Quantum Theory of Galactic Dynamics Cosmological Mass Accumulations Described by s p d f g h i... Symmetry Quantised Gravity and Mass Spectra Within the Lambda Based Dust Universe Model

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1 Abstract

Much of the introductory section of this paper is devoted to displaying some previously obtained formulae, incorporating a change of notation and variables and giving some explanation of the relation of the work to Newtonian gravitation theory. This section all refers to a quantisation of gravity concentrated on and limited to galaxies with totally spherically symmetric cores and halos. Only the radial variable r is involved and the emphasis is on the dark matter concept. All the following sections are devoted to generalising the theory to additionally incorporate a dependence of galactic structure on the θ and ϕ spherical angular coordinates. The theory is derived using Schrödinger quantum theory in much the same way as it was used in developing the theory of atomic structure. The theoretical structure to be developed in this papers is a hybrid formulation involving three fundamental theoretical facets, general relativity, Schrödinger quantum mechanics and a new theoretical version of isothermal gravity self equilibrium. The combined structure has only become possible because of the discovery of an infinite discrete set of equilibrium states associated with this later theory, the l parameter states. The *configuration space* structure of these states has been found to be available in Schrödinger theory from a special inverse square law potential which appears to supply an inverse cube self attraction to the origin that maintains galaxies in an isolated steady state self gravity quantum condition. The *arbitrary* numerical coefficients of these

Schrödinger states can also depend on l and are appropriately imported from the isothermal equilibrium theory. The work discussed here is much about how these l states can be interleaved with with the usual Schrödinger parameter for angular momentun which I call l' to avoid confusion. The l values have been found to be two possible cases of infinite subsets of the l' values, a D set for the usual mass density distributions in galaxies and an P set for Einstein's extra pressure term density $3P/c^2$. However these identifications are just a working hypothesis. The usual atomic electron theory approach of separation of variables is used to solve the general gravitational Schrödinger equation and it turns out to be rather simpler than the atomic electronic situation. Two version of adapting the Schrödinger equation to hold the isothermal l states are given. The first I call a transplant operation that in fact is a replacement of appropriate Schrödinger l' angular momentum state representations with isothermal l state representations. The second version is in the conclusions section and involves simply displaying restricted Schrödinger representations that describe various gravitational situations. Also in this section, it is made clear that each of the one component Schrödinger representations can be replaced with an equivalent two component representation consisting of a Laplace equation together with a quantised energy equation. Finally, I display the mapping of the angular symmetry defining letters from atomic theory into the quantum theory structure of the isothermal l states. The main products of the theory are a *quantisation* of the gravitational field with explicitly a refined collections of mass accumulation spectra and a generalisation of Newtonian gravitation theory based on general relativity.

Keywords: Dust Universe, Dark Energy, Dark Matter,
Newton's Gravitation Constant, Einstein's Cosmological Constant,
Cosmological Mass Spectra, Quantised Gravity

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2 Introduction

This paper is a follow up of papers, [48], [49], [50], [52] and [55] of similar titles on the problem of formulating the equation that describes the equilibrium of a gaseous material in a self gravitational equilibrium condition in the galaxy modelling context, [47], see also, appendix 2 of ([35]). In previous papers I have applied this new theory to examining the rotation curves for galactic star motions. That work established that the velocity curves for these quantized dark matter halos are decisivel flat. That theory also implied a precise formula that can give many possible mass spectra each of which can give a discrete infinity of spectral lines determined by a quantum parameter l with integral values, $1, 2, 3, \dots, \infty$, starting at unity and extending up to integral ∞. Individual spectra are determined by three *free* parameters, t_b, r_ϵ, β. Thus there is a triple continuous infinity number of possibilities when choosing the right spectra for any specific application. The mass spectra generating function is displayed below

$$M_{l+}(r_\epsilon, \beta, t_b) =$$

$$\frac{c^2\Lambda s(t_b)}{G}\left(\frac{\beta^{2l}(2l-1)^{4l}2lr_\epsilon^{3-4l}}{3(4l-3)} + \frac{\beta^{4l-1}(2l-1)^{8l-2}(4l-1)r_\epsilon^{5-8l}}{(8l-5)}\right). \tag{2.1}$$

This can be represented as the sum of two parts arising from a general relativity mass density contribution and the corresponding general relativity pressure distribution as shown below with D and P subscripts consecutively.

$$M_{l+D}(r_\epsilon, \beta, t_b) = \frac{c^2\Lambda s(t_b)}{G}\left(\frac{\beta^{2l}(2l-1)^{4l}2lr_\epsilon^{3-4l}}{3(4l-3)}\right) \tag{2.2}$$

$$M_{l+P}(r_\epsilon, \beta, t_b) = \frac{c^2\Lambda s(t_b)}{G}\left(\frac{\beta^{4l-1}(2l-1)^{8l-2}(4l-1)r_\epsilon^{5-8l}}{(8l-5)}\right), \tag{2.3}$$

where $s(t)$ is used to denote the function $\sinh^{-2}((3\Lambda)^{1/2}ct/2)$ from the general relativity dust universe model. I shall use the rest of this introduction in the form of a subsection to make some changes of notation in the original theory to avoid conflict with the next stage and also improve clarity, before going on to the spherical angles θ and ϕ generalised theory in section (3).

2.1 Notational Improvement

It has turned out that these last two mentioned functions are easier to use and manipulate if the parameter β with dimensions, m^2, is replaced with a dimensionless parameter θ_0, where

$$\theta_0 = \beta r_e^{-2}, \tag{2.4}$$

with r_ϵ remaining unchanged with the dimension length, m. θ_0 is dimensionless and is identical in meaning to the plain θ used in the previous paper. I have had to make this last change in order to use plain θ later, in its usual angular variable context of spherical coordinates.. The discarded symbol β will be used later in this paper for a dimensionless version of the radius scalar variable r. The two functions re-expressed then become

$$M_{l+D}(r_\epsilon, \theta_0, t_b) = \frac{c^2 \Lambda s(t_b)}{G} \left(\frac{(2l-1)^{4l} 2l \theta_0^{2l} r_\epsilon^3}{3(4l-3)} \right) \quad (2.5)$$

$$M_{l+P}(r_\epsilon, \theta_0, t_b) = \frac{c^2 \Lambda s(t_b)}{G} \left(\frac{(2l-1)^{8l-2}(4l-1)\theta_0^{4l-1} r_\epsilon^3}{(8l-5)} \right), \quad (2.6)$$

If we make the same variable change in the function which gives the total actual mass, $M_l(r)$, associated with a galaxy up to a radius r from its centroid,

$$\begin{aligned} M_l(r) &= \frac{c^2 \Lambda s(t_b) \beta^{2l} (2l-1)^{4l}}{2G(4l-3)} \left(\frac{4l r_\epsilon^{3-4l}}{3} - r^{3-4l} \right) + \\ &\quad \frac{3c^2 \Lambda s(t_b) \beta^{4l-1} (2l-1)^{8l-2}}{2G(8l-5)} \left(\frac{r_\epsilon^{5-8l}(8l-2)}{3} - r^{5-8l} \right) + \\ &\quad \frac{c^2 \Lambda}{3G} r^3, \end{aligned} \quad (2.7)$$

we get the replacement

$$\begin{aligned} M_l(r) &= \frac{c^2 \Lambda s(t_b) \theta_0^{2l} r_\epsilon^3 (2l-1)^{4l}}{2G(4l-3)} \left(\frac{4l}{3} - \left(\frac{r}{r_\epsilon} \right)^{3-4l} \right) + \\ &\quad \frac{3c^2 \Lambda s(t_b) \theta_0^{4l-1} r_\epsilon^3 (2l-1)^{8l-2}}{2G(8l-5)} \left(\frac{(8l-2)}{3} - \left(\frac{r}{r_\epsilon} \right)^{5-8l} \right) + \\ &\quad \frac{c^2 \Lambda r_\epsilon^3}{3G} \left(\frac{r}{r_\epsilon} \right)^3 . \end{aligned} \quad (2.8)$$

I emphasize that the above formula is the *total* mass as indicated by the plus sign before the dark energy mass, as opposed to *total effective* mass when there would be a minus sign before the dark energy mass within the galaxy domain.

