Invertible dynamic systems

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Abstract: - In this paper we introduce the notion of invertible dynamic system, we indicate a very general method to determine the inverse of such a system and we give evidence of the numerous applications of the subclass of dynamic systems defined by this notion.

Key-Words: - Invertible dynamic system, the inverse of a dynamic system, Lagrange interpolation polynomial.

1 Introduction

Let us start from the general definition of finite dimensional dynamic system formulated only in terms of input-output. If by $p_1,..,p_n$ we denote the inputs, and by $r_1,..,r_n$ the outputs of such a system, then it can be mathematically abstracted by a set of n equations

$$\Sigma:\begin{cases} r_1(q)=f_1(p_1,..,p_n,q) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ r_n(q)=f_n(p_1,..,p_n,q) \end{cases}, \qquad (1)$$

in which the parameter q expresses the possible states of the system or the operating mode in which it finds itself when it receives the impulses $p_1,..,p_n$. The parameter q can be, depending on the nature of the system that it characterizes, a scalar or a vector. Just as in the case of some algebraic system with n equations and n unknowns, we can ask ourselves whether in the case of the dynamic system (1) the inputs $p_1,..,p_n$ could be determined in terms of the outputs $r_1,..,r_n$. Such a question does not only have a speculative base, but also a base imposed by the concrete situations in which the dynamic systems are used to model different types of processes from surrounding world.

For instance, when we want to determine the constitutive parameters (or the quantitative characteristics) $p_1,..,p_n$ of a physical entity (or of some natural phenomenon), that we can only perceive through its interactions $r_1,..,r_n$ with a certain system of detection Σ. Under such a hypothesis, the problem that must be solved is that of indicating a way by which the numerical values $r_1(q),..,r_n(q)$ of the outputs of the system Σ (which, by the time of the experiment is in the state q) to be converted into the numerical values $p_1(q),..,p_n(q)$ of the searched parameters. Thus, from mathematical point of view we have to determine a set of n functions

$$\begin{cases} p_1(q) = g_1(r_1,..,r_n,q) \\ .... , \\ p_n(q) = g_n(r_1,..,r_n,q) \end{cases} \quad (2)$$

that would invert in a certain sense the functions of set (1).

In case the set of relations (2) can be determined, let us notice that this defines at its turn a new dynamic system in which the inputs are the outputs and the outputs are the inputs of the system (1). The dynamic system defined by the relations (2) will be denoted symbolically by Σ^{-1} and by definition will be named the inverse of the system Σ.

Remarks : 1) If, in the case of a dynamic system of the form (1), the expressions of the functions

$$f_1 = f_1(p_1,..,p_n,q),.., f_n = f_n(p_1,..,p_n,q)$$

would be known, then, the problem enounced earlier would reduce itself to a concrete algebra problem. Nevertheless, even so, the problem of solving the system (1) with respect to the unknowns $p_1,..,p_n$, might not be possible (the existence or the non-existence of the solutions of this system can be established with the help of the implicit function theorem). For this reason, in order to differentiate between the two possible cases we will use the terms of "solvable system" and "non-solvable system".
2) It is very important to keep in mind that the problem we want to approach in this paper has in view the general and very abstract case of dynamic systems used for modelling the phenomena and the processes occurring in nature, that is expressly, exactly those cases in which the functions

$$f_1 = f_1(p_1,..,p_n,q),.., f_n = f_n(p_1,..,p_n,q),$$

are not explicitly known. For instance, the monetary policies reflect themselves onto the economical systems through the effects they produce, but this fact does not allow us to actually know the internal mechanisms that are producing these effects (formally expressed by an sequence of functions $f_1,..,f_n$), although we assume that they exist.

For this reasons, the problem of finding the inverse of a dynamic system cannot be faced in a general way. The main result of this paper consists in offering practitioners a concrete method of approximation no matter how accurate the inverse of a solvable dynamic system in all its generality is.

