+4(q-1)}}{2}$$

Graphs which are both sum and product eigen-pair balanced are said to be *eign-bi-balanced* with respect to the pair a,b.

Note that the largest eigenvalues occurs in the eigen-pair associated with some classes of graphs discussed above. Also, the only regular eigen-pair balanced graphs on 2 and 3 vertices are

$K_2; K_3$

Respectively.

The 4-cycle is the same as the complete bipartite graph

$K_{2,2}$

Which is sum and product eigen-pair balanced, while the only other regular graph on 4 vertices is:

K_4

The 5-cycle has eigenvalues:

$(2)^1$; $(\frac{-1+\sqrt{5}}{2})^2$; $(\frac{-1-\sqrt{5}}{2})^2$

Which is not eigen-pair sum or product balanced when the largest eigenvalue is included in the eigen pair. The only other regular graph on 5 vertices is:

K_5

Thus we have the following theorem:

THEOREM 1

The only regular graphs on n vertices, where

$2 \leq n \leq 5$

Belonging to eigen-pair balanced classes of graphs, where the eigen pair contains the largest eigenvalue, are:

$K_2; K_3; K_4; K_5 \text{ and } K_{2,2}$

1.4 Eigen-bi-balanced graphs – criticality, ratios, asymptotes, density, areas and energy

If a class of graphs $\Im$ are *both* sum and product eigen-pair balanced with respect to the eigen-pair a,b , have been defined above as *eigen-bi-balanced* with respect to a,b.

Complete graphs G are eigen-bi-balanced with the property that the removal of any vertex G from G results in another class of eigen-bi-balanced graphs. The same holds for completer bi-partite graphs except for star graphs. Such graphs are said to be *stable* eigen-bi-balanced. If G belongs to a class $\Im$ of eigen-bi-balanced graphs, and there exists a vertex v of G, such that G-v belongs to another class $\Im'$ of graphs which is *not* eigen-bi-balanced, we say that $\Im$ is *critically* eigen-bi-balanced with respect to v. Wheels on p spokes are eigen-bi-balanced and the removal of their center results in p-cycles, which are not eigen-bi-balanced, so that they are critically eigen-bi-balanced with respect to their centre – revealing that this vertex is essential to the eign-bi-balanced characteristic of wheels.

The reciprocals of eigenvalues are connected to the idea of *robustness* or *tightness* of graphs (see [30).

Since a and b are non zero, the sum of their reciprocals is defined, and we define the *eigen-balanced ratio* of the structure (with respect to the eigen-pairs) as:

$$\frac{1}{b}+\frac{1}{a}=\frac{a+b}{a.b}=r(a\Im b)$$

(The product is never zero).

If this ratio is a function f(n) of the size n of the graph, and has an asymptote, we call this asymptote the *asymptotoic eigen-balanced ratio* with respect to the eigen-pair a,b and denoted by:

$$r(a\Im b)^{\infty} \;\; or \;\; asymp(r)$$

This asymptote can be seen as the behavior of the ratio as the size of the graph becomes increasingly large.

Eigenvalues have been associated with the *expansion* of graphs (see [3]) which motivates the idea of *areas* associated with the ratio of eigen-bi-balanced graphs. Since we can integrate this ratio, if it is a function of n, the size of a graph, on m edges, we defne the *eigen-balanced ratio area* of the class with respect to the pair a,b:

$$A(\Im)^{a,b} = \frac{2m}{n}\left|\int \frac{a+b}{ab}\,dn\right| \quad for \;\; a+b \neq 0$$

or

$$= \frac{2m}{n}\left|\int_a^b dn\right| = \frac{2m}{n}|2b| \quad for \;\; a=-b$$

Where A=0 when n=0, 1 or 2.

If there is more than one pair giving rise such area, then the area of the the class is:

$$\max A(\Im)^{a_i,b_i} \;\; for\; all \;\; pairs\; a_i, b_i$$

If there is only one eigen-pair associated with the class of graphs that gives rise to the ratio, then the ratio is *unique.*

1. For example, the complete graph on n vertices has the unique eigen-balanced ratio of:

   $$r((n-1)K_n(-1)) = \frac{n-2}{1-n}$$

   Which depends on the size of the graph and has the unique asymptotic

   eigen-   balanced ratio:

   $$r((n-1)K_n(-1))^{\infty} = -1$$

   And eigen-balanced ratio area:

$$A(K_n)^{-1,n-1} = \frac{2m}{n}\left|\int \frac{n-2}{1-n}dn\right| = (n-1)\left|\int [\frac{2}{n-1} - \frac{n}{n-1}]dn\right|$$

$$= (n-1).B; \;\; B = \left|\int \frac{2}{n-1} - \frac{n-1}{n-1} - \frac{1}{n-1}\right| dn = n - \ln|n-1| + c$$

When n=0 we have A=0 so c=0 so that its area is:

$$(n-1)(n - \ln(n-1)) = (n-1)\mathbf{B}$$

Note that the length of the longest path for the complete graph is n-1, so that B in the above expression can be regarded as the *height* of the graph. Also, the term log(n-1) occurs as part of the upper bound of the *diameter* of a graph involving the second largest eigenvalue (see[3]). Is this area the maximum for all classes of eigen-bi-balanced graphs?

2. The complete bipartite graph

$$K_{p,k}$$

on p+k vertices has the unique eigen-balanced ratio of

$$\frac{\sqrt{pq} - \sqrt{pq}}{-pq} = 0$$

Which is independent of the size of the graph. Its area is:

$$\frac{2pq}{p+q} 2\sqrt{pq} = 4\frac{(pq)^{\frac{3}{2}}}{p+q}$$

It attains its maximum when:

$p = q = \dfrac{n}{2}$ then the graph (the *split complete bi-partite* graph on n vertices) is p-regular and:

The area is:

$$A(K_{p,q})^{-n/2,n/2} = \frac{n}{2}\left|\int_{-n/2}^{n/2} dn\right| = \frac{n^2}{2}$$

3. Wheels on p spokes have eigen-balance ratio of

   $$\frac{2}{-p}$$

   Which depends on the size of the graph so that they have an asymptotic eigen-balanced ratio of 0 and eigen-balanced area of:

   $$\frac{2m}{n}\left|\int \frac{2}{1-n}dn\right| = \frac{4p}{p+1}(\log|n-1| + c) = \frac{4(n-1)}{n}(\log|n-1| + c)$$

   A=0 when n=2 so that c=0.

4. Star graphs with p rays of length 2 have eigenvalues see [spectra]:

   $$0, -1, 1, \pm\sqrt{p+1}$$

   Using the pair -1,1 we obtain the ratio 0/-1, while using the pair:

   $$a = \sqrt{p+1}\ ; b = -\sqrt{p+1}$$

   We have the ratio 0/-(p+1) so that such a class of graphs do not have a unique eigen-balanced ratio- although the asymptotic eigen-balanced ratio can be taken as 0.

   Their area with respect to the pair -1,1 is:

   $$\frac{4p}{2p+1}(2)$$

   With respect to the second pair is:

$$\frac{4p}{2p+1}2\sqrt{p+1}$$

Where p=(n-1)/2, so that areas are, respectively:

$$\frac{4(n-1)}{n};\ \ and\ \ \frac{2\sqrt{2}(n-1)}{n}\sqrt{n+1}$$

The bigger of the two give the area of the class of graphs.

5. The p-regular hypercube has ratio (k fixed, n varying, p different from 2k; k not 0):

$$\frac{2p-2k}{p^2-2pk}=2\frac{\frac{\ln n}{\ln 2}-k}{\frac{\ln^2 n}{\ln^2 2}-2k\frac{\ln n}{\ln 2}}=2\ln 2\frac{\ln n-k\ln 2}{\ln^2 n-2k\ln 2\ln n}$$

$$p=\ln n/\ln 2$$

Which has asymptote 0.

Its area using pair p and p-2 (k=1) involves the integral of this ratio (multiplied by p):

$$\int\frac{2p-2}{p^2-2p}d(2^p)=2\ln 2\int\frac{\ln n-\ln 2}{\ln^2 n-2\ln 2\ln n}dn\ \ (*)$$

$$u=\ln^2 n-2\ln 2\ln n\Rightarrow du=[\frac{2\ln n}{n}-\frac{2\ln 2}{n}]dn$$

$$=\frac{2}{n}[\ln n-\ln 2]dn$$

$$\ln^2 n-2\ln 2\ln n-u=0\Rightarrow \ln n=\frac{2\ln 2\pm\sqrt{(2\ln 2)^2+4u}}{2}$$

$$\Rightarrow \ln n=\ln 2+\sqrt{\ln^2 2+u}\Rightarrow n=e^{\ln 2+\sqrt{\ln^2 2+u}}$$

So that (*) becomes:

$$\ln 2\int [\frac{n}{u}]du = \ln 2\int \frac{n}{u}du = \ln 2\int \frac{e^{\ln 2+\sqrt{\ln^2 2+u}}}{u}du$$

$$u = \ln^2 2\sinh^2 t;\ \ t > 0 \Rightarrow \ln 2\int \frac{e^{\ln 2+\cosh t\ln 2}}{\sinh^2 t}2\sinh t\cosh t dt =$$

$$2\ln 2 e^{\ln 2}\int \frac{e^{\cosh t\ln 2}}{\sinh t}\cosh t dt$$

$$put\ \ v = \ln 2\cosh t; \Rightarrow v > \ln 2;\ \ dt = \frac{dv}{\ln 2\sinh t} \Rightarrow$$

$$[2\ln 2]e^{\ln 2}\int \frac{e^v}{(\sinh t)}\frac{v}{(\ln 2)}\frac{dv}{(\ln 2\sinh t)}$$

$$= W\int e^v \frac{v}{v^2 - \ln^2 2}dv;\ \ if\ \ (*)\ \ v \geq \frac{1+\sqrt{1+\ln^2 2}}{2} \approx 1.1\ \ (*)\ \ then\ \ v^2 - \ln^2 2 \geq v$$

$$\Rightarrow W\int e^v \frac{v}{v^2 - \ln^2 2}dv \leq W\int e^v dv$$

$$= [2\ln 2]e^{\ln 2}e^{\ln 2\cosh t} + c = [2\ln 2]e^{\ln 2}e^{\sqrt{\ln^2 2+u}} + c$$

$$= (2\ln 2)n + c' \Rightarrow with : average\ \ \deg = p = \frac{\ln n}{\ln 2}$$

$$\Rightarrow area \leq 2n\ln n + c$$

Provided:

$$\ln^2 n - 2\ln 2\ln n = u = v^2 - \ln^2 2\ \ \geq 0.73\ \ subject\ \ to\ \ (*) \Rightarrow n \geq 8 = 2^3$$

6. The q-cliqued graphs discussed below have the (unique) eigen-balanced ratio of

   $$\frac{-1}{1-q}$$

   Which depends on q (its size is n=q.q+1), and hence the size of the graph, and which tends to 0 as the size of the structure becomes increasingly large so that its asymptotic ratio is 0 and its eigen-balanced area is:

   $$A(G)^{-1,1-q} = q.\left|\int \frac{1}{\sqrt{n-1}-1}dn\right| = q\left|\int \frac{2udu}{u-1}\right| =$$
   $$q\int [2\frac{u-1}{u-1} + \frac{2}{u-1}]du = \sqrt{n-1}(2\sqrt{n-1} + 2\ln\left|\sqrt{n-1}-1\right| + c)$$

   When n=1 we have A=0 so that c=0.

CONJECTURE 1

The only class of regular graphs which are neither sum nor product eigen-pair balanced are cycles.

THEOREM 2

If a class of graphs are eigen-bi-balanced with respect to the pair a,b, which are conjugate pairs arising from the quadratic:

$$\lambda^2 + s\lambda + t'$$

with at least one of a,b positive and of the form n+c (and integer with c negative), and the ratio r is a function of n, then t' is negative and the eigen-pair asymptote lies on the interval [-1,0].