The Newtonian gravitational potential, $V_l(r)$, at distance r from the origin produced by this *actual* total mass is given by

$$\begin{aligned} V_l(r) &= \frac{M_l(r)G}{r} = \frac{c^2\Lambda s(t_b)\theta_0^{2l} r_\epsilon^3 (2l-1)^{4l}}{2(4l-3)} \left(\frac{4l}{3r} - \frac{r^{2-4l}}{r_\epsilon^{3-4l}}\right) + \\ & \frac{3c^2\Lambda s(t_b)\theta_0^{4l-1} r_\epsilon^3 (2l-1)^{8l-2}}{2(8l-5)} \left(\frac{(8l-2)}{3r} - \frac{r^{4-8l}}{r_\epsilon^{5-8l}}\right) + \\ & \frac{c^2\Lambda r_\epsilon^3}{3}\left(\frac{r^2}{r_\epsilon^3}\right). \end{aligned} \tag{2.9}$$

The gravitational force produced by this potential per unit mass of subjected particle is given by the gradient, $\hat{\mathbf{r}} \cdot \nabla V_l(r)$, of the potential with respect to r in the positive radial direction, $\hat{\mathbf{r}}$,

$$\begin{aligned} \hat{\mathbf{r}} \cdot \nabla V_l(r) &= \frac{c^2\Lambda s(t_b)\theta_0^{2l} r_\epsilon^3 (2l-1)^{4l}}{2(4l-3)} \left(-\frac{4l}{3r^2} + \frac{(4l-2)r^{1-4l}}{r_\epsilon^{3-4l}}\right) + \\ & \frac{3c^2\Lambda s(t_b)\theta_0^{4l-1} r_\epsilon^3 (2l-1)^{8l-2}}{2(8l-5)} \left(-\frac{(8l-2)}{3r^2} + \frac{(8l-4)r^{3-8l}}{r_\epsilon^{5-8l}}\right) + \\ & \frac{c^2\Lambda r_\epsilon^3}{3}\left(\frac{2r}{r_\epsilon^3}\right). \end{aligned} \tag{2.10}$$

$$\begin{aligned} M_l(r) &= \frac{c^2\Lambda s(t_b)\theta_0^{2l} r_\epsilon^3 (2l-1)^{4l}}{2G(4l-3)} \left(\frac{4l}{3} - \left(\frac{r}{r_\epsilon}\right)^{3-4l}\right) + \\ & \frac{3c^2\Lambda s(t_b)\theta_0^{4l-1} r_\epsilon^3 (2l-1)^{8l-2}}{2G(8l-5)} \left(\frac{(8l-2)}{3} - \left(\frac{r}{r_\epsilon}\right)^{5-8l}\right) + \\ & \frac{c^2\Lambda r_\epsilon^3}{3G}\left(\frac{r}{r_\epsilon}\right)^3. \end{aligned} \tag{2.11}$$

If this is evaluated at $r = r_\epsilon$, the total core mass $M_l(r_\epsilon)$ is expressed as

$$\begin{aligned} M_l(r_\epsilon) &= \frac{c^2\Lambda s(t_b)\theta_0^{2l} r_\epsilon^3 (2l-1)^{4l}}{2G(4l-3)} \left(\frac{4l}{3} - 1\right) + \\ & \frac{3c^2\Lambda s(t_b)\theta_0^{4l-1} r_\epsilon^3 (2l-1)^{8l-2}}{2G(8l-5)} \left(\frac{(8l-2)}{3} - 1\right) + \\ & \frac{c^2\Lambda r_\epsilon^3}{3G}. \end{aligned} \tag{2.12}$$

$$
\begin{aligned}
M_l(r) &= \frac{c^2\Lambda s(t_b)\theta_0^{2l} r_\epsilon^3 (2l-1)^{4l}}{2G(4l-3)} \left(\frac{4l}{3} - \left(\frac{r}{r_\epsilon}\right)^{3-4l} \right) + \\
&\quad \frac{3c^2\Lambda s(t_b)\theta_0^{4l-1} r_\epsilon^3 (2l-1)^{8l-2}}{2G(8l-5)} \left(\frac{(8l-2)}{3} - \left(\frac{r}{r_\epsilon}\right)^{5-8l} \right) + \\
&\quad \frac{c^2\Lambda r_\epsilon^3}{3G} \left(\frac{r}{r_\epsilon}\right)^3 . \qquad (2.13)
\end{aligned}
$$

the ratio, excluding dark energy, of total mass to core mass is given by

$$
\begin{aligned}
M_{l,T}(\infty) &= \frac{2c^2\Lambda s(t_b)\theta_0^{2l} r_\epsilon^3 (2l-1)^{4l} l}{3G(4l-3)} + \\
&\quad \frac{c^2\Lambda s(t_b)\theta_0^{4l-1} r_\epsilon^3 (2l-1)^{8l-2}(4l-1)}{G(8l-5)} . \qquad (2.14)
\end{aligned}
$$

$$
\begin{aligned}
M_{l,C}(r_\epsilon) &= \frac{c^2\Lambda s(t_b)\theta_0^{2l} r_\epsilon^3 (2l-1)^{4l}}{6G} + \\
&\quad \frac{c^2\Lambda s(t_b)\theta_0^{4l-1} r_\epsilon^3 (2l-1)^{8l-2}}{2G} . \qquad (2.15)
\end{aligned}
$$

$$
\begin{aligned}
n_T(l,\theta) = \frac{M_{l,T}(\infty)}{m(l, r_\epsilon, t_b, \theta_0)} &= \frac{4l}{(4l-3)} + \frac{6\theta_0^{2l-1}(2l-1)^{4l-2}(4l-1)}{(8l-5)} \qquad (2.16) \\
n_C(l,\theta_0) = \frac{M_{l,C}(r_\epsilon)}{m(l, r_\epsilon, t_b, \theta_0)} &= 1 + 3\theta_0^{2l-1}(2l-1)^{4l-2} \qquad (2.17) \\
m(l, r_\epsilon, t_b, \theta_0) &= (2l-1)^{4l} \frac{c^2\Lambda s(t_b)\theta_0^{2l} r_\epsilon^3}{6G} . \qquad (2.18)
\end{aligned}
$$

Thus the ratio, $R_{T/C}(l,\theta_0)$, of $M_{l,T}(\infty)$ to $M_{l,C}(r_\epsilon)$ is given by

$$
R_{T/C}(l,\theta_0) = \frac{1}{(1 + 3\theta_0^{2l-1}(2l-1)^{4l-2})} \left(\frac{4l}{(4l-3)} + \frac{6\theta_0^{2l-1}(2l-1)^{4l-2}(4l-1)}{(8l-5)} \right) . \qquad (2.19)
$$

The gravitational acceleration field at distance r from the galactic mass centroid is found as follows. Consider the total actual gravitating mass and its potential

$$
\begin{aligned}
V_l(r) = \frac{M_l'(r)G}{r} &= \frac{M_{l+}'G}{r} + \frac{M_{l-}'G}{r}. && (2.20)\\
M_{l-}'(r) &= A_l\left(-r^{3-4l}\right) + B_l\left(-r^{5-8l}\right) && (2.21)\\
M_{l+}'(r) &= M_{l+}(r) + C_l r^3. && (2.22)
\end{aligned}
$$