2 The approximation of the inverse of a solvable dynamic system

Before describing the method by which we can approximate the inverse of a dynamic system of the form (1), we must define the conditions in which this thing is possible.

2.1 The working hypothesis

We suppose that the minimum and maximum values of the parameters p_1 and r_1 are respectively b_1 and B_1, or in other words that the range of parameters p_1 and r_1 is the interval $[b_1, B_1]$. We make the same assumption for the all parameters, namely for parameter p_2 and r_2 we will suppose that they varies within the interval $[b_2, B_2]$,…, and for p_n and r_n in the interval $[b_n, B_n]$. These hypotheses are not essential to reaching the set goal, but considerably simplify the way in which theory is presented.

We will also require that the correspondence

$$[b_1,B_1]\times\cdots\times[b_n,B_n]\ni(p_1(q),..,p_n(q))\leftrightarrow$$

$$\leftrightarrow(r_1(q),..,r_n(q))\in[b_1,B_1]\times\cdots\times[b_n,B_n], \quad (3)$$

be one-to-one, separately, for each value of parameter q. This requirement constitutes the hypothesis which will guarantee the compatibility of the mathematical model we are about to build.

2.2 The method of approximation

We will divide the interval $[b_1, B_1]$ in m_1 equidistant subintervals (the requirement that the subintervals be equally distanced is not necessary for solving the problem, it only simplifies its presentation) of ends $p_1^{j_1} = b_1 + h_1 j_1$, $j_1 = 0,1,2,..,m_1$, where $h_1 = \frac{B_1 - b_1}{m_1}$. We will do the same with the other intervals $[b_2, B_2]$, .., $[b_n, B_n]$: for example

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the last interval $\left[b_n, B_n\right]$ will be divided into a number m_n of equidistant subintervals of ends $p_n^{j_n} = b_n + h_n j_n$, $j_n = 0, 1, 2,.., m_n$, where $h_n = \frac{B_n - b_n}{m_n}$.

If within the system (1) we make the following substitutions $p_1 = p_1^{j_1}$, $p_2^{j_2} = p_2$,..., $p_n = p_n^{j_n}$, for each of the index value $j_1 = 0,1,2,..,m_1$, $j_2 = 0,1,2,..,m_2$, .., $j_n = 0,1,2,..,m_n$, as a result we will obtain the following correspondence

$$\left(p_1^{j_1},..,p_n^{j_n},q\right) \to \left(r_1^{j_1 \dots j_n}(q),..,r_n^{j_1 \dots j_n}(q)\right), \qquad (4)$$

where $\left(r_1^{j_1 \dots j_n}(q),..,r_n^{j_1 \dots j_n}(q)\right)$ represent the output of the system Σ functioning in working mode q, to the input $\left(p_1^{j_1},..,p_n^{j_n}\right)$.

Under these circumstances, due to the hypothesis accepted in the previous subparagraph, for each q, we can discuss about the inverted application

$$\left(r_1^{j_1 \dots j_n}(q),..,r_n^{j_1 \dots j_n}(q)\right) \to \left(p_1^{j_1},..,p_n^{j_n},q\right). \qquad (5)$$

So, if we assume the functions $g_1 = g_1\left(r_1,.., r_n, q\right),.., g_n = g_n\left(r_1,.., r_n, q\right)$ from the inverse system (2) to be known then, we should have fulfilled the conditions

$$\begin{cases} p_1^{j_1}(q) = g_1\left(r_1^{j_1 .. j_n}(q),..,r_n^{j_1 .. j_n}(q),q\right) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ p_n^{j_n}(q) = g_n\left(r_1^{j_1 .. j_n}(q),..,r_n^{j_1 .. j_n}(q),q\right) \end{cases}, \qquad (6)$$