Proof

Let the pair a,b arise from the quadratic:

$$\lambda^2 + s\lambda + t';\ \ t' > 0 \Rightarrow roots : a,b = \frac{-s \pm \sqrt{s^2 - 4t'}}{2};\ \ s \geq 2\sqrt{t'} \Rightarrow t', s \geq 0$$

$$\Rightarrow a,b\ \ both < 0;\ \ contrad.\ \ let\ \ t' = -t\ \ ; t > 0 \Rightarrow$$

$$a,b \leq n-1; a+b = -s; ab = -t \Rightarrow r = \frac{s}{t}$$

If a=-b then the ratio a+b/ab is 0. If a and b are both fixed constants then the ratio is not a function of n. From above, the ratio is s/t. If t =f(n) =O(n), and s is a fixed constant, then the asymptote is 0. Since s =a+b is a function of n so will t be a function of n. If both a and b are functions of n then a+b has

$$O(n^p)\ \ and\ \ ab\ \ has\ \ O(n^q) : q \geq p \Rightarrow asymp\ \ (r) = 0$$

Thus a=n+c>0 and b =k, k negative. Then:

$$If\ \ s = n + c';\ \ t = kn + c'';\ \ (k<0);\ \ c', c''\ \ are\ const. \Rightarrow asymp\ \ (r) = \frac{1}{k} < 0$$

.

Since a is an integer, b=k must be an integer too so that:

$$k \leq -1 \Rightarrow asymp(r) \geq -1 \Rightarrow asymp(r) \in [-1,0] = I$$

We have equality at the end the left hand end point of the interval for the complete graph (-1) since the quadratic for the complete graph is (see theorem 5):

$$\lambda^2 - (n-2)\lambda - (n-1) \Rightarrow \lambda = \frac{(n-2) \pm \sqrt{(n-2)^2 + 4(n-1)}}{2}$$

The interval [-1,0] is more convenient if it is a positive interval: we define the *eigen-pair density* of a class of eigen-bi-balanced graphs with asymptote r as:

$$\Omega_r(\Im) = |asymp(r)|$$

So that the complete graph has eigen-pair density 1, which we propose is the *largest* density of all possible eigen-bi-balanced graphs (the maximum denity of a class of graphs will be the largest of its densities over all its possible ratios) :

CONJECTURE 2

The density of eigen-bi-balanced classes of graphs lie on the interval [0,1] with the largest density that of complete graphs, which equals 1.

THEOREM 3

The eigen-balanced ratio areas of complete bipartite graphs, wheel graphs, the star graph with rays of length 2, are each bounded above by the area of the complete graph.

Proof

For the complete graph and the split complete bipartite graph, the areas are, respectively:

$$(n-1)(n-\ln(n-1)); \ \frac{n^2}{2}$$

Replacing n with:

$$e^s + 1 = n \ \ for \ \ some \ \ s$$

In the former yields:

$$(e^s)(e^s - (s-1)) = e^{2s} - e^s(s-1)$$

Now:

$$s - 1 \le \frac{e^{2s}}{2} - e^s - \frac{1}{2} \Rightarrow$$

$$(e^s)(e^s - (s-1)) = e^{2s} - e^s(s-1) \ge e^{2s} - (\frac{e^{2s}}{2} - e^s - \frac{1}{2}) = \frac{(e^s+1)^2}{2}$$

Which proves our result.

If the eigenvectors:

$$v_1, v_2$$

associated with the eigen-pair:

a,b

giving rise to the sum and product eigen-balanced ratios, have *unit length* then we have the *matrix eigen-balanced ratio equation:*

$$\frac{v_1{}^t A v_1 + v_2{}^t A v_2}{v_1{}^t A v_1 v_2{}^t A v_2} = \frac{a+b}{ab}$$

Note that cycles are neither sum nor product eigen-pair balanced, while the class of dumbbell graphs consisting of two copies of n-cylces joined by a single edge, fall into this same category.

THEOREM 4

If a class of graphs has eigen-balanced ratio

$\frac{a+b}{a.b} = r(a\Im b) = r$ then

$ar \neq 1 \;\; and \;\; br \neq 1$

So that if r is non-zero, the elements of the eigen pair a, b cannot both be 1/r.

Proof.

Let:

$$\frac{a+b}{a.b} = r(a\Im b) = r$$
$$\Rightarrow a+b = rab$$

If we let ab=y we get:

$$a+b = rab = ry \;\; and$$

$$a + \frac{y}{a} = ry \Rightarrow a^2 + y = ray$$

$$\Rightarrow y = \frac{-a^2}{1-ar} = ab$$

$$\Rightarrow b = \frac{a}{ar-1}$$

Thus $ar \neq 1$; swopping the roles of a and be we get the desired result.

Taking the join of the complement of the complete graph on 2 vertices and the complete graph on n vertices, we see from [I[12]] that this resulting graph has the conjugate pair of eigenvalues:

$$\frac{(n-1)\pm\sqrt{(n-1)^2+8n}}{2}$$

So that their eigen-balanced ratio is

$$\frac{n-1}{-2n}$$

Which tends to -1/2.

The following theorem can be derived from [12]:

THEOREM 5

Define the class of graphs:

$$\Im=\overline{K_k}\oplus K_n$$

Where m is fixed, and n, which varies and is greater than 1. Then this class has eigen-pair $\frac{(n-1)\pm\sqrt{(n-1)^2+4nk}}{2}$ with asymptotic eigen-balanced ratio:

$\frac{-1}{k}$ and area:

$$\left(\frac{n(n-1)+2kn}{n+k}\right)\left(\frac{n}{k}-\frac{1}{k}\ln(n+1)\right)$$

Proof

The eigenvalue conjugate pair associated with this join is:

$$\frac{(n-1) \pm \sqrt{(n-1)^2 + 4nk}}{2}$$

The sum is (n-1) and product is –kn which yields the result as n becomes increasingly large. Their eigen-balanced area is (with average degree D):

$$D\left|\int \frac{1-n}{kn} d(n+k)\right| = \frac{n(n-1)+2kn}{n+k}\left|\int [\frac{1}{kn} - \frac{1}{k} d(n)]\right| = D(\frac{n}{k} - \frac{1}{k}\ln n + c)$$

With k=1, the area must be that of the complete graph on n+1 vertices which is (n)((n+1)-ln(n)) so that c=1- hence its area is:

$$(\frac{n(n-1)+2kn}{n+k})(\frac{n}{k} - \frac{1}{k}(\ln n) + 1)$$

Alternatively, we could have formed the join (with n vertices):

$$\Im = \overline{K_k} \oplus K_{n-k}$$

which has conjugate pairs: $\dfrac{n-k-1 \pm \sqrt{(n-k-1)^2 + 4k(n-k)}}{2}$

with ratio: $\dfrac{k+1-n}{k(n-k)} = \dfrac{-(n-k)+1}{k(n-k)}$ which has asymptote as before: $-\dfrac{1}{k}$

The area is: $D\left|\int [-\dfrac{1}{k} dn + \dfrac{1}{k(n-k)}] dn\right| = D(\dfrac{n}{k} - \dfrac{1}{k}\ln(n-k) + c)$

$$(\frac{(n-k)(n-k-1)+2k(n-k)}{n})(\frac{n}{k} - \frac{1}{k}\ln(n-k) + c)$$

When k=1 we must get area of complete graph so that c=0.

CONJECTURE 3

The maximum eigen-balanced ratio area of classes of graphs on at least 6 vertices is that of the complete graph: n-ln(n-1)

Note that for wheels with n spokes, the eigen-pair is:

$a = \frac{2+\sqrt{4+4n}}{2}$; $b = \frac{2-\sqrt{4+4n}}{2}$ so that;

$|a+b| + |a.b| = n+2$

For the join of two cycles of length n, there exist the pair of eigenvalues: (see [12])

$2 \pm n$ so that:

$|a+b| + |a.b| = n^2$

Also, for q-cliqued graphs discussed below with eigen-pair :

$a = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}; b = \frac{-1-\sqrt{1+4(q-1)}}{2};$

$|a+b| + |a.b| = q$

This suggests the following:

CONJECTURE 4

If a class of non-complete graphs, is eigen-bi-balanced with associated eigen-pair a,b of a member of the class, the member having maximum(minimum) degree m(n) respetively, then

$(i)\ if\ a+b \neq 0$ then

$$|a+b|+|a.b|\le mn$$

$(ii)\ if\ \ a+b=0$ then

$$\frac{|a.b|}{n}\le m+1$$

There is much research on the *energy* of a graph -it is related to the total π-electron energy in a molecule represented by a (molecular) graph.
The *energy of a graph* with adjacency matrix A with eigenvalues $\lambda_1^A \ge \lambda_2^A \ge ... \ge \lambda_n^A$ is:

$$E^A = \sum_{i=1}^{n} |\lambda_i^A|$$

How does the energy of a graph behave with respect to the asymptote associated with eigen-pair a,b ?

The *r-asymptotic eigen-balanced matrix* $C_r^\infty = (c_{ij})$,associated with the adjacency matrix $A=(a_{ij})$ of G on n vertices with an asmptotoic eigen-balanced ratio r, is defined as:

$$c_{ij} = \left\langle \begin{array}{l} a_{ij}; i \ne j \\ \deg(a_j) + r; i = j \end{array} \right.$$

If G is k-regular and A has eigenvalues: $k=\lambda_1 \ge \lambda_2 \ge ... \ge \lambda_n$,

then the eigenvalues of C_r^∞ are:

$$2k+r=\lambda_1 \ge \lambda_2 + k + r \ge ... \ge \lambda_n + k + r$$

In particular, if r=0 the C_0^∞ is the Signless Laplacian matrix (see [spectra]).

The *energy* of the *r-asymptotic eigen-balanced matrix* $C_r^{\infty} = C$, associated with the graph G on n vertices and m edges, with eigenvalues:

$\lambda_1^C \geq \lambda_2^C \geq \ldots \geq \lambda_n^C$ is (see[14]) is:

$$E^{C_r^{\infty}} = \sum_{i=1}^{n} \left| \lambda_i^C - \frac{2m}{n} \right|$$

If r=0 then we get the energy of the Signless Laplacian matrix.

If r is not zero, such as the complete graph G on n vertices and $\frac{n(n-1)}{2}$ edges, then its (-1)-asymptotic eigen-balanced energy is found as follows: the eigenvalues of G are:

$(n-1)^1;(-1)^{n-1}$ so that the eigenvalues of C_{-1}^{∞} are:

$$(n-1+(n-1)-1)^1;(-1+(n-1)-1)^{n-1}$$

$(2n-3)^1;(n-3))^{n-1}$ so that the r-asymptotic eigen-balanced energy of G (with eigen pair a,b) is:

$$E^{C_{-1}^{\infty}} = \sum_{i=1}^{n} \left| \lambda_i^C - \frac{2m}{n} \right|$$

$$|n-2| + (n-1)|-2| = 3n-4$$

This energy is greater than the normal energy 2n-2 of a complete graph on a large number of vertices. This asymptotic energy can be regarded as the eigen-pair balanced energy associated with the graph G as its size becomes increasingly large.