The acceleration per unit mass caused by this potential at distance r from the origin is

$$
\begin{aligned}
\hat{\mathbf{r}}\cdot\nabla V_l(r) &= \frac{c^2\Lambda s(t_b)\theta_0^{2l} r_\epsilon^3 (2l-1)^{4l}}{2(4l-3)}\left(-\frac{4l}{3r^2} + \frac{(4l-2)r^{1-4l}}{r_\epsilon^{3-4l}}\right) + \\
&\quad \frac{3c^2\Lambda s(t_b)\theta_0^{4l-1} r_\epsilon^3 (2l-1)^{8l-2}}{2(8l-5)}\left(-\frac{(8l-2)}{3r^2} + \frac{(8l-4)r^{3-8l}}{r_\epsilon^{5-8l}}\right) + \\
&\quad \frac{c^2\Lambda r_\epsilon^3}{3}\left(\frac{2r}{r_\epsilon^3}\right), && (2.23)
\end{aligned}
$$

where the dimensioned parameter β has been replaced by the dimensionless parameter $\theta_0 = \beta/r_\epsilon^2$ to clarify the dimensionality of the various contributions. Thus all the last bracketed quantities become dimensionally inverse square but not all variably inverse square. All the coefficients of the large brackets have dimensions $m^3 s^{-2}$. Thus all the terms are accelerations. Notably, Newton's gravitation constant G does not occur. In fact, G is replaced by Λ. This quantized gravitational expression is clearly a substantial generalisation of Newton's law of gravitation. However, we can identify main inverse square law forms as the first terms in the first two large brackets. Both of these terms have minus signs and so represent the usual Newtonian gravitational law of attraction towards the origin. However, both of the large brackets contain also many possible positive signed terms of inverse form determined by the quantum state parameter l. They thus represent repulsions from the origin. Clearly the last positive term above represents the repulsive effect of twice Einstein's dark energy term. The two first large brackets originate in the galactic context, from the galactic mass density and the Einstein pressure term mass density from general relativity respectively. The inverse repulsive terms in the first two brackets with their positive signs appear to go along with the negative gravity of the last term. They are the terms which simulate negative mass by contributing repulsion and actually exist outside the reference sphere of radius r. I mention one more effect from the correction. The negative gravitating term contributed by Einstein's dark energy, the last term above, was left out when I calculated the rotation curve for the small galaxy on the grounds that for a small galaxy it would only make a negligible contribution on account of the smallness of Λ. However, if is used in such calculations under the corrected version of this theory it would contribute a small positive addition to the rotation curve gradient formula for large galaxies. For sufficiently large galaxies the rotation curves would eventually curve up from their flat condition at very large distances from the origin. There has been mention of observations to this effect. The integer parameter l is closely related to the isotropic index n and was derived in a new version of gravity self equilibrium theory. In the following sections, I shall show how it is related to the quantum angular momentum parameter, also usually denote by l and which to avoid confusion, I shall here denote by l'. Atomic states are usually described, using my changed notation by l', and a second Parameter, m, sometimes called the z component of angular momentum. With these two angular related parameters the comprehensive theory of atomic state structure has been developed. In the follow pages, I shall show how my new l and the same old m can be used to describe the geometrical state mass distributions of galaxies in much the same way that the angular parameters l' and m are used in the atomic context where it can give considerable graphical and pictorial dressing. Thus I shall take the purely spherically symmetric structure from my theory of galaxies in earlier papers to a much more realistic mathematical representation that can describe many of the possible galactic shapes that are seen.

3 Quantum States of Cycling Mass Within a Galaxy

In Newtonian mechanics, the energy, E, and its equation associated with a particle of mass μ moving in a potential field $U(r)$ with velocity $\mathbf{v}$ is given by

$$E = \frac{P^2}{2\mu} + U(r) \tag{3.1}$$

$$P = |\mathbf{P}| = \mu\,|\mathbf{v}| \tag{3.2}$$

The essential step in progressing from this so called classical dynamics equation into a quantum mechanics version was the replacement of the two classical dynamical variables energy and momentum, E and P, with operators and necessarily to add to the system something for them to operate on, a state function ψ, say.

$$E \to \hat{H} = i\hbar\frac{\partial}{\partial t} \tag{3.3}$$

$$P_i \to \hat{P}_i = -i\hbar\frac{\partial}{\partial x_i} \tag{3.4}$$

$$\mathbf{P} \to \hat{\mathbf{P}} = -i\hbar\nabla. \tag{3.5}$$

$$\mathbf{P}^2 \to \hat{\mathbf{P}}^2 = -\hbar^2\nabla^2. \tag{3.6}$$

Thus in non-relativistic quantum theory the dynamical states ψ of a particle of mass μ moving in a spherically symmetric field, centred at position $r = 0$, can be described by a Schrödinger equation, the last equation at, (3.7) which is just an operator version of (3.1) acting on the state function ψ. The energy E becomes the Hamiltonian operator, (3.8),

$$E = \frac{P^2}{2\mu} + U(r) \to i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\mu}\nabla^2\psi + \hat{U}(r)\psi \tag{3.7}$$

$$\hat{H} = i\hbar\frac{\partial}{\partial t} = -\frac{\hbar^2}{2\mu}\nabla^2 + \hat{U}(r) \tag{3.8}$$

$$\nabla = \frac{\mathbf{e_1}\,\partial}{\partial r} + \frac{\mathbf{e_2}}{r}\frac{\partial}{\partial\theta} + \frac{\mathbf{e_3}}{r\sin(\theta)}\frac{\partial}{\partial\phi} \tag{3.9}$$

$$\nabla^2 = \frac{\partial}{r^2\partial r}\left(\frac{r^2\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\left(\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin(\theta)}\frac{\partial^2}{\partial\phi^2}\right) \tag{3.10}$$

$$= \frac{2\partial}{r\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left(\frac{\cot(\theta)\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right) \tag{3.11}$$

$$= \frac{2\partial}{r\partial r} + \frac{\partial^2}{\partial r^2} - \frac{\lambda}{r^2},\ \ say \tag{3.12}$$

$$\hat{\mathrm{L}} = -\left(\frac{\cot(\theta)\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right) \tag{3.13}$$

$$\hat{\mathrm{L}}\psi_\lambda = \lambda\psi_\lambda \tag{3.14}$$

$$x = r\sin(\theta)\cos(\phi) \tag{3.15}$$

$$y = r\sin(\theta)\sin(\phi) \tag{3.16}$$

$$z = r\cos(\theta), \tag{3.17}$$

if the particle is subjected to the influence of a potential field $\hat{U}(r)$ when it is at radial distance r from the centre of force. At equation (3.4), the momentum vector is shown in component operator form in Cartesian coordinates and in the following line it is shown as a vector operator. However, in discussing systems in three dimensions with a central potential only depending on r, they are best represented in terms of spherical polar coordinates, equations (3.11) etc. The operator, ∇, is given at equation (3.9) and its square is given at equation (3.10) both in spherical polar coordinates. The square of the momentum operator is obtained from this by multiplication through by $-\hbar^2$. Under these substitutions of operators replacing classical physical functions to generate quantum theory, the angular momentum $\mathbf{L}$ of a particle which was initially defined as the vector product $\mathbf{r} \wedge \mathbf{P}$ is converted, using the above transformations, as

$$
\begin{aligned}
\mathbf{L} = \mathbf{r} \wedge \mathbf{P} \rightarrow \hat{\mathbf{L}} &= -i\hbar \mathbf{r} \wedge \nabla && (3.18) \\
\hat{L}_x &= +\ i\hbar(\sin(\phi)\frac{\partial}{\partial\theta} + \cot(\theta)\cos(\phi)\frac{\partial}{\partial\phi}) && (3.19) \\
\hat{L}_y &= -\ i\hbar(\cos(\phi)\frac{\partial}{\partial\theta} - \cot(\theta)\sin(\phi)\frac{\partial}{\partial\phi}) && (3.20) \\
\hat{L}_z &= -i\hbar\frac{\partial}{\partial\phi} && (3.21) \\
\mathbf{L}^2 = (\mathbf{r} \wedge \mathbf{P})^2 \rightarrow \hat{\mathbf{L}}^2 &= -\hbar^2(\mathbf{r} \wedge \nabla)^2 \\
&= \frac{-\hbar^2}{\sin(\theta)}\left(\cos(\theta)\frac{\partial}{\partial\theta} + \sin(\theta)\frac{\partial^2}{\partial\theta^2}\right) - \\
&\quad \frac{\hbar^2}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2} = \hbar^2\hat{\mathrm{L}} = \hbar^2\lambda && (3.22)
\end{aligned}
$$

and apart from the $-\hbar^2$ appearing here this last expression also importantly appears as the last term divided by r^2 in the expression for ∇^2 in the Hamiltonian expression at (3.10). If we are interested in describing a system in which a particle moves in a planar orbit, There is substantial simplification if we take the direction of the z axis as the same direction as the normal to the plane of motion of the particle which will be the same as the direction of the z-component of angular momentum. The Schrödinger equation at (3.7) or (3.23), expanded in terms of these coordinates takes the form (3.24)

$$
i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\mu}\nabla^2\psi + \hat{U}(r)\psi. \tag{3.23}
$$

$$\left(\frac{2\partial}{r\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\left(\frac{\cot(\theta)\partial}{\partial\theta}+\frac{\partial^2}{\partial\theta^2}+\frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right)\right)\psi$$
$$+\frac{2\mu}{\hbar^2}(\hat{H}-\hat{U}(r))\psi=0. \tag{3.24}$$

If we can find a solution of the product form

$$\psi(r,\theta,\phi,t)=\mathtt{R}(r)\Theta(\theta)\Phi(\phi)e^{-\frac{iEt}{\hbar}} \tag{3.25}$$

in which each variable appears alone in its own function, separation of the variables will have the solved the differential equation for ψ. Substituting this ψ into the equation we get

$$\left(\frac{2\partial}{r\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\left(\frac{\cot(\theta)\partial}{\partial\theta}+\frac{\partial^2}{\partial\theta^2}+\frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right)\right)\mathtt{R}(r)\Theta(\theta)\Phi(\phi)e^{-\frac{iEt}{\hbar}}$$
$$+\frac{2\mu}{\hbar^2}(i\hbar\frac{\partial}{\partial t}-\hat{U}(r))\mathtt{R}(r)\Theta(\theta)\Phi(\phi)e^{-\frac{iEt}{\hbar}}=0. \tag{3.26}$$

If we look at equation (3.26), we see that there is a possible easy to solve differential equation in ϕ, if $\frac{\partial^2}{\partial\phi^2}$ is taken to be a constant $-m^2$, say, because this would give, $e^{im\phi}$, or $\sin(m\phi)$ and $\cos(m\phi)$ solutions. Thus we can write down the differential equation (3.34) as *part of* a possible solution to equation (3.26). Then taking a further look at equation (3.26), if we have come across spherical harmonics in terms of θ, we may recognise a second equation in terms of θ that can be solved if we put

$$-\lambda=\left(\frac{\cot(\theta)\partial}{\partial\theta}+\frac{\partial^2}{\partial\theta^2}-\frac{m^2}{\sin^2(\theta)}\right)=-l'(l'+1), \tag{3.27}$$

giving as a second possible solution equation (3.33). We can further use this λ expression in the original equation to give, a third possibly solvable radial variable equation including the time factor at (3.28),

$$\left(\frac{2\partial}{r\partial r}+\frac{\partial^2}{\partial r^2}-\frac{\lambda}{r^2}+\frac{2\mu}{\hbar^2}(i\hbar\frac{\partial}{\partial t}-\hat{U}(r))\right)\mathtt{R}(r)e^{-\frac{iEt}{\hbar}}=0. \tag{3.28}$$

In this work I use a *special* form of what is usually called the *external* potential function $\hat{U}(r)$ by what in fact is a *feed back* function of form

$$\hat{U}(r)=E_{l,m}+\frac{A_l}{r^2}. \tag{3.29}$$

The $E_{l,m}$ is a function of l and now including the extra angular related parameter m is constant in that it does not depend on r. The second term is also a constant dependent on l but additionally it appears divided by r^2 and so is an inverse square law potential. The $E_{l,m}$ term is a constant total quantity of energy associated with a mass accumulation in self gravitating isothermal equilibrium. The term involving r^2 is a *quantum* potential that keeps the whole mass assembly in a quantum steady state condition. Thus if this type of potential is used in (3.28) we get

$$\left(\frac{2\partial}{r\partial r} + \frac{\partial^2}{\partial r^2} - \frac{\lambda}{r^2} + \frac{2\mu}{\hbar^2}(i\hbar\frac{\partial}{\partial t} - E_{l,m} - \frac{A_l}{r^2})\right)\mathtt{R}(r)e^{-\frac{iEt}{\hbar}} = 0 \quad (3.30)$$

Thus now if we use the relation

$$(i\hbar\frac{\partial}{\partial t} - E)e^{-\frac{iEt}{\hbar}} = 0 \quad (3.31)$$

and identify the energy $E = E_{l,m}$, then we will have separated the radial function with time factor, (3.30), into the two equations (3.31 and (3.32).

$$\left(\frac{2\partial}{r\partial r} + \frac{\partial^2}{\partial r^2} - \frac{\lambda}{r^2} - \frac{2\mu}{\hbar^2}\frac{A_l}{r^2}\right)\mathtt{R}(r) = 0 \quad (3.32)$$

$$\left(\frac{\partial^2}{\partial\theta^2} + \frac{\cot(\theta)\partial}{\partial\theta} + \lambda - \frac{m^2}{\sin^2(\theta)}\right)\Theta(\theta) = 0 \quad (3.33)$$

$$\left(\frac{\partial^2}{\partial\phi^2} + m^2\right)\Phi(\phi) = 0. \quad (3.34)$$

$$\lambda = l'(l'+1) \quad (3.35)$$

The original differential equation (3.23) is now expressed in terms of the four equations (3.31), (3.32), (3.33) and (3.34) completely separating the variables t, r, θ, ϕ. The last four just mentioned equations are a parameterisation of equation (3.26) and whatever m and λ are they can be eliminated between these equations to give (3.26) with $E = E_{l,m}$. Thus they are entirely equivalent to the one original equation. We have seen that the last two mentioned equations have known solutions. I have shown in previous papers that the quantization of galactic dynamics require the inverse *square* law potential introduced above. This will be discussed in the next section.

4 Quantum Gravity Inverse Square Law Potential

The mass densities that I am using in developing this theory are tightly defined as arising from Einstein's cosmological constant. All mass density in this theory, positively or negatively gravitating, arises from Λ. It has recently becoming more apparent that so called *dark matter* mass is the greatly dominant constituent of most of the positively gravitating mass in the universe. Dark energy on the other hand is *positive* mass which apparently is negatively gravitating and is present uniformly every where but at extremely low density. There are two basic mass densities involved in this galaxy modelling project, Einstein's general relativity mass density and also the additional mass density from his pressure term $3P/c^2$, the pair actually obtained by deriving a polytropic gas equation from a new version formula for describing gravitational self equilibrium. The two densities are respectively at (4.2) and (4.5),

$$\begin{aligned}
\grave{\rho}_l(r) &= \left(\frac{-2a(2l-1)^2}{\pi}\right)^{2l}\left(\frac{r}{r_0}\right)^{-4l} = \theta_0^{2l}(2l-1)^{4l}\left(\frac{r}{r_\epsilon}\right)^{-4l} && (4.1)\\
\rho_l(r) &= \rho(t_b)\theta_0^{2l}(2l-1)^{4l}\left(\frac{r}{r_\epsilon}\right)^{-4l} && (4.2)\\
\grave{\rho}_{P,l}(r) &= 3(\grave{\rho}_l(r))^{\frac{4l-1}{2l}} && (4.3)\\
&= 3\theta_0^{4l-1}(2l-1)^{8l-2}\left(\frac{r}{r_\epsilon}\right)^{2-8l} && (4.4)\\
\frac{3P(r)}{c^2} &= \rho(t_b)\grave{\rho}_{P,l}(r) = 3\rho(t_b)\theta_0^{4l-1}(2l-1)^{8l-2}\left(\frac{r}{r_\epsilon}\right)^{2-8l} && (4.5)\\
\rho(t) &= (3/(8\pi G))(c/R_\Lambda)^2\sinh^{-2}(3ct/(2R_\Lambda)). && (4.6)
\end{aligned}$$