2.THE CONSRTUCTION OF Q-CLIQUED GRAPHS

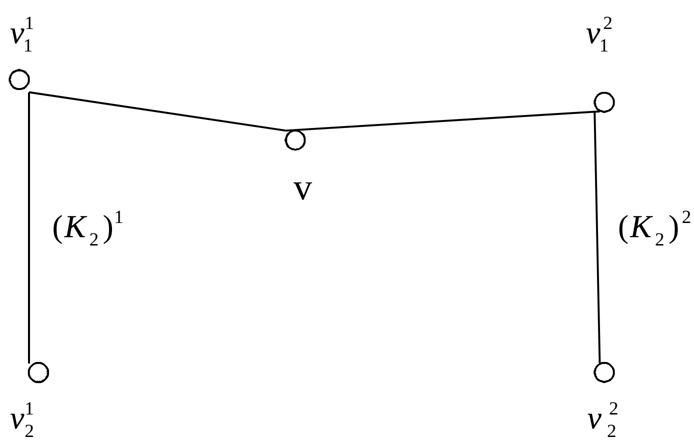
Construction of Adjacency matrix for q-clique design graphs, ${G_{Kq}}^*$, for q $\geq$2 :

For q=2, take 2 copies of K_2, namely $(K_2)^i;\ i=1,2.$ together with a single vertex v. Join v to $v_1^i;\ i=1,2$, so that v has degree 2.

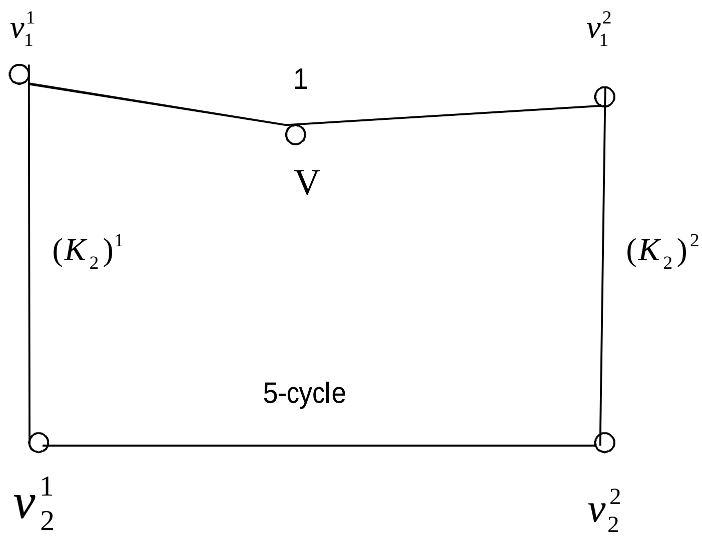
Generally:

Label the vertices of each copy of $(K_q)^i$ as $v_1^i, v_2^i, \ldots, v_q^i$

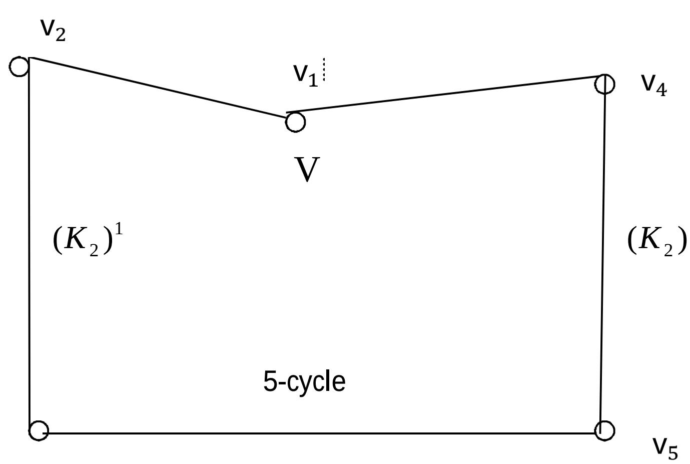
Join v to $v_1^i;\ i=1,2,\ldots q.$ so that v has degree q generally.



i.e. join vertices v_2^1 *and* v_2^2 of $(K_2)^1$ and $(K_2)^2$ to form a 5-cycle.



Label vertex v as vertex v_1, and then for each sub-clique, label the vertices starting from $v_1^1 = v_2, v_2^1 = v_3$, and $v_1^2 = v_4, v_2^2 = v_5$.



v3

Then the 5x5 adjacency matrix of ${G_{K_2}}^*$, where the rows are $v_1 \ldots v_5$ and the columns are $v_1 \ldots v_5$ is:

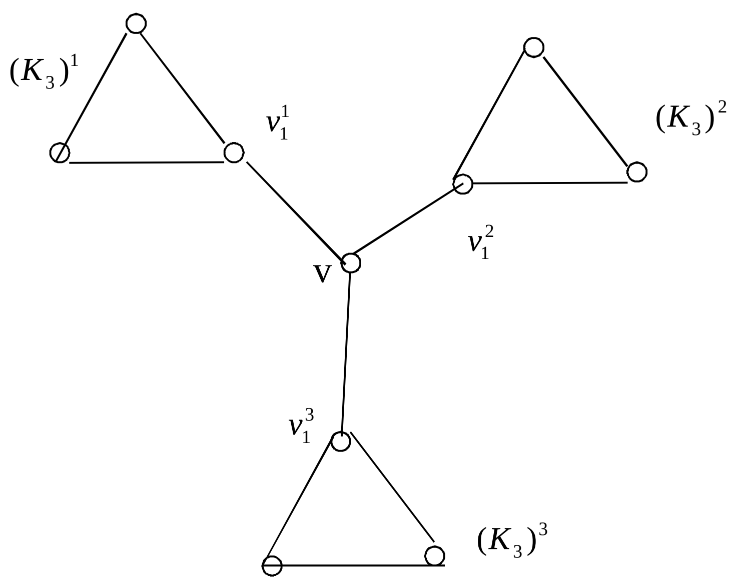
$$\begin{bmatrix} 0 & 1 & 0 & 1 & \\ 1 & 0 & 1 & 0 & \\ & 1 & 0 & & 1 \\ 1 & & & 0 & 1 \\ & & 1 & 1 & 0 \end{bmatrix}$$

The polynomial of ${G_{K_2}}^*$ is

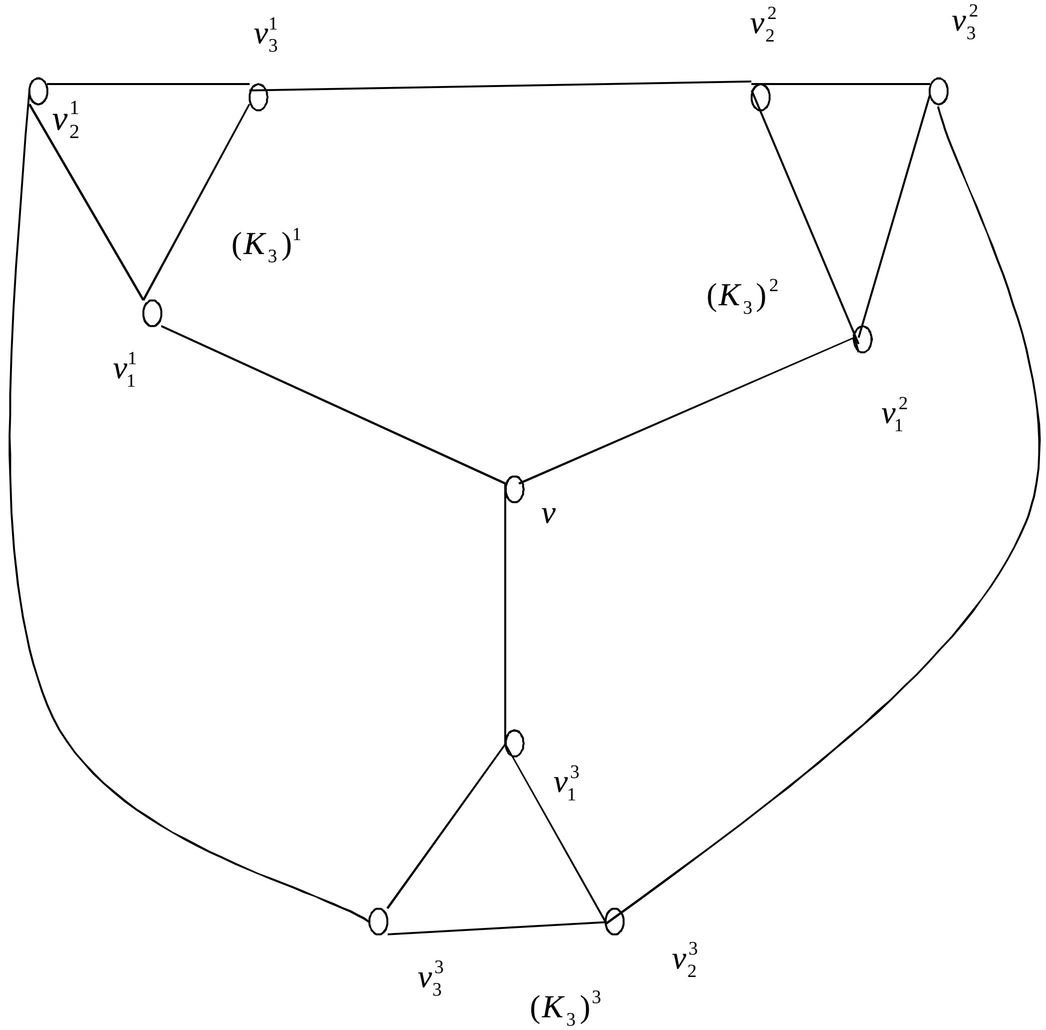
$$-\lambda^5 + 5\lambda^3 - 5\lambda + 2$$

The eigenvalues of this adjacency matrix are: 2 once; $(\frac{-1+\sqrt{5}}{2})^2$; $(\frac{-1-\sqrt{5}}{2})^2$

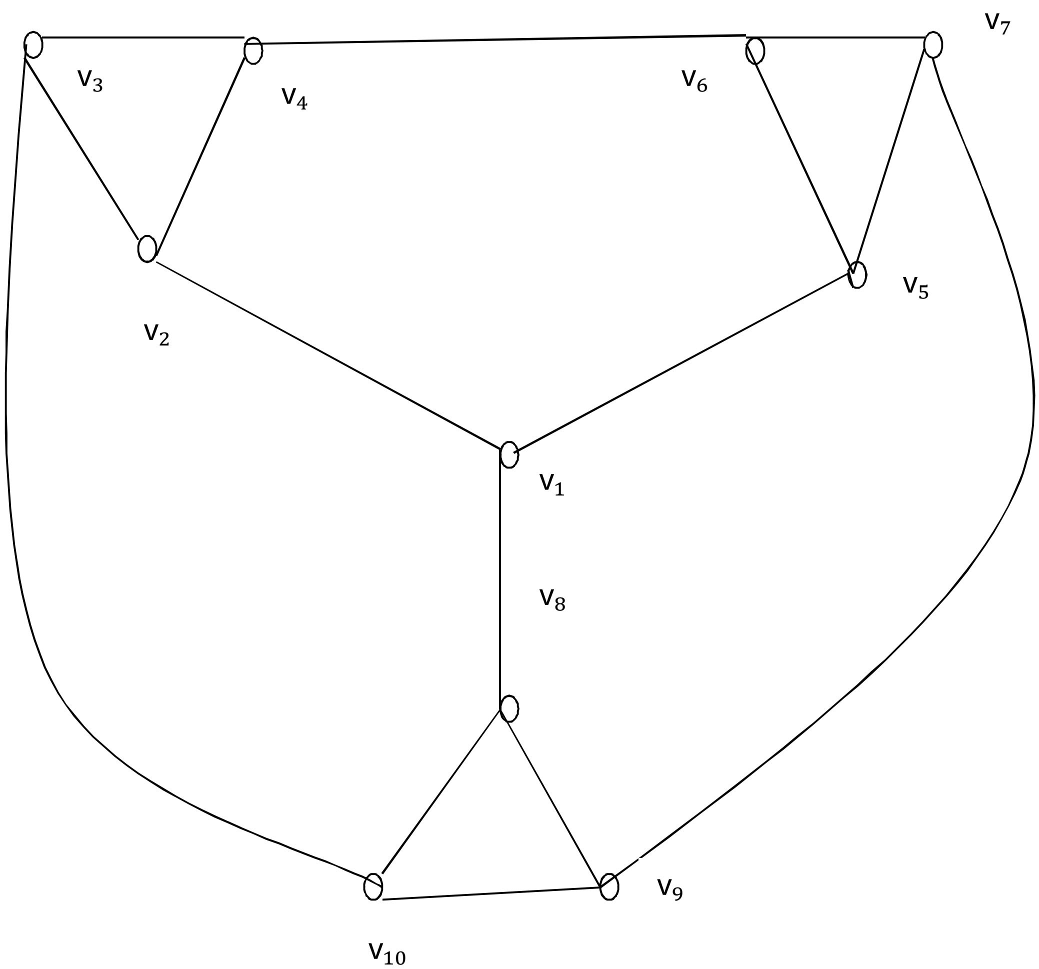
For q=3, ${G_{K_3}}^*$: we take 3 copies of K_3, namely $(K_3)^1$, $(K_3)^2$, and $(K_3)^3$ together with a single centre vertex v. Join v to v_1^i; $i = 1,2,3.$:



Join the remaining vertices of the 3 copies of K_3 to form 3 5-cycles. i.e. v_2^1 *and* v_2^2 ; v_3^2 *and* v_2^3; v_3^3 *and* v_3^1



Label vertex v as vertex v_1, and then for each sub-clique, label the vertices starting from $v_1^1 = v_2, v_2^1 = v_3, v_3^1 = v_4$ and $v_1^2 = v_5, v_2^2 = v_{6,} v_3^2 = v_7$ and $v_1^3 = v_8, v_2^3 = v_{9,} v_3^3 = v_{10}$.