The same two densities at (4.1) and (4.4) respectively are dimensionless versions in not having the mass per unit volume pre-multiplier, $\rho(t_b)$ which comes from general relativity. $\rho(t)$ is the general relativity positively gravitating mass density of the substratum at epoch t. $\rho(t_b)$ is the time constant density with which the galaxy is born and retains for life. The dimensionless versions are indicated with the top grave accent. If we form two new functions $\psi_{D,l}$ and $\psi_{P,l}$ from (4.1) and (4.4) by taking their square roots and in addition give them a steady state time dependence with a factor

$\exp(-\frac{iE_{D,l}t}{\hbar})$ or $\exp(-\frac{iE_{P,l}t}{\hbar})$, we obtain

$$\psi_{D,l}(r) = \theta_0^l (2l-1)^{2l} \left(\frac{r}{r_\epsilon}\right)^{-2l} \exp(-\frac{iE_{D,l}t}{\hbar}) \tag{4.7}$$

$$\psi_{P,l}(r) = 3\theta_0^{2l-1/2}(2l-1)^{4l-1} \left(\frac{r}{r_\epsilon}\right)^{1-4l} \exp(-\frac{iE_{P,l}t}{\hbar}). \tag{4.8}$$

Mathematically, they can in fact be regarded as the solution of an *unusual* classical eigen-value problem expressed as follows. Find the *eigen-potentials* $V_{D,l}(\mathbf{r})$ and $V_{P,l}(\mathbf{r})$ and steady state energy wave functions $\psi_{D,l}$ and $\psi_{P,l}$ that must be operative if the classical Newtonian energy equation is replaced by what might be called a potential function operator version of Schrödinger shape for a system with two distinguishable parts density, D and pressure, P.

$$\hat{V}(\mathbf{r}) = i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2\mu}\nabla^2 \tag{4.9}$$

$$\hat{V}(\mathbf{r})\psi_{D,l}(\mathbf{r},t)) = V_{D,l}(\mathbf{r})\psi_{D,l}(\mathbf{r},t) \tag{4.10}$$

$$\hat{V}(\mathbf{r})\psi_{P,l}(\mathbf{r},t)) = V_{P,l}(\mathbf{r})\psi_{P,l}(\mathbf{r},t). \tag{4.11}$$

Thus we have essentially two associated fundamental quantum gravity eigen-potentials. The steady state energies $E_{D,l}$ and $E_{P,l}$ are obtainable from the formalism and so the eigen-potentials are easily calculated from the last two equations to be

$$V_{D,l}(\mathbf{r}) = E_{D,l} + \frac{\hbar^2 l(2l-1)}{\mu r^2} = E_{D,l} + \frac{\hbar^2 q_{D,l}}{\mu r^2} \tag{4.12}$$

$$V_{P,l}(\mathbf{r}) = E_{P,l} + \frac{\hbar^2(2l-1)(4l-1)}{\mu r^2} = E_{P,l} + \frac{\hbar^2 q_{P,l}}{\mu r^2} \tag{4.13}$$

$$q_{D,l} = l(2l-1) \tag{4.14}$$

$$q_{P,l} = (2l-1)(4l-1). \tag{4.15}$$

The galactic model discussed earlier had both of these fields present so that a mass moving under the combination of these two quantum potential will experience a local *quantum* gravitational potential which is just their sum as

$$U_l(\mathbf{r}) = E_l + \frac{\hbar^2(2l-1)(5l-1)}{\mu r^2} \tag{4.16}$$

$$= E_l + \frac{\hbar^2 q_l}{\mu r^2}, say \tag{4.17}$$

$$q_l = (2l-1)(5l-1). \tag{4.18}$$

The densities and potentials just discussed in this section arose in earlier work on this theory in the special case that they had no angular orientation dependence. That is to say they only depended on a radial variable r and so definite *pure* spherical symmetry only was involved. Clearly this, although being a substantial advance on previous galactic structure theory, is not adequate to describe actual galactic structure which is known to take a variety of non-pure spherical geometric forms just from observation. In this earlier theory, the total masses associated with such density distributions accumulations were easily calculate just by integrating over r. From these integrations the steady state energies E_l, which only depended on the one quantum parameter l, needed to go with the Schrödinger description were calculated. In the next section, we shall incorporate angular variation into the structure so that steady state masses or energies will now have to be recalculated and will then depend on the angular structure or angular parameters of the galactic model. This implies that the mass spectra derived in the earlier work will take on a substantially refined form in then depending on additional angular parameters. Let us now return to the radial equation for the spherical quantum state, (3.28) and check the result of using $V_{D,l}$ to represent the potential $\hat{U}(r)$,

$$\left(\frac{2\partial}{r\partial r} + \frac{\partial^2}{\partial r^2} - \frac{\lambda}{r^2} + \frac{2\mu}{\hbar^2}(E - (E_{D,l} + \frac{\hbar^2 q_{D,l}}{\mu r^2}))\right) \mathtt{R}(r) e^{-\frac{iEt}{\hbar}} = 0 \tag{4.19}$$

Or when the following condition holds

$$\lambda = l'(l'+1) \tag{4.20}$$

this becomes

$$\left(\frac{2\partial}{r\partial r}+\frac{\partial^2}{\partial r^2}-\frac{l'(l'+1)}{r^2}+\frac{2\mu}{\hbar^2}(E_{l'}-E_{D,l}-\frac{\hbar^2 q_{D,l}}{\mu r^2})\right)\mathtt{R}(r)e^{-\frac{iEt}{\hbar}}=0. \tag{4.21}$$

Let us compare equation (4.21 with the radial equation in the case of an electron in orbit under the Coulomb electric potential

$$\left(\frac{2\partial}{r\partial r}+\frac{\partial^2}{\partial r^2}-\frac{l'(l'+1)}{r^2}+\frac{2\mu}{\hbar^2}\left(E+\frac{e^2Z}{r}\right)\right)\mathtt{R}(r)=0. \tag{4.22}$$

Simply now by putting $E_{l'}-E_{D,l}=E$ these equations become much alike except for the, *oppositely signed*, inverse quadratic potential in the one and the inverse linear potential in the other and the additional time factor in the first. Thus, if we can find solutions to equation (4.21), we will be able to describe massive objects in orbital motion in a galaxy in the same style as electron are quantum mechanically describable in orbital motion in an atom. However, we see also that at this juncture we have parted company with the quantized elctron theory because, we have chosen an extra equation (3.31) that removes the term $E=E_{l'}-E_{D,l}$ and the factor $e^{-\frac{iEt}{\hbar}}$ from equation (4.21) in order to achieve a full separation of variables.