Then the 10x10 adjacency matrix of G_{K3}^{*}, where the rows are $v_1 \ldots v_{10}$ and the columns are $v_1 \ldots v_{10}$ is:

$$\begin{bmatrix} 0 & 1 & 0 & & 1 & & & 1 & & \\ 1 & 0 & 1 & 1 & & & & & & \\ & 1 & 0 & 1 & & & & & & 1 \\ & 1 & 1 & 0 & & 1 & & & & \\ 1 & & & & 0 & 1 & 1 & & & \\ & & & 1 & 1 & 0 & 1 & & & \\ & & & & 1 & 1 & 0 & & 1 & \\ 1 & & & & & & & 0 & 1 & 1 \\ & & & & & & 1 & 1 & 0 & 1 \\ & & 1 & & & & & 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation of the adjacency matrix for q=3 is:

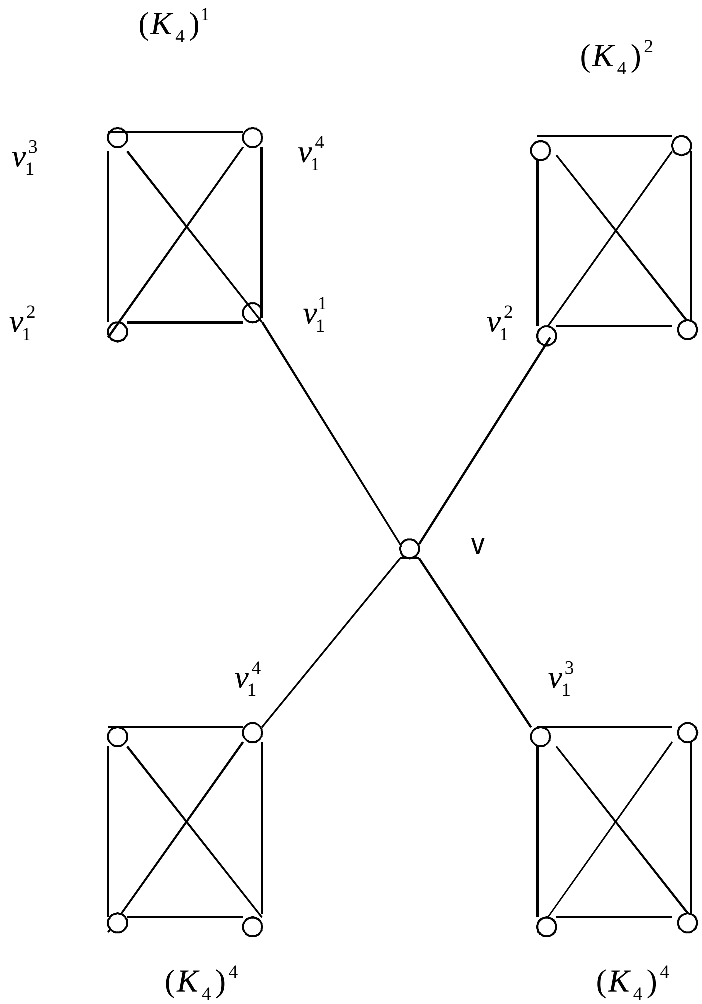
$$\lambda^{10} - 15\lambda^{8} - 6\lambda^{7} + 75\lambda^{6} + 48\lambda^{5} - 144\lambda^{4} - 114\lambda^{3} + 75\lambda^{2} + 68\lambda + 12$$

The eigenvalues of this adjacency matrix are:

3, 1, -2, -2, 1.879, 1.879, -0.347, -0.347, -1.532, -1.532

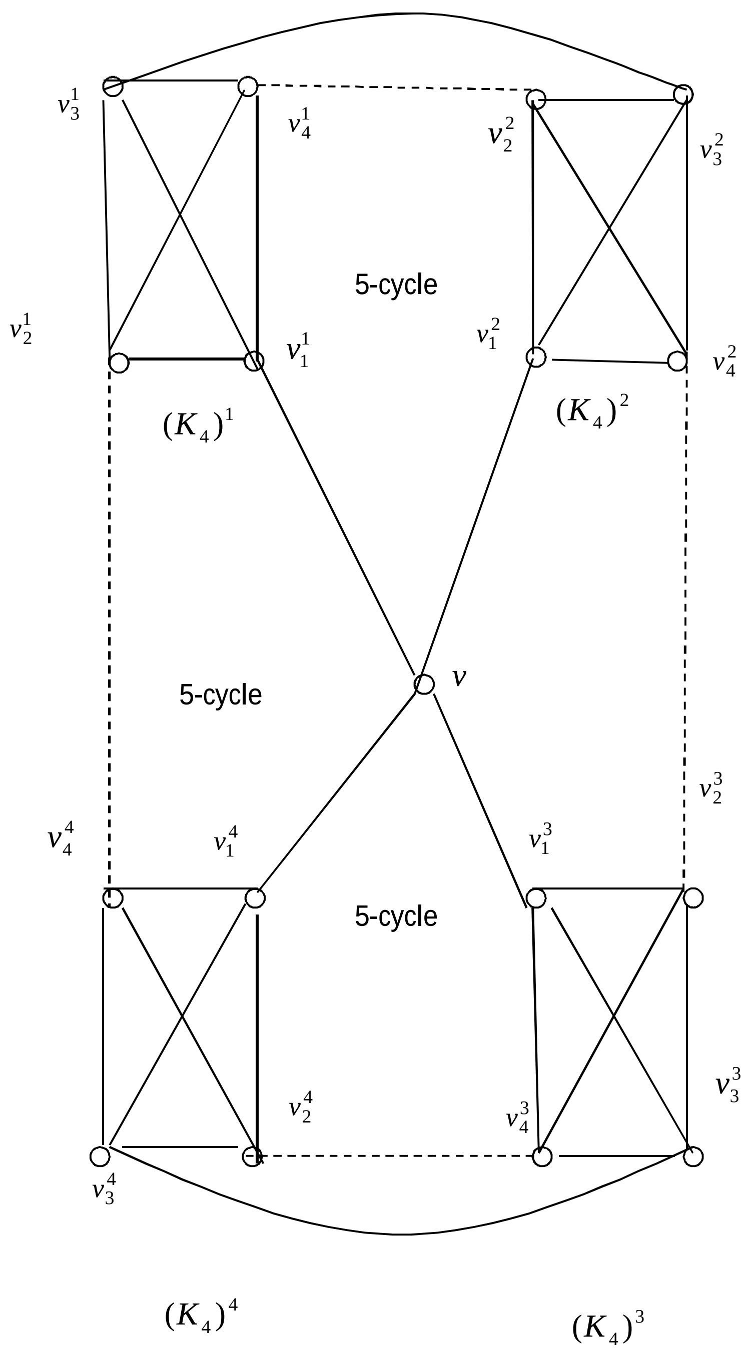
$$(\frac{-1+\sqrt{9}}{2})^2; \ (\frac{-1-\sqrt{9}}{2})^2$$

The graph ${G_{K_4}}^*$: For q=4, take 4 copies of K_4, namely $(K_4)^1$, $(K_4)^2$, $(K_4)^3$,and $(K_4)^4$ together with a single centre vertex v. Join v to $v_1^i;\ i = 1,2,3,4.$ Label each of the vertices within each copy of K_4 anti-clockwise, starting with $v_1^i; v_2^i; v_3^i; v_4^i; i = 1,2,3,4.$

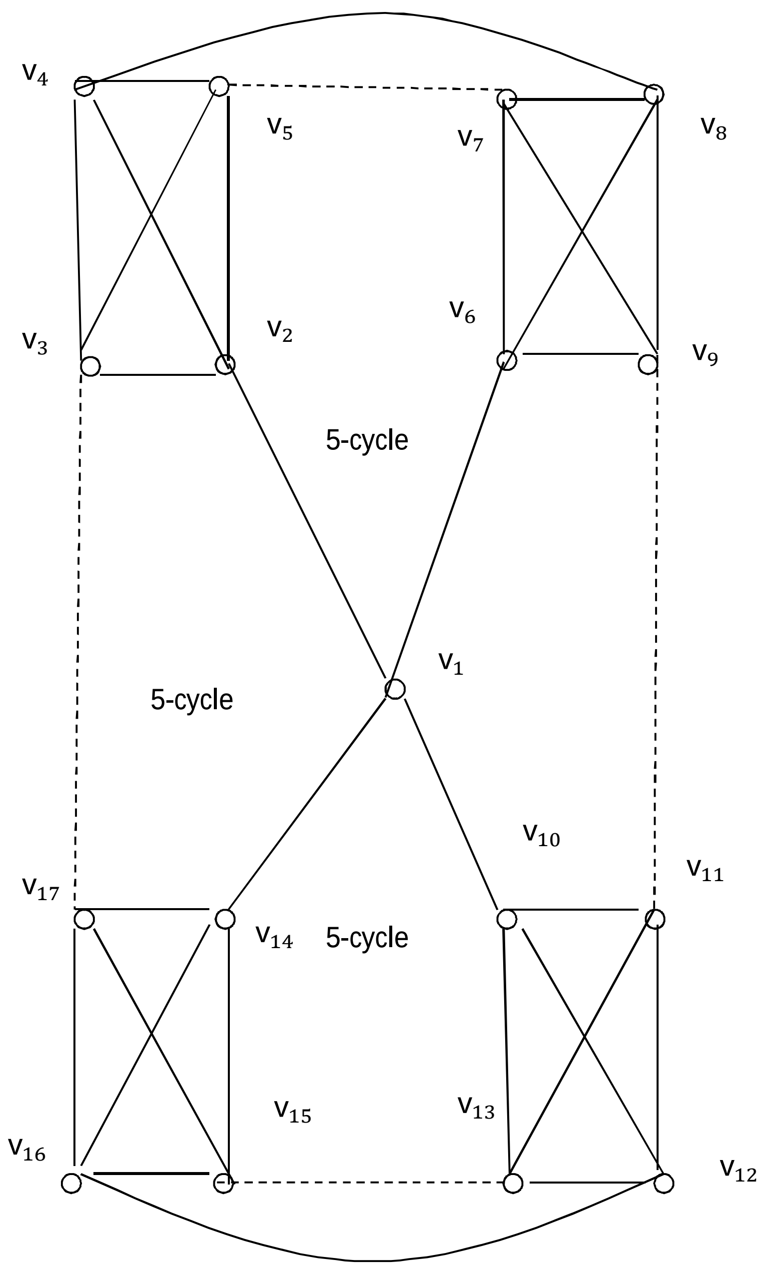

$(K_4)^1$
$(K_4)^2$
v_1^3
v_1^4
v_1^2
v_1^1
v_1^2
v
v_1^4
v_1^3
$(K_4)^4$
$(K_4)^4$

Join vertices v_i^4 to v_{i+1}^2 for 1≤i≤n, where $v_{n+1}^2 = v_1^2$, to form 4 5-cycles.