4.1 Radial Equation Solution

We can now see solving equation (4.21) as a purely mathematics problem by making it a little simpler and non-dimensional by the dimensionless substitution β for the dimensioned quantity r. Because the equation to be solved is so much like the hydrogen atom equation, we can follow much the same route as is used to solve the hydrogen quantum radial equation, although a distinct deviation from that route is required. Thus, making the substitutions

$$r = \beta\frac{\hbar}{\mu c} \tag{4.23}$$

$$E_{l'}-E_{D,l} = 0, \tag{4.24}$$

where $\frac{\hbar}{\mu c}$ is the Compton wavelength of the mass μ divided by 2π. The result is

$$\left(\frac{2\partial}{\beta\partial\beta}+\frac{\partial^2}{\partial\beta^2}-\frac{l'(l'+1)}{\beta^2}-2\frac{q_{D,l}}{\beta^2}\right)\mathtt{R}(\beta)=0. \tag{4.25}$$

5 Beyond The Gravitational S State

Various equations are collected below for ease of reference. If we inspect equation (4.21) repeated below at (5.1), we see two parameter contributions in the inverse β^2 term, $l'(l'+1)$ and $2l(2l-1)$. The first of these terms comes from the ∇^2 of the Schrödinger kinetic energy and the second of these terms comes from the external quantum potential energy. If an s state $l'=0$ is chosen, the first of these terms would be zero and only the second term would appear. This would correspond to the earlier work when the case of no dependence on angular variables was involved and it was necessary to introduce the eternal potential involving the l parameter to access the isothermal thermal equilibrium states. It thus becomes apparent that the formula would be exactly the same if the spatially dependant part external potential term had not been introduced but rather the state dependence on the l' parameter was restricted to the form $l'=2l-1$ to match the l term and where the values of l' there are restricted by range of value by its l dependence, the same as in the external potential term. Effectively $l'(l'+1)\rightarrow(2l-1)2l$ so matching the l term only the order of this commuting product is changed. Thus we can transplant the isothermal gravitational equilibrium states into the full angular dependent schrödinger equation by agreeing to restrict the angular momentum states of that equation in a specific way to D type quantum gravity states. Thus we start with the following three equations.

$$\left(\frac{2\partial}{\beta\partial\beta}+\frac{\partial^2}{\partial\beta^2}-\frac{l'(l'+1)}{\beta^2}-\frac{2l(2l-1)}{\beta^2}\right)\mathtt{R}(\beta)=0. \tag{5.1}$$

$$\left(\frac{\partial^2}{\partial\theta^2}+\frac{\cot(\theta)\partial}{\partial\theta}+l'(l'+1)-\frac{m^2}{\sin^2(\theta)}\right)\Theta(\theta)=0 \tag{5.2}$$

$$\left(\frac{\partial^2}{\partial\phi^2}+m^2\right)\Phi(\phi)=0 \tag{5.3}$$

and bearing in mind the extra parameter m, we have

$$\begin{aligned}
V_{D,l,m}(\mathbf{r}) &= E_{D,l,m}+\frac{\hbar^2 l(2l-1)}{\mu r^2}=E_{D,l,m}+\frac{\hbar^2 q_{D,l}}{\mu r^2} && (5.4)\\
V_{P,l,m}(\mathbf{r}) &= E_{P,l,m}+\frac{\hbar^2(2l-1)(4l-1)}{\mu r^2}=E_{P,l,m}+\frac{\hbar^2 q_{P,l}}{\mu r^2} && (5.5)\\
q_{D,l} &= l(2l-1) && (5.6)\\
q_{P,l} &= (2l-1)(4l-1). && (5.7)
\end{aligned}$$

It is necessary that when the substitution $l'=2l-1$ into equation (5.1) is made it is also made into the θ angle determining equation following, (5.2). Consequently, I have made the changes in two steps repeating the relevant three equations at each step so that it is clear that these, rather unusual manipulations, are clearly seen to be mathematically and physically correct.

The first step take $l'=2l-1$ in the relevant equations

$$\left(\frac{2\partial}{\beta\partial\beta}+\frac{\partial^2}{\partial\beta^2}-\frac{2l(2l-1)}{\beta^2}-\frac{2l(2l-1)}{\beta^2}\right)\mathtt{R}(\beta)=0. \tag{5.8}$$

$$\left(\frac{\partial^2}{\partial\theta^2}+\frac{\cot(\theta)\partial}{\partial\theta}+(2l-1)2l-\frac{m^2}{\sin^2(\theta)}\right)\Theta(\theta)=0. \tag{5.9}$$

$$\left(\frac{\partial^2}{\partial\phi^2}+m^2\right)\Phi(\phi)=0. \tag{5.10}$$

Second step is to remove the spatially determined part of the external potential in equation (5.8). This step simply changes to zero the term $\frac{2l(2l-1)}{\beta^2}$ of the external potential because effectively it becomes part of the normal Schödinger structure. The result is

$$\left(\frac{2\partial}{\beta\partial\beta}+\frac{\partial^2}{\partial\beta^2}-\frac{2l(2l-1)}{\beta^2}\right)\mathtt{R}(\beta)=0. \tag{5.11}$$

$$\left(\frac{\partial^2}{\partial\theta^2}+\frac{\cot(\theta)\partial}{\partial\theta}+(2l-1)2l-\frac{m^2}{\sin^2(\theta)}\right)\Theta(\theta)=0 \qquad (5.12)$$

$$\left(\frac{\partial^2}{\partial\phi^2}+m^2\right)\Phi(\phi)=0. \qquad (5.13)$$

Solutions of equation (5.12) are the Legendre functions,

$$\Theta_{l',m}(\theta)=P_{l'}^{m}(\cos(\theta))=\sin^m(\theta)T_{l'-m}^{m}(\cos(\theta)), \qquad (5.14)$$

where the T functions are the tesseral harmonics. The Legendre functions are finite over the range $0\leq\theta\leq\pi$ only when l is an integer such that $-(2l-1)=-l'\leq m\leq +l'=+(2l-1)$ in the D case and l is an integer such that $-(4l-2)=-l'\leq m\leq +l'=+(4l-2)$ in the P case. Solutions to equation (5.13) are of the form

$$\Phi_m(\phi)=A\exp(im\phi), \qquad (5.15)$$

where A is an arbitrary constant. In the rotating electron quantum context m is called the magnetic quantum number. In the gravitation context of uncharged rotating mass being discussed in this paper, that name for m is not appropriate. the D type field with angular variations is

$$\psi_D(\beta,\theta,\phi,t)=\mathtt{R}_{2l-1}(\beta)\Theta_{D,2l-1,m}(\theta)\Phi_m(\phi)e^{-\frac{iE_Dt}{\hbar}}. \qquad (5.16)$$

We see that the second step makes no difference to the θ equation and no difference to the ϕ equation. The last equation displayed above is the final solution of D type after the changes. The parameter change in the P case is $l'\rightarrow 4l-2$ in order for $l'(l'+1)$ to become $(4l-2)(4l-1)=2q_{P,l}$.

6 Refined Quantum Mass Spectra

The theory being developed here arises from three distinct areas of study, general relativity, isothermal self-gravity equilibrium and quantum mechanics. The theory leads to a quantum theory of gravity based on Einstein's Λ, itself involving, the use of a mass spectra system, also based on Λ, which can supply the source mass accumulations and their distant gravitational influence. The emphasis here and earlier has been on the gravitational field from galaxies which has been shown to agree with the modern dark matter interpretation of the galactic structure. The mass spectra that have been identified in the structure are, in a sense a side effect of the theory and are strongly dependent on the isothermal self gravity aspect of the structure in that arbitrary coefficients in the quantum structure described in earlier sections have to be imported from self gravity equilibrium theory. Another important feature of the mass densities involved in the theory is that in their raw state they are divergent at the $r = 0$ origin of spherical coordinates. To bring this aspect into line with physical reality a minimum value for r, r_ϵ, is introduced within in which radius the densities are taken to have the value they have at $r = r_\epsilon$. This section will complete these aspect for the more general case of variable angular dependence that is being developed in this paper.