Join vertex v_i^3 to v_{i+1}^3 for 1≤i≤n, where i is odd.



Label vertex v as vertex v_1, and then for each sub-clique, label the vertices clockwise for each sub-clique, starting from $v_1^1 = v_2, v_2^1 = v_3, v_3^1 = v_4, v_4^1 = v_5$ and $v_1^2 = v_6, v_2^2 = v_7, v_3^2 = v_8, v_4^2 = v_9$ and $v_1^3 = v_{10}, v_2^3 = v_{11}, v_3^3 = v_{12}, v_4^3 = v_{13}$, and $v_1^4 = v_{14}, v_2^4 = v_{15}, v_3^4 = v_{16}, v_4^4 = v_{17}$,



Then the 17x17 adjacency matrix of $G_{K4}{}^{*}$, where the rows are $v_1 \dots v_{17}$ and the columns are $v_1 \dots v_{17}$ is:

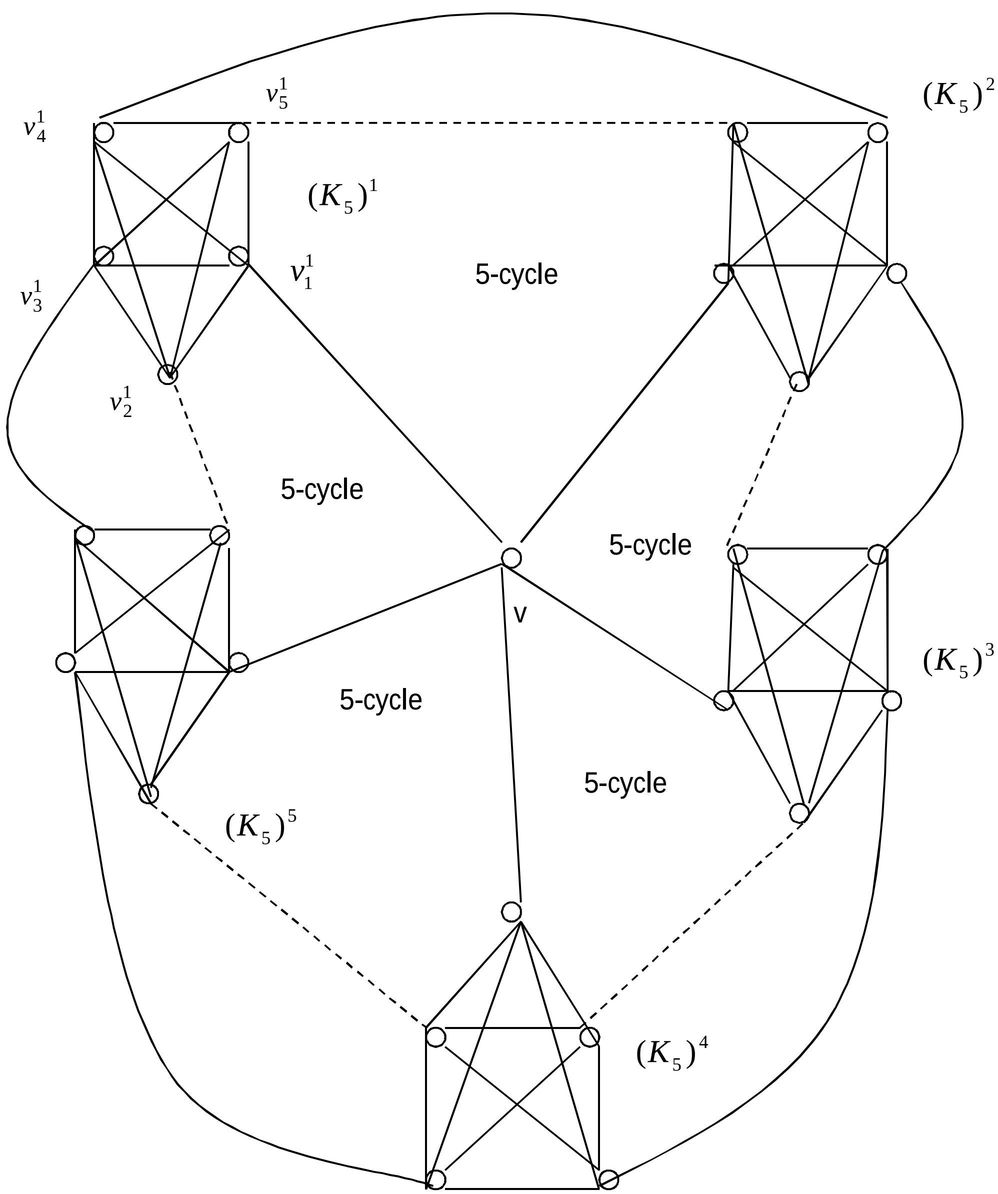
$$\begin{bmatrix}
0 & 1 & & & & 1 & & & & 1 & & & & 1 & & & \\
1 & 0 & 1 & 1 & 1 & & & & & & & & & & & & \\
 & 1 & 0 & 1 & 1 & & & & & & & & & & & & 1 \\
 & 1 & 1 & 0 & 1 & & & 1 & & & & & & & & & \\
 & 1 & 1 & 1 & 0 & & 1 & & & & & & & & & & \\
1 & & & & & 0 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 1 & 1 & 0 & 1 & 1 & & & & & & & & \\
 & & & 1 & & 1 & 1 & 0 & 1 & & & & & & & & \\
 & & & & & 1 & 1 & 1 & 0 & & 1 & & & & & & \\
1 & & & & & & & & & 0 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 1 & 1 & 0 & 1 & 1 & & & & \\
 & & & & & & & & & 1 & 1 & 0 & 1 & & & 1 & \\
 & & & & & & & & & 1 & 1 & 1 & 0 & & 1 & & \\
1 & & & & & & & & & & & & & 0 & 1 & 1 & 1 \\
 & & & & & & & & & & & & 1 & 1 & 0 & 1 & 1 \\
 & & & & & & & & & & & 1 & & 1 & 1 & 0 & 1 \\
 & & 1 & & & & & & & & & & & 1 & 1 & 1 & 0
\end{bmatrix}$$

All blank entries are zero. The eigenvalues for this adjacency matrix are:

-2.303, 1.303, 4 , 3.403, 2.935, 2.303, -0.463, -0.684, -1.303, -1.719, -1.473, -2, -2, -2, 0, 0, 0

$$(\frac{-1+\sqrt{13}}{2})^2; \quad (\frac{-1-\sqrt{13}}{2})^2$$

For q=5, $G_{K_5}{}^*$ is :


v_5^1
$(K_5)^2$
v_4^1
$(K_5)^1$
v_1^1
5-cycle
v_3^1
v_2^1
5-cycle
5-cycle
v
$(K_5)^3$
5-cycle
5-cycle
$(K_5)^5$
$(K_5)^4$

Label vertex v as vertex v_1, and then for each sub-clique, label the vertices clockwise starting from $v_j^1, 1 \leq j \leq 5$. The resultant 26x26 adjacency matrix of G_{K5}^*, where the rows are $v_1 \ldots v_{26}$ and the columns are $v_1 \ldots v_{26}$ is:

$$
\left[\begin{array}{cccccccccccccccccccccccccc}
0 & 1 & & & & & 1 & & & & & 1 & & & & & 1 & & & & & 1 & & & & \\
1 & 0 & 1 & 1 & 1 & 1 \\
& 1 & 0 & 1 & 1 & 1 & & & & & & & & & & & & & & & & & & & & 1 \\
& 1 & 1 & 0 & 1 & 1 & & & & & & & & & & & & & & & & & & & 1 \\
& 1 & 1 & 1 & 0 & 1 & & & 1 \\
& 1 & 1 & 1 & 1 & 0 & & 1 \\
1 & & & & & & 0 & 1 & 1 & 1 & 1 \\
& & & & & 1 & 1 & 0 & 1 & 1 & 1 \\
& & & & 1 & & 1 & 1 & 0 & 1 & 1 \\
& & & & & & 1 & 1 & 1 & 0 & 1 & & & 1 \\
& & & & & & 1 & 1 & 1 & 1 & 0 & & 1 \\
1 & & & & & & & & & & & 0 & 1 & 1 & 1 & 1 \\
& & & & & & & & & & 1 & 1 & 0 & 1 & 1 & 1 \\
& & & & & & & & & 1 & & 1 & 1 & 0 & 1 & 1 \\
& & & & & & & & & & & 1 & 1 & 1 & 0 & 1 & & & 1 \\
& & & & & & & & & & & 1 & 1 & 1 & 1 & 0 & & 1 \\
1 & & & & & & & & & & & & & & & & 0 & 1 & 1 & 1 & 1 \\
& & & & & & & & & & & & & & & 1 & 1 & 0 & 1 & 1 & 1 \\
& & & & & & & & & & & & & & 1 & & 1 & 1 & 0 & 1 & 1 \\
& & & & & & & & & & & & & & & & 1 & 1 & 1 & 0 & 1 & & & 1 \\
& & & & & & & & & & & & & & & & 1 & 1 & 1 & 1 & 0 & & 1 \\
1 & & & & & & & & & & & & & & & & & & & & & 0 & 1 & 1 & 1 & 1 \\
& & & & & & & & & & & & & & & & & & & & 1 & 1 & 0 & 1 & 1 & 1 \\
& & & & & & & & & & & & & & & & & & & 1 & & 1 & 1 & 0 & 1 & 1 \\
& & & 1 & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & 0 & 1 \\
& & 1 & & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & 1 & 0
\end{array}\right]
$$

All blank entries are zero.

Eigenvalues q=5: -2,562;1,562;5,00;4,381;4,381;3,447;3,447-1,662;-1,662;-1,272;-1,272;-0,719;-0,719;-0,174;-0,174;0;0;0;0;0;-2;-2;-2;-2; -2.

$(\frac{-1+\sqrt{17}}{2})^2$; $(\frac{-1-\sqrt{17}}{2})^2$

The general construction of the $(1 + n^2)$ x $(1 + n^2)$ adjacency matrix of $G_{Kn}{}^{*}$, where the rows are v_1 ... v_{1+n^2} and the columns are v_1 ... v_{1+n^2} is as follows:

$a_{i,i} = 0$, $1 \le i \le (n^2 + 1)$

Join v to v_i^1:

$a_{1,1+\lambda n+1} = 1$, $0 \le \lambda \le n-1$

$a_{1+\lambda n+1,1} = 1$, $0 \le \lambda \le n-1$

Sub-cliques:

$a_{1+\lambda n+k,1+\lambda n+l} = 0$, $0 \le \lambda \le n-1$, k=1,2,…, n, l= 1,2,…,n, k=l

$a_{1+\lambda n+k,1+\lambda n+l} = 1$, $0 \le \lambda \le n-1$, k=1,2,…, n, l= 1,2,…,n, k≠l

v_i^n joins to v_{i+1}^2:

$a_{1+\lambda n+n,1+(\lambda+1)n+2} = 1$, $0 \le \lambda \le n-1$

$a_{1+(\lambda+1)n+2,1+\lambda n+n} = 1$, $0 \le \lambda \le n-1$

v_i^j joins to v_{i+1}^{j-1}: 4≤j≤n-1, j even:

$a_{1+\lambda n+j,1+(\lambda+1)n+(j-1)} = 1$, $0 \le \lambda \le n-1$, 4≤j≤n-1, j even

$a_{1+(\lambda+1)n+(j-1),1+\lambda n+j} = 1$, $0 \le \lambda \le n-1$, 4≤j≤n-1, j even

v_i^j joins to v_{i+1}^j: j=n-1, n even, i is odd:

$a_{1+\lambda n+j,1+(\lambda+1)n+j}$= 1,   0 ≤λ≤ n-1, j=n-1, n even, λ even

$a_{1+(\lambda+1)n+j,1+\lambda n+j}$= 1, 0 ≤λ≤ n-1, j=n-1, n even, λ even

If for $a_{i,j}$ i>(n^2 + 1), then i=i-n^2, and if for $a_{i,j}$ j>(n^2 + 1), then j=j-n^2

$a_{i,j}$ = 0, 1≤i≤ (n^2 + 1), 1≤j≤ (n^2 + 1), otherwise.

3.THE CONJUGATE EIGEN-PAIR OF Q-CLIQUED GRAPHS

In this section we show that the cubic:

$\lambda^3 - \lambda^2(q-1) - q\lambda + q(q-1) - \lambda(q-1)$ Is a factor of the characteristic equation determined by $A\underline{x} = \lambda\underline{x}$ where A is the adjacency matrix of a q-cliqued graph constructed in section 2.

The conjugate pairs arise out if the “tightness” of the connection between two adjacent cliques – for convention we chose the first and last clique:

3.1 Vertex notation convention:

First vertex (central vertex), second vertex, anchor vertex (vertex of last clique joined to first), {generating vertices on second last clique=vertices of second last clique adjacent to vertices in the last clique}, last vertex, ,third vertex and switching vertices (third and second last vertices whose sum is 0) = (respectively):

$$x_1, x_2, x_a, \{x_{k1}; x_{k2}, ..., x_{kt}\,(t = \frac{q-1}{2}; q\ odd\ \ or\, t = \frac{q}{2}; q\ even)\}, x_l, x_3, x, x'$$

Generating set =

$$S = T \cup T^{'} = \{x_1, x_2\} \cup \{x_{k1}; x_{k2}, ..., x_{kt}; (t = \frac{q-1}{2}; q\ odd\ \ or\, t = \frac{q}{2}; q\ even)\}$$

If $S = \{x_1, x_2, ..., x_k\}$; we define: $\sum S = \sum_1^k x_i$

3.2 The two main equations that generate the conjugate eigen-pairs

We use the relationship $A\underline{x} = \lambda \underline{x}$ to create the two equations:

$$\lambda \sum S = (q-1)\sum S + (q-1)x_l$$
$$\Rightarrow \sum S = \frac{(q-1)x_l}{(\lambda-(q-1))}(*)$$

$\sum S = \lambda^2 x_l - q x_l$ $(**)$ which yield the following:

$$\frac{(q-1)\lambda x_l}{\lambda-(q-1)} = \lambda^2 x_l - q x_l; \;\; \lambda \neq q-1$$

So that:

$$(q-1)\lambda = \lambda^2(\lambda-(q-1)) - q(\lambda-(q-1))$$
$$\Rightarrow \lambda^3 - \lambda^2(q-1) - q\lambda + q(q-1) - \lambda(q-1) = 0$$
$$\Rightarrow (\lambda - q)(\lambda^2 + \lambda - (q-1)) = 0$$

This gives us 3 eigenvalues: q and our conjugate pair:

$$\frac{-1 \pm \sqrt{1+4(q-1)}}{2}$$

With sum -1 and product 1-q.

3.3 The case q=2

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$x_2 + x_4 = \lambda x_1$$

$$x_1 + x_3 = \lambda x_2$$

$$x_2 + x_5 = \lambda x_3$$

$$x_1 + x_5 = \lambda x_4$$

$$x_3 + x_4 = \lambda x_5$$

$$x_3 + x_4 = \lambda (x_5)$$

Taking the neighbours of $x_3$and x_4, we get

$$(x_2 + x_5) + (x_1 + x_5) = \lambda (x_3 + x_4)$$

$$(x_1 + x_2) + 2x_5 = \lambda (\lambda x_5)$$

$$(x_1 + x_2) = \lambda^2 x_5 - 2x_5 \ (**)$$

Taking the neighbours of $x_1$and x_2, we get

$$(x_2 + x_4) + (x_1 + x_3) = \lambda(x_1 + x_2)$$

$$(x_1 + x_2) + (x_3 + x_4) = \lambda(x_1 + x_2)$$

$$(x_1 + x_2) + \lambda x_5 = \lambda(x_1 + x_2)$$

$$(\lambda - 1)(x_1 + x_2) = \lambda x_5$$

$$(x_1 + x_2) = \frac{\lambda}{\lambda - 1} x_5 \ (*)$$

Substitute (*) into (**) to get

$$\frac{\lambda x_5}{\lambda - 1} = \lambda^2 x_5 - 2x_5; \ \ \lambda \neq 1$$

$$\lambda = \lambda^2(\lambda - 1) - 2(\lambda - 1)$$

$$\lambda^3 - \lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda^2 + \lambda - 1)$$

Eigenvalue of 2 (degree of graph) and our conjugate pairs!

$$\frac{-1 \pm \sqrt{1+4}}{2}$$

3.4 The case q =3

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix}.$$

$$= \begin{bmatrix} x_2 + x_5 + x_8 \\ x_1 + x_3 + x_4 \\ x_2 + x_4 + x_{10} \\ x_2 + x_3 + x_6 \\ x_1 + x_6 + x_7 \\ x_4 + x_5 + x_7 \\ x_5 + x_6 + x_9 \\ x_1 + x_9 + x_{10} \\ x_7 + x_8 + x_{10} \\ x_3 + x_8 + x_9 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix}$$

$$x_3 + x_8 + x_9 = \lambda x_{10}$$

$$(x_2 + x_4 + x_{10}) + (x_1 + x_9 + x_{10}) + (x_7 + x_8 + x_{10}) = \lambda(\lambda(x_{10})$$

$$x_1 + x_2 + x_4 + x_7 + (x_8 + x_9) = \lambda^2 x_{10} - 3x_{10} \ (**)$$

$$Set \ x_4 = -x_8; x_9 = 0$$

$$Let \ S = (x_1, x_2, and \ x_7)$$

Then the neighbor set S'' of S is a subset of the lhs of (**)

$$(x_2 + x_5 + x_8) + (x_1 + x_3 + x_4) + (x_5 + x_6 + x_9) = \lambda(x_1 + x_2 + x_7)$$

$Set \ x_4 = -x_8, x_9 = 0 \ \ (from \ above)$ and set

$$x_3 = x_1; x_2 = 0; x_6 = 2x_7; x_5 = \lambda x_{10}$$

$$(2x_1 + 2x_2 + 2x_7 + 2x_5) = \lambda(x_1 + x_2 + x_7)$$

$$2(x_1 + x_2 + x_7) + 2\lambda x_{10} = \lambda(x_1 + x_2 + x_7)$$

$$x_1 + x_2 + x_7 = \frac{2\lambda x_{10}}{\lambda - 2} (*)$$

Substitute (*) into (**) to get

$$\frac{2\lambda x_{10}}{\lambda - 2} = \lambda^2 x_{10} - 3x_{10}; \ \ \lambda \neq 1$$

$$2\lambda = \lambda^2(\lambda-2)-3(\lambda-2)$$

$$\lambda^3-2\lambda^2-5\lambda+6=0$$

$$(\lambda-3)(\lambda^2+\lambda-2)$$

Two eigenvalues 3 and

$$\frac{-1\pm\sqrt{1+2.4}}{2}=\frac{-1\pm\sqrt{9}}{2}$$

Eigenvector is:

$$= \begin{bmatrix} x_2+x_5+x_8 \\ x_1+x_3+x_4 \\ x_2+x_4+x_{10} \\ x_2+x_3+x_6 \\ x_1+x_6+x_7 \\ x_4+x_5+x_7 \\ x_5+x_6+x_9 \\ x_1+x_9+x_{10} \\ x_7+x_8+x_{10} \\ x_3+x_8+x_9 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda x_{10}+x_8 \\ 2x_1-x_8 \\ -x_8+x_{10} \\ x_1+2x_7 \\ x_1+3x_7 \\ -x_8+\lambda x_{10}+x_7 \\ 2x_7+\lambda x_{10} \\ x_1+x_{10} \\ x_7+x_8+x_{10} \\ x_1+x_8 \end{bmatrix} = \lambda \begin{array}{c} \begin{bmatrix} x_1 \\ 0 \\ x_1 \\ -x_8 \\ \lambda x_{10} \\ 2x_7 \\ x_7 \\ x_8 \\ 0 \\ x_{10} \end{bmatrix} \end{array} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}$$

we use the generating set $S'=\{x_1,x_2,x_7\} with\ their\ sum = G$ which we found from using equation 10. Now:

$\lambda(x_1+x_2+x_7)=$ using equations 1,2 and 7, noting that the variable x_2 is 0:

$$\lambda x_{10}+x_8+2x_1-x_8+2x_7+\lambda x_{10}=2x_1+2.0+2x_7+2\lambda x_{10}=2G+2\lambda x_{10}=\lambda G$$

$$\Rightarrow G=\frac{2\lambda x_{10}}{\lambda-2}\,(*)$$

We used λx_2 even though it is 0.

Now we get the next equation (this should be first in construction above). Using equation 10:

$\lambda\lambda x_{10} = \lambda x_1 + \lambda x_8$ using 3 (not 1!!!) and 8 in RHS:

$-x_8 + x_{10} + x_1 + x_{10}$ (11) we need an x_7 so we use equation 9 which gives us:

$-x_8 = x_7 + x_{10}$ (12) substituting (12) into (11) gives us:

$$x_1 + x_7 + 3x_{10} = \lambda^2 x_{10} \Rightarrow x_1 + x_2 + x_7 + 3x_{10} = \lambda^2 x_{10} \Rightarrow G = \lambda^2 x_{10} - 3x_{10} (**)$$

We take x_2 =0 here as we do not need λx_2.