Suppose that we have obtained the complete form for the now angular dependant quantum theory *amplitude* for a possible mass accumulation in terms of all it's parameters as discussed above, $\Psi(r, t, l', \theta, \phi, m)$, say, with only the main parameters being explicitly mentioned. The amount of accumulated mass that this implies will have the form

$$\int\int\int_{all\ 3\ space} \Psi(r, t, l', \theta, \phi, m)\Psi^*(r, t, l', \theta, \phi, m)r^2 \sin(\theta)drd\theta d\phi, \quad (6.1)$$

where

$$\begin{aligned}\Psi(r, t, l', \theta, \phi, m) &= \mathtt{R}_{l'}(r)\Theta_{l',m}(\theta)\Phi_m(\phi)e^{-iE_{l',m}t/\hbar} \\ &= \mathtt{R}_{l'}(r)Ae^{im\phi}\sin^m(\theta)T^m_{l'-m}(\cos(\theta))e^{-iE_{l',m}t/\hbar} \\ &= \mathtt{R}_{l'}(r)Y_{1,l'm}(\theta, \phi)e^{-iE_{l',m}t/\hbar} \quad (6.2)\end{aligned}$$

where

$$\begin{aligned}Y_{1,l'm} &= Ae^{im\phi}P^m_{l'}(\cos(\theta)) \quad (6.3)\\ \dot{Y}_{l'm} &= N_{l'm}e^{im\phi}P^m_{l'}(\cos(\theta)) \quad (6.4)\end{aligned}$$

are firstly the non-normalised spherical harmonics then followed by the usual spherical harmonics with a possible normalisation factor $N_{l'm}$. Thus the total accumulated mass, (6.1), becomes

$$\int_0^\infty (\mathtt{R}_{l'}(r))^2r^2dr \int\int_{angles} |Y_{1,l',m}(\theta, \phi)|^2 \sin(\theta)d\theta d\phi \quad (6.5)$$

where

$$\int_0^{2\pi} d\phi \int_0^{\pi} |Y_{1,l',m}(\theta,\phi)|^2 \sin(\theta) d\theta = \frac{4\pi A^2}{\epsilon_m(2l'+1)} \left(\frac{(l'+m)!}{(l'-m)!}\right) \tag{6.6}$$

and using the possibly normalised case of the Ys denoted by $\dot{Y}$.

$$\begin{aligned}\int_0^{2\pi} d\phi \int_0^{\pi} |\dot{Y}_{l',m}(\theta,\phi)|^2 \sin(\theta) d\theta &= \frac{4\pi R_{l',m}^2}{\epsilon_m(2l'+1)} \left(\frac{(l'+m)!}{(l'-m)!}\right) = 1 \quad (6.7)\\ \implies R_{l',m} &= \pm\left(\frac{\epsilon_m(2l'+1)}{4\pi}\left(\frac{(l'-m)!}{(l'+m)!}\right)\right)^{1/2} \quad (6.8)\end{aligned}$$

giving the usual nomalisation value and in both of the cases above

$$\epsilon_0 = 1,\ \epsilon_m = 2,\ (|m| = 1,2,3,\ldots,l'). \tag{6.9}$$

In this version of this paper modulus signs about the m parameter in the three formulae (6.6)→(6.8) have been removed to avoid a local distortion of the mass spectrum that they induced. The earlier form with the mod signs was not valid for negative m. We notice that in the formula (6.6), if we take the arbitrary constant from now on to have the value $A = 1$, when $m = 0$

$$\int_0^{2\pi} d\phi \int_0^{\pi} |Y_{1,l',0}(\theta,\phi)|^2 \sin(\theta) d\theta = \frac{4\pi}{(2l'+1)} \tag{6.10}$$

as $\epsilon_0 = 1$. However when there is no angular dependence, $l' = 0$, the double angular integral has the value 4π, the solid angle subtended by a spherical surface at its centre. In the formation of the original mass spectra functions (2.5) and (2.6) this 4π was included from the start so that to convert those mass quantities into the new angular dependent mass quantities, they need only be multiplied by the function

$$A(l',m) = \frac{1}{\epsilon_m(2l'+1)} \left(\frac{(l'+m)!}{(l'-m)!}\right) \tag{6.11}$$

with the appropriate value for l' for the D and P cases. In the D case $l' = 2l-1$. In the P case $l' = 4l-2$. The two cases being

$$A(2l-1,m) = \frac{1}{\epsilon_m(4l-1)} \left(\frac{(2l-1+m)!}{(2l-1-m)!}\right) \tag{6.12}$$

$$A(4l-2,m) = \frac{1}{\epsilon_m(8l-3)}\left(\frac{(4l-2+m)!}{(4l-2-m)!}\right). \tag{6.13}$$

Thus the mass spectra function for the D and P cases are respectively

$$\begin{aligned} M_{l,m,D}(r_\epsilon,\theta_0,t_b) &= \frac{c^2\Lambda s(t_b)}{G}\left(\frac{(2l-1)^{4l}2l\theta_0^{2l}r_\epsilon^3}{3(4l-3)}\right)A(2l-1,m) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.14) \\ M_{l,m,P}(r_\epsilon,\theta_0,t_b) &= \frac{c^2\Lambda s(t_b)}{G}\left(\frac{(2l-1)^{8l-2}(4l-1)\theta_0^{4l-1}r_\epsilon^3}{(8l-5)}\right) \\ &\qquad \times A(4l-2,m). \qquad\qquad (6.15) \end{aligned}$$

The mass quantity for the amount of mass from $r = 0$ up to a radius r including both the mass density distribution and the Einstein pressure distribution for use in forming the total gravitational potential effective at radius r for a galaxy is given by

$$\begin{aligned} M_{l,m}(r) = \frac{c^2\Lambda s(t_b)\theta_0^{2l}r_\epsilon^3(2l-1)^{4l}A(2l-1,m)}{2G(4l-3)}\left(\frac{4l}{3}-\left(\frac{r}{r_\epsilon}\right)^{3-4l}\right) \\ + \quad \frac{3c^2\Lambda s(t_b)\theta_0^{4l-1}r_\epsilon^3(2l-1)^{8l-2}A(4l-2,m)}{2G(8l-5)}\left(\frac{(8l-2)}{3}-\left(\frac{r}{r_\epsilon}\right)^{5-8l}\right). \end{aligned} \tag{6.16}$$

The first two of the last three equations can now be seen as a refinement of the original mass spectra formula in that it now depend on the m parameter and from its derivation it is clear that l can now be interpreted as a value associated with a quantised angular momentum of the galaxy. These same remarks apply to the last term which now gives the mass $M_{l,m}(r)$ to be used in calculating the gravitational potential contributed by a not necessarily spherical galaxy at distance r from its centre.

7 Conclusions

This paper fundamentally is about a quantum theory for the dynamics of accumulations of mass that are in a state of self gravitating equilibrium. An obvious class of examples of matter in such a state is the galactic family. The structure of this theory and its mathematics has much in common with the mathematical theory of quantised atomic structure. In this paper, the original theory which only involved *spherically symmetric* mass distribution, only r dependence, has been substantially extended to include mass distributions that depend on the three variables of three dimensional polar coordinates r, θ, and ϕ and so making the theory more realistic. However, this is not a theory about the dynamics of arbitrary given quantities of mass. The mass accumulation quantities involved are themselves derived from general relativity theory supplemented with a new theory for self gravitating assembles of mass all based on Einstein's cosmological constant Λ. Thus there are two immediate results from the theory, mass spectra of the quantised mass distributions and the gravitation equivalent of Newton's inverse square law of gravitation that these quantised masses exert on other distant masses. In previous papers, I have shown that these mass accumulations generate *flat velocity-radius* curves on neighbouring matter in bound motion so complying with recent observations. In the title of this paper, I have mentioned the atomic quantum states and its s p d f g h i... classification of atomic states. This classification can be taken over into the quantum theory of galactic mass accumulations using the theory given above. To do this I will first give an alternative version of the *transplant* version of placing isothermal gravitation equilibrium states into the Schrödinger equation given above. From (3.23) and (3.24), we see that the general Schrödinger equation operator with its operand ψ can be written in the form