3.5 The Case q=4

For q=4:

$$x_3 + x_{14} + x_{15} + x_{16} = \lambda x_{17}$$

$$(x_2 + x_4 + x_5 + x_{17}) + (x_1 + x_{15} + x_{16} + x_{17}) + (x_{13} + x_{14} + x_{16} + x_{17})$$
$$+ (x_{12} + x_{14} + x_{15} + x_{17}) = \lambda(x_3 + x_{14} + x_{15} + x_{16}) \quad (\#)$$

$$x_1 + x_2 + x_4 + x_5 + 2(x_{14} + x_{15} + x_{16}) + x_{12} + x_{13} + 4x_{17} = \lambda(\lambda x_{17})$$

Set $x_4 = x_5 = x_{14} = 0; x_{15} = -x_{16}$

We take the set $S^{'} = \{x_1, x_2, x_{12}, x_{13}\}$, which is a subset of the lhs of (#) and the neighbours of S' to be S''. Then S' is a subset of S''.

$$x_1 + x_2 + x_{12} + x_{13} + 4x_{17} = \lambda^2 x_{17};$$

$$\Rightarrow x_1 + x_2 + x_{12} + x_{13} = \lambda^2 x_{17} - 4x_{17} \quad (*)$$

Main: $x_1 + x_2 + x_{12} + x_{13}$ - neighbors yield:

$$(x_2 + x_6 + x_{10} + x_{14}) + (x_1 + x_3 + x_4 + x_5) +$$

$$(x_{10} + x_{11} + x_{13} + x_{16}) + (x_{10} + x_{11} + x_{12} + x_{15}) = \lambda(x_1 + x_2 + x_{12} + x_{13})$$

$$From\ \ above, x_4 = x_5 = x_{14} = 0; x_{15} = -x_{16}$$

$$x_2 + x_6 + x_{10} + x_1 + x_3 + x_{10} + x_{11} + x_{13} + x_{10} + x_{11} + x_{12} = 0$$

$$x_1 + x_2 + x_3 + x_6 + 3x_{10} + 2x_{11} + x_{12} + x_{13} = 0$$

$$Set\ \ x_{10} = \lambda x_{17};$$

$$Set\ \ x_3 = 2x_1$$

$$Set\ \ x_{11} = x_2$$

$$x_{12} = 0$$

$$Set\ \ 2x_{13} = x_6$$

$$x_1 + x_2 + 2x_1 + 3\lambda x_{17} + 2x_2 + 3x_{12} + 3x_{13} = \lambda(x_1 + x_2 + x_{12} + x_{13})$$

$$3(x_1 + x_2 + x_{12} + x_{13}) + 3\lambda x_{17} = \lambda(x_1 + x_2 + x_{12} + x_{13})$$

$$x_1 + x_2 + x_{12} + x_{13} = \frac{3\lambda x_{17}}{\lambda - 3} \quad (**)$$

(**) substitute in (*)

$$\frac{3\lambda x_{17}}{\lambda - 3} = \lambda^2 x_{17} - 4x_{17}$$

$$\lambda^2(\lambda - 3)x_{17} - 4(\lambda - 3)x_{17} = 3\lambda x_{17}$$

$$\lambda^3 x_{17} - 3\lambda^2 x_{17} - 4\lambda x_{26} + 12x_{17} - 3\lambda x_{17} = 0$$

$$\lambda^3 x_{17} - 3\lambda^2 x_{17} - 7\lambda x_{26} + 12x_{17} = 0 \Rightarrow (\lambda - 4)(\lambda^2 + \lambda - 3)x_{17} = 0 \Rightarrow \lambda = 4;$$

$$\lambda = \frac{-1 \pm \sqrt{1 - (4. - 3)}}{2} = \frac{-1 \pm \sqrt{13}}{2}$$

The eigenvector is:

$$\begin{bmatrix} x_2 + x_6 + x_{10} + x_{14} \\ x_3 + x_4 + x_5 + x_1 \\ x_2 + x_4 + x_5 + x_{17} \\ x_3 + x_5 + x_2 + x_8 \\ x_2 + x_3 + x_4 + x_7 \\ x_1 + x_7 + x_8 + x_9 \\ x_5 + x_6 + x_8 + x_9 \\ x_4 + x_6 + x_7 + x_9 \\ x_6 + x_7 + x_8 + x_{11} \\ x_1 + x_{11} + x_{12} + x_{13} \\ x_9 + x_{10} + x_{12} + x_{13} \\ x_{10} + x_{11} + x_{13} + x_{16} \\ x_{10} + x_{11} + x_{12} + x_{15} \\ x_1 + x_{15} + x_{16} + x_{17} \\ x_{13} + x_{14} + x_{16} + x_{17} \\ x_{12} + x_{14} + x_{15} + x_{17} \\ x_3 + x_{14} + x_{15} + x_{16} \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ x_{16} \\ x_{17} \end{bmatrix} = \left[\begin{bmatrix} x_2 + 2x_{13} + \lambda x_{17} + 0 \\ 2x_1 + 0 + 0 + x_1 \\ x_2 + 0 + 0 + x_{17} \\ 2x_1 + 0 + x_2 + x_8 \\ x_2 + 2x_1 + 0 + x_7 \\ x_1 + x_7 + x_8 + x_9 \\ 0 + 2x_{13} + x_8 + x_9 \\ 0 + 2x_{13} + x_7 + x_9 \\ 2x_{13} + x_7 + x_8 + x_2 \\ x_1 + x_2 + x_{13} \\ x_9 + \lambda x_{17} + x_{13} \\ \lambda x_{17} + x_2 + x_{13} + x_{16} \\ \lambda x_{17} + x_2 + 0 - x_{16} \\ x_1 + 0 + x_{17} \\ x_{13} + 0 + x_{16} + x_{17} \\ -x_{16} + x_{17} \\ 2x_1 + 0 + 0 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ 2x_1 \\ 0 \\ 0 \\ 2x_{13} \\ x_7 \\ x_8 \\ x_9 \\ \lambda x_{17} \\ x_2 \\ 0 \\ x_{13} \\ 0 \\ -x_{16} \\ x_{16} \\ x_{17} \end{bmatrix} \right] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \end{bmatrix}$$

Equations 1,2,12 and 13 yield:

$$3x_1 + 3x_2 + 3 + 3x_{12} + 3x_{13} + \lambda x_{17} +$$

$\Rightarrow 3(x_1 + x_2 + x_{12} + x_{13}) + 3\lambda x_{17} = \lambda(x_1 + x_2 + x_{12} + x_{13})$ our first equation.

Next: equation 17 and 3 gives:

$\lambda\lambda x_{17} = \lambda(2x_1) = x_2 + x_{17} + 0 + 0 + 0 + 0$ equation 14 gives:

$x_1 + x_{17} = 0$ and 15 +16 gives: $x_{13} + 2x_{17} = 0$

So that: $\lambda^2 x_{17} = x_1 + x_2 + x_{12} + x_{13} + 4x_{17}$

3.6 The Case q=5

Step 1 – write down first equation using last vertex:

$x_3 + x_{22} + x_{23} + x_{24} + x_{25} = \lambda x_{26}$

Expand left hand side with their neighbors has vertices belonging to set S:

$$x_2 + x_4 + x_5 + x_6 + x_{26} + x_1 + x_{23} + x_{24} + x_{25} + x_{26} + x_{21} + x_{22} + x_{24} + x_{25} + x_{26}$$

$$+ x_{20} + x_{22} + x_{23} + x_{25} + x_{26} + x_4 + x_{22} + x_{23} + x_{24} + x_{26}$$

$$\begin{bmatrix} x_2+x_7+x_{12}+x_{17}+x_{22} \\ x_1+x_3+x_4+x_5+x_6 \\ x_2+x_4+x_5+x_6+x_{26} \\ x_2+x_3+x_5+x_6+x_{25} \\ x_2+x_3+x_4+x_6+x_9 \\ x_2+x_3+x_4+x_5+x_8 \\ x_1+x_8+x_9+x_{10}+x_{11} \\ x_6+x_7+x_9+x_{10}+x_{11} \\ x_5+x_7+x_8+x_{10}+x_{11} \\ x_7+x_8+x_9+x_{11}+x_{14} \\ x_7+x_8+x_9+x_{10}+x_{13} \\ x_1+x_{13}+x_{14}+x_{15}+x_{16} \\ x_{11}+x_{12}+x_{14}+x_{15}+x_{16} \\ x_{10}+x_{12}+x_{13}+x_{15}+x_{16} \\ x_{12}+x_{13}+x_{14}+x_{16}+x_{19} \\ x_{12}+x_{13}+x_{14}+x_{15}+x_{18} \\ x_1+x_{18}+x_{19}+x_{20}+x_{21} \\ x_{16}+x_{17}+x_{19}+x_{20}+x_{21} \\ x_{15}+x_{17}+x_{18}+x_{20}+x_{21} \\ x_{17}+x_{18}+x_{19}+x_{21}+x_{24} \\ x_{17}+x_{18}+x_{19}+x_{20}+x_{23} \\ x_1+x_{23}+x_{24}+x_{25}+x_{26} \\ x_{21}+x_{22}+x_{24}+x_{25}+x_{26} \\ x_{20}+x_{22}+x_{23}+x_{25}+x_{26} \\ x_4+x_{22}+x_{23}+x_{24}+x_{26} \\ x_3+x_{22}+x_{23}+x_{24}+x_{25} \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ x_{16} \\ x_{17} \\ x_{18} \\ x_{19} \\ x_{20} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \\ x_{25} \\ x_{26} \end{bmatrix} =$$

Step 2: put $x_{25} = -x_{24}$ (second and third largest have opposite signs) – this guarantees no 24 and 25 not in S- called *switching pair.*

$$x_1 + x_2 + 2x_4 + x_5 + x_6 + x_{20} + x_{21} + 3x_{22} + 3x_{23} + 5x_{26} \ (R)$$

Step 3: put $x_4 = x_5 = 0; x_6 = -3x_{22}; x_{23} = 0$

R will become (**) after we have generating set.

Step 4: select generating set S' as (n= total number vertices): . Put all vertices in S' that belong to the second last clique and are neighbors of the last clique– in this case

$$x_{20}; x_{21}$$

Add the first 2:

$$x_1; x_2$$

So with q=5, generating set is:

$$S' = \{x_1, x_2, x_{20}, x_{21}\}$$

Write down neighbors of S'

$$x_2 + x_7 + x_{12} + x_{17} + x_{22} + x_1 + x_3 + x_4 + x_5 + x_6$$

$$+ x_{17} + x_{18} + x_{19} + x_{21} + x_{24} + x_{17} + x_{18} + x_{19} + x_{20} + x_{23}$$

$$= \lambda(x_1 + x_2 + x_{20} + x_{21})$$

$$x_{24} = -x_{25} \quad switching \ \ pair$$.

$$x_2 + x_7 + x_{12} + x_{17} + x_{22} + x_1 + x_3 + x_4 + x_5 + x_6$$

$$+x_{17}+x_{18}+x_{19}+x_{21}+x_{24}+x_{17}+x_{18}+x_{19}+x_{20}+x_{23}$$

$$x_1+x_2+x_3+x_4+x_6+x_5+x_7+x_{12}+3x_{17}+2x_{18}+2x_{19}+x_{20}$$
$$+x_{21}+x_{22}-x_{25}+x_{23}$$

$$x_1+x_2+x_3+x_6+x_7+x_{12}+3x_{17}+2x_{18}+2x_{19}+x_{20}$$
$$+x_{21}+x_{22}-x_{25}+$$

From R:

$$x_6=-3x_{22};x_{25}=-2x_{22}=x_{24}$$

Must watch 2 switching pair equations so that last vertex x_{26} is not 0:

Put: $3x_1=x_7;x_{12}=4\lambda x_{26}$

$$x_3=3x_2$$

$$x_{18}=\frac{3}{2}x_{20};\quad x_{19}=\frac{3}{2}x_{21}$$

$$x_4=x_5=x_{17}=0$$

We have from R:

$$x_{23}=0;x_{24}=-x_{25}\quad \mathit{switching\ \ pair}$$

Now get (*)

$$4(x_1+x_2+x_{20}+x_{21})+4\lambda x_{26}$$
$$=\lambda(x_1+x_2+x_{20}+x_{21})$$

The eigenvector is:

$$
\begin{bmatrix}
x_2 + 3x_1 + 4\lambda x_{26} + x_{22} \\
x_1 + 3x_2 - 3x_{22} \\
x_2 - 3x_{22} + x_{26} \\
\\
\\
\\
\\
\\
\\
\\
\\
\\
\\
\\
\\
\\
\\
\\
x_{15} + 0 + \frac{3}{2}x_{20} + x_{20} + x_{21} \\
\frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{21} - 2x_{22} \\
\frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{20} + 0 \\
x_1 + 0 + x_{26} \\
x_{21} + x_{22} + 0 + x_{26} \\
x_{20} + x_{22} + 2x_{22} + x_{26} \\
0 + x_{22} - 2x_{22} + x_{26} \\
x_3 + x_{22} + 0 + 0 + 0
\end{bmatrix}
= \lambda
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ x_{16} \\ x_{17} \\ x_{18} \\ x_{19} \\ x_{20} \\ x_{21} \\ x_{22} \\ 0 \\ -x_{25} \\ x_{25} \\ x_{26}
\end{bmatrix}
=
\begin{bmatrix}
1 \\ 2 \\ 3 \\ \\ \\ 6 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \\ 25 \\ 26
\end{bmatrix}
$$