$$\hat{S}_{l',m}\psi = i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2\mu}\nabla^2\psi - \hat{U}(r)\psi \quad = \tag{7.1}$$

$$\frac{\hbar^2}{2\mu}\left(\frac{2\partial}{r\partial r} + \frac{\partial^2}{\partial r^2} + \frac{A_{l',m}(\theta,\phi)}{r^2}\right)\psi + (\hat{H} - \hat{U}(r))\psi \quad = \quad 0 \tag{7.2}$$

$$A_{l',m}(\theta,\phi)\psi = \left(\frac{\cot(\theta)\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{B_m(\phi)}{\sin^2\psi(\theta)}\right)\psi \quad = \quad -l'(l'+1)\psi \tag{7.3}$$

$$B_m(\phi)\psi = \frac{\partial^2\psi}{\partial\phi^2} = -m^2\psi \tag{7.4}$$

provided that ψ is a solution in the separated parameter product form. It will be useful to express the three dimensional Laplace operator squared, (3.10), in a similar form as the Schrödinger operator above,

$$
\begin{aligned}
\hat{L}_{l',m}\psi = \nabla^2\psi &= \quad (7.5)\\
\left(\frac{2\partial}{r\partial r} + \frac{\partial^2}{\partial r^2} + \frac{A_{l',m}(\theta,\phi)}{r^2}\right)\psi &= 0 \quad (7.6)\\
A_{l',m}(\theta,\phi)\psi = \left(\frac{\cot(\theta)\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{B_m(\phi)}{\sin^2\psi(\theta)}\right)\psi &= -l'(l'+1)\psi \quad (7.7)\\
B_m(\phi)\psi = \frac{\partial^2\psi}{\partial\phi^2} = -m^2\psi & \quad (7.8)
\end{aligned}
$$

The quantum theory of galactic structures has, in earlier papers and in this paper, identified four *fundamental* types of solution each with its own Schrödinger equation $V(\mathbf{r})$ as follows

$$
\begin{aligned}
V_{Ds,l}(\mathbf{r}) &= E_{Ds,l} + \frac{\hbar^2 l(2l-1)}{\mu r^2} = E_{Ds,l} + \frac{\hbar^2 q_{Ds,l}}{\mu r^2} \quad (7.9)\\
V_{Ps,l}(\mathbf{r}) &= E_{Ps,l} + \frac{\hbar^2(2l-1)(4l-1)}{\mu r^2} = E_{Ps,l} + \frac{\hbar^2 q_{Ps,l}}{\mu r^2} \quad (7.10)\\
V_{D,l,m}(\mathbf{r}) &= E_{D,l,m} \quad (7.11)\\
V_{P,l,m}(\mathbf{r}) &= E_{P,l,m}. \quad (7.12)
\end{aligned}
$$

From (7.1), the four Schrödinger equation operators that go with these potentials are respectively

$$\begin{aligned}
\hat{S}_{0,0,D}\psi &= i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2\mu}\nabla^2\psi - V_{Ds,l}(\mathbf{r})\psi = \frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{\hbar^2 q_{Ds,l}}{\mu r^2}\psi \\
&= \frac{\hbar^2}{2\mu}\hat{L}_{2l-1,0}\psi, \qquad i\hbar\frac{\partial}{\partial t}\psi = E_{D,l}\psi \qquad (7.13)\\
\hat{S}_{0,0,P}\psi &= i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2\mu}\nabla^2\psi - V_{Ps,l}(\mathbf{r})\psi = \frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{\hbar^2 q_{Ps,l}}{\mu r^2}\psi \\
&= \frac{\hbar^2}{2\mu}\hat{L}_{4l-2,0}\psi, \qquad i\hbar\frac{\partial}{\partial t}\psi = E_{P,l}\psi \qquad (7.14)\\
\hat{S}_{2l-1,m,D}\psi &= i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2\mu}\nabla^2\psi - V_{D,l}(\mathbf{r})\psi,\ V_{D,l}(\mathbf{r}) = E_{D,l,m} \\
&= \frac{\hbar^2}{2\mu}\hat{L}_{2l-1,m}\psi, \qquad i\hbar\frac{\partial}{\partial t}\psi = E_{D,l,m}\psi \qquad (7.15)\\
\hat{S}_{4l-2,m,P}\psi &= i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2\mu}\nabla^2\psi - V_{P,l}(\mathbf{r})\psi,\ V_{P,l}(\mathbf{r}) = E_{P,l,m} \\
&= \frac{\hbar^2}{2\mu}\hat{L}_{4l-2,m}\psi, \qquad i\hbar\frac{\partial}{\partial t}\psi = E_{P,l,m}\psi. \qquad (7.16)
\end{aligned}$$

These are the basic Schrödinger equations for the D and P type solutions. More complex solutions such as the galactic solutions which involve simultaneously a sum of D type for Einstein's mass density term and a P for his pressure term come from superposition of solutions. In the list of Schrödinger operators and operands above each Schrödinger case is followed, one step down, by two equations, an l' restricted Laplace operator with operand and a quantised energy equation which together are equivalent to the preceding Schrödinger equation.

The first two cases above are from the pre-angular dependent work only involve s states as indicated by the s and $l' = 0$ subscript on the Schrödinger operator. However, for these first two cases, all the isothermal gravitational equilibrium states map into an s state giving the mass spectra found in earlier work, (2.5) and (2.6). The first two cases above look very much like the second two cases above. However, for the first two cases the angular momentum content of the Laplace versions of these two cases are picked up from the *non-angular* related external potential term in l whereas, the angular momentum content of the Laplace equation for the second two cases is picked up from the *angular* content of the l' term at (7.3). The last two cases above D and P map the quantum lettered states, $l' = 2l - 1$, and $l' = 4l - 2$ respectively into values of the isothermal gravitational equilibrium states $(l : 1, 2, 3, 4, \dots)$ as follows; For D states $l' = 2l - 1$

$$
\begin{aligned}
l &= 1 \rightarrow l' = 1 \equiv p && (7.17)\\
l &= 2 \rightarrow l' = 3 \equiv f && (7.18)\\
l &= 3 \rightarrow l' = 5 \equiv h && (7.19)\\
l &= 4 \rightarrow l' = 7 \equiv k && (7.20)\\
l &= 5 \rightarrow l' = 9 \equiv m && (7.21)\\
\vdots &= \quad \dots && (7.22)
\end{aligned}
$$

For P states $l' = 4l - 2$

$$
\begin{aligned}
l &= 1 \rightarrow l' = 2 \equiv d && (7.23)\\
l &= 2 \rightarrow l' = 6 \equiv i && (7.24)\\
l &= 3 \rightarrow l' = 10 \equiv m && (7.25)\\
l &= 4 \rightarrow l' = 14 \equiv t && (7.26)\\
l &= 5 \rightarrow l' = 18 \equiv x && (7.27)\\
\vdots &= \quad \dots \quad . && (7.28)
\end{aligned}
$$

It is noticeable that with advancing l values of the isothermal equilibrium states the P solutions advance along the letter classification of the angularly involved states much faster than do the D solutions. There may well be other and better ways to get and express the results of this paper. The course I have chosen is, I think, likely to be the simplest. The reason for emphasising and using the *transplant* idea is that it seems to expose something very fundamental about what I have identified as the inverse cube force that seems to assist gravity in keeping large mass accumulations stable. This force arises from an inverse square law potential. From the work above and in the gravitational context this force can be seen as arising from the Schrödinger equation ∇^2 term's angular momentum part. It is as though this force gravitationally counters rotational *centrifugal* force. This facet seems to be a special aspect of the gravitational equilibrium for astrophysical mass accumulations.

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