Adding equations 1,2,20 and 21 yield

$$4(x_1 + x_2 + x_{20} + x_{21}) + 4\lambda x_{26} = \lambda(x_1 + x_2 + x_{20} + x_{21})$$

24 +25 yield $x_{20} + 2x_{22} + 2x_{26} = 0 \ (27)$

22 and 3 from 26 yield:

$$x_1 + 0 + x_{26} + x_2 - 3x_{22} + x_{26} = \lambda^2 x_{26} \ (28)$$

And 23 gives: $x_{21} + x_{22} + 0 + x_{26} = 0 \ (29)$

27,28 and 29 yield:

$$(x_1 + x_2 + x_{20} + x_{21}) = \lambda^2 x_{26} - 5x_{26}$$

3.7 General Construction.

Vertex notation convention reminder:

First vertex, second vertex, anchor vertex (vertex of last clique joined to first), {generating vertices on second last clique}, last vertex, ,third vertex and switching vertices (third and second last vertices whose sum is 0) = (respectively):

$$x_1, x_2, x_a, \{x_{k1}; x_{k2}, ..., x_{kt} (t = \frac{q-1}{2}; q \ odd \ \ or \, t = \frac{q}{2}; q \ even)\}, x_l, x_3, x, x'$$

Generating set =

$$S = T \cup T^{'} = \{x_1, x_2\} \cup \{x_{k1}; x_{k2}, ..., x_{kt}; (t = \frac{q-1}{2}; q \ odd \ \ or \, t = \frac{q}{2}; q \ even)\}$$

$neighbors\ x_l : x_3 + x_a + x_{a+1} + x_{a+2} + \ldots x_{l-3} + x + x'$;

all vertices in $Q = \{x_{a+1}, x_{a+2}, \ldots, x_{l-3}\}$:which give “0” equations

$(x_l; x; x'\quad all \neq 0)$

$neighbors\ x_a : x_1 + x + x' + x_l$ all others 0 from Q

$$\begin{aligned}
&\lambda^2 x_l \\
&= \lambda(\lambda x_l) \\
&= x_1 + x_2 + x_4 + x_5 + \ldots + x_q + x_{q+1} + \\
&(q-2)x_a + (q-2)x_{a+1}, (q-2)x_{a+2} + \ldots + (q-2)x_{a+t} +,.. + (q-2)x_{l-3} \\
&+ (q-2)x + (q-2)x' + qx_l + (x_{k_1} + x_{k_2} + \ldots + x_{k_t}) + (x_{r_1} + x_{r_2} + \ldots + x_{r_s})
\end{aligned}$$

Put $x_4 = x_5 = \ldots = x_q = 0;\ Q = \{0,0,\ldots,0\};\ x_{q+1} = -(q-2)x_a; x' = -x,$

$$\begin{aligned}
&\lambda^2 x_l = x_1 + x_2 + 0 + 0 + \ldots + 0 - (q-2)x_a + \\
&(q-2)x_a + 0 + 0 + \ldots + 0 + 0 \\
&+ (q-2)x - (q-2)x + qx_l + (x_{k_1} + x_{k_2} + \ldots + x_{k_t}) + (0 + 0 + \ldots + 0)
\end{aligned}$$

$$\lambda^2 x_l = x_1 + x_2 + qx_l + (x_{k1} + x_{k2} + \ldots + x_{kt})$$

$$\lambda^2 x_l - qx_l = x_1 + x_2 + (x_{k1} + x_{k2} + \ldots + x_{kt})$$

This gives equation (**)

Now we look at the neighbors of the generating set S:

$$S = T \cup T' = \{x_1, x_2\} \cup \{x_{k1}; x_{k2}, \ldots, x_{kt}; (t = \frac{q-1}{2}; q\ odd\ \ or\ t = \frac{q}{2}; q\ even)\}$$

$x_1 : x_2, x_{2+q}, x_{2+2q}, \ldots., x_{2+q(q-1)} = x_a$

$x_2 : x_1, x_3, x_{4,}, \ldots, x_{q+1}$

$$Neighbours\ of\ \ T^{'} = (t-1)T^{'} + t(P) + Q'$$

where

- P= t vertices from 2nd last clique, other than T', excluding , $x_{2+q(q-2)}$
- Q' is a subset of Q, whose vertices join backwards to vertices of T'

Let the sum of the elements of S be Z:

Therefore the sum of the neighbors of the elements of S:

$$\lambda Z$$
$$= (x_2 + x_{2+q} + x_{2+2q} + ... + x_{2+q(q-1)}(= x_a)) + (x_1 + x_3 + x_4 + ... + x_q + x_{q+1}) +$$
$$+ (t-1)T^{'} + t(P) + Q'$$

From before:

Put $x_4 = x_5 = ... = x_q = 0;\ \ Q = \{0,0,...,0\};\ \ x_{q+1} = -(q-2)x_a; x' = -x,$

$$\lambda Z =$$
$$(x_2 + x_{2+q} + x_{2+2q} + ... + x_{2+q(q-1)}) + (x_1 + x_3 + 0 + ... + 0 - (q-2)x_a) +$$
$$+ (t-1)T^{'} + t(P) + x_{l-2}$$

$$= x_1 + x_2 + x_3 - (q-2)x_a + x_{2+q} + x_{2+2q} + ... + x_{2+q(q-1)} +$$
$$+ (t-1)T^{'} + t(P) + x_{l-2}$$

Set

$$x_3 = (q-2)x_2;$$
$$x_{2+q} = (q-2)x_1$$
$$x_{2+2q} = (q-1)\lambda x_l$$
$$x_{a-q} = x_{2+q(q-2)} = 0$$
$$x_{l-2} = (q-3)x_a = -x_{l-1}$$

$$\lambda Z$$
$$= x_1 + x_2 + (q-2)x_2 - (q-2)x_a + (q-2)x_1 + (q-1)\lambda x_l + ... + 0 + x_a +$$
$$+ (t-1)T' + t(P) + (q-3)x_a$$

$$= (q-1)x_1 + (q-1)x_2 + (q-1)\lambda x_l + (t-1)T' + t(P)$$

Set

$$x_{p_1} = \frac{q-t}{t} x_{k_1};$$
$$x_{p_2} = \frac{q-t}{t} x_{k_2}$$
$$\ldots$$
$$x_{p_t} = \frac{q-t}{t} x_{k_t}$$

$$\Rightarrow \lambda Z$$
$$= (q-1)x_1 + (q-1)x_2 + (q-1)\lambda x_l +$$
$$(t-1)(x_{k_1} + x_{k_{2+}} + \ldots + x_{k_t}) + t\left[\frac{q-t}{t}(x_{k_1} + x_{k_{2+}} + \ldots + x_{k_t})\right]$$

$$= (q-1)x_1 + (q-1)x_2 + (q-1)\lambda x_l + (q-1)(x_{k_1} + x_{k_{2+}} + \ldots + x_{k_t})$$

$$= (q-1)(x_1 + x_2 + x_{k_1} + x_{k_{2+}} + \ldots + x_{k_t}) + (q-1)\lambda x_l$$

3.8 General Eigenvector

For (**) last equation $l: x_3 + x_a$

Second last equation: $l-1: x_a - (q-3)x_a + x_l$

Third last equation: $l-2 := a+t : -(l-1) : x_a + (q-3)x_a + x_\alpha + x_l$

where $x_\alpha \in T'$ *and is connected to switching vertex* x_{l-2}

Will have q-4-(t-1) equations equal to 0 which have

$$x_a + x_l + x + x'$$

Zero equations (obtained from all vertices in the last clique, who connect backwards to the (q-1) clique, ie to the vertices of $T'\backslash\{x_\alpha\}$ (t-1) of these such equations

$$: x_a + x_{\beta_1} + (x_{l-1} + x_{l-2}) + x_l = x_a + x_{k_\beta} + 0 + x_l \ (t-1) \ such\ quations$$

$$where\ 1 \le \beta \le t, and\ x_{k_\beta} \ne x_\alpha$$

Anchor vertex

$l-(q-1) = a : x_1 + x_{a+1} + x_{a+2} + ... + x_{a+t} + ... + x_{l-2} + x_{l-1} + x_l = x_1 + x_l$

Sum of generating set T' without x_α: $x_{k_i} : (t-2)T'\backslash\{x_{k_i}\} + (t-1)P + (t-1)x_{a-q};$

Equation for x_α in generation set: $x_\alpha : T'\backslash\{x_\alpha\} + P + x_{a-q} + x_{l-2};$

Third vertex: $3: x_2 + x_4 + x_5 + x_6 + ... + x_{q+1} + x_l$

Second vertex: $2: x_1 + x_3 + x_4 + ... + x_{q+1}$

First vertex: $1: x_2 + x_{2+q} + x_{2+2q} + ... + x_{a-q} + x_a$

3.9 The Final General Equations

We need to create the two equations:

$$\lambda \sum S = (q-1)\sum S + (q-1)x_l$$
$$\Rightarrow \sum S = \frac{(q-1)x_l}{(\lambda-(q-1))}(*)$$

$\sum S = \lambda^2 x_l - qx_l$ (**) which yield the following:

The last vertex equation for x_l yields using equations for x_a, x_3:

$x_1 + x_{a+1} + x_{a+2} + ... + x_{a+t} + ... + x_{l-2} + x_{l-1} + x_l + x_2 + x_4 + x_5 + x_6 + ... + x_{q+1} + x_l$
yields: $\lambda^2 x_l = x_1 + x_2 + x_{q+1} + 2x_l$

Switching vertices yields: $2x_a + 2x_l + x_{a-t} = 0$

Adding the 0 equations yields:: $(t-1)x_a + (t-1)x_l + x_{a-1} + x_{a-2} + .. + x_{a-(t-1)}$

Other 0 equations yield q-4-(t-1) of $x_a + x_l$

All yields :

$$\lambda_2 x_l = x_1 + x_2 + x_{q+1} + 2x_l + 2x_a + 2x_l$$
$$+ x_{a-t} + (t-1)x_a + (t-1)x_l + x_{a-1} + x_{a-2} + .. + x_{a-(t-1)} + (q-4-(t-1))[x_a + x_l]$$
$=$sum of elements from generating set = $(q-2)x_a + qx_l$

$x_{q+1} = -(q-2)x_a$ yields our equation (**).

Using equations from S we get equation (*) by assigning each vertex, respectively, in P the value: $\frac{t+1}{t}x_k$ where $x_k \in T'$, and 0 to each vertex $x_4, x_5, ..., x_q$ and each vertex adjacent to the first except x_a, x_{2+q}, x_{2+2q}, and making

$x_3 = (q-1)x_2; x_{2+q} = (q-1)x_1; x_{2+2q} = \lambda(q-1)x_l$, $x_{q+1} = -(q-2)x_a$.

$x_{l-2} = (q-3)x_a = -x_{l-1}$.

We have now proved the first part of the following theorem. The second and third part are proved in section 1 above and the last part is easily verifiable.

THEOREM 6

The class of q-cliqued graphs :

(a) Is sum*(-1)* eigen-pair and product*(1-q) eigen-pair balanced with respect to the eigen-pair:

$$\frac{-1 \pm \sqrt{1+4(q-1)}}{2}$$

(b) Has ratio: $\frac{-1}{1-q}$, asymptote 0 and density 0.

(c) Has area: $\sqrt{n-1}(2\sqrt{n-1} + 3\ln\left|\sqrt{n-1}-1\right|)$

(d) Is critically eigen-bi-balanced with respect to the central vertex.

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