for any set of values $\left(j_1,..,j_n\right)$, $0 \le j_1 \le m_1$,.., $0 \le j_n \le m_n$. Since the found relations (6), are the only known data of the problem, it is clear that the problem of finding the functions $g_1 = g_1\left(r_1,.., r_n, q\right),.., g_n = g_n\left(r_1,.., r_n, q\right)$ from the relations set (2) must be solved based only on these. In mathematics this type of problem is solved using one of the known interpolation techniques. We propose to express the unknown functions $g_1,..,g_n$ through Lagrange type interpolation polynomials, of n independent variables, which satisfy the condition set (6). To this end, for each fixed values set $\left(j_1,..,j_n\right)$, $0 \le j_1 \le m_1$,.., $0 \le j_n \le m_n$, we will define the polynomial

$$L_q^{j_1 \dots j_n}\left(r_1,..,r_n\right) = \prod_{i=1}^{n} \left(\prod_{r_i^{k_1 \dots k_n}(q) \neq r_i^{j_1 \dots j_n}(q)} \frac{r_i - r_i^{k_1 \dots k_n}(q)}{r_i^{j_1 \dots j_n}(q) - r_i^{k_1 \dots k_n}(q)} \right) \qquad (7)$$

Polynomials thus built have the properties:

$$L_q^{j_1 \dots j_n}\left(r_1^{k_1 \dots k_n}(q),.., r_n^{k_1 \dots k_n}(q)\right) = \begin{cases} 1, & \left(k_1,..,k_n\right) = \left(j_1,..,j_n\right) \\ 0, & \left(k_1,..,k_n\right) \neq \left(j_1,..,j_n\right) \end{cases}. \qquad (8)$$

Observation: Due to the one-to-one correspondence between

$$\left(r_1^{k_1 \dots k_n}(q),..,r_n^{k_1 \dots k_n}(q)\right)$$

and

$$\left(p_1^{k_1},..,p_n^{k_n},q\right) \equiv \left(p_1^{k_1}(q),..,p_n^{k_n}(q)\right),$$

and between

$$\left(p_1^{k_1}(q),..,p_n^{k_n}(q)\right) \text{ and } \left(k_1,..,k_n\right),$$

where parameter q is considered fixed, and $\left(k_1,..,k_n\right) \in \{0,1,..,m_1\} \times \dots \times \{0,1,.., m_n\}$, we deduce that

$$\left(r_1^{k_1 \dots k_n}(q),..,r_n^{k_1 \dots k_n}(q)\right) = \left(r_1^{j_1 \dots j_n}(q),..,r_n^{j_1 \dots j_n}(q)\right), \qquad (9)$$

if and only if $\left(k_1,..,k_n\right) = \left(j_1,..,j_n\right)$. With the help of polynomials $L_q^{j_1 \dots j_n}$, $0 \le j_1 \le m_1,...,0 \le j_n \le m_n$, thus built, and of the values $p_i^{j_i} = b_i + h_i j_i$, $i = 1, 2,.., n$, used to calculate the values $r_1^{j_1 \dots j_n}(q)$,.., $r_n^{j_1 \dots j_n}(q)$ we will define the functions:

$$g_i\left(r_1,..,r_n,q\right) = \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} L_q^{j_1 \dots j_n}\left(r_1,..,r_n\right) \cdot p_i^{j_1 \dots j_n}, \; i = 1,2,..,n, \qquad (10)$$

where $p_i^{j_1 \dots j_n}$ represents the element $p_i^{j_i}$ uniquely determined by the one-to-one correspondence

$$\left(r_1^{j_1\ldots j_n}(q),..,r_n^{j_1\ldots j_n}(q)\right) \leftrightarrow \left(p_1^{j_1},.., p_n^{j_n}, q\right)$$

earlier established.

It is easy to observe that the set of functions (10) verifies the conditions (6) and thus they are a solution of the problem we had to solve.

Remarks: 1) The studied problem can be solved through other interpolation schemes as well.

2) It should be noted that these interpolation schemes are more precise as the number of the test inputs $p_1^{j_1},..,p_n^{j_n}$, $0 \le j_1 \le m_1,..,$ $0 \le j_n \le m_n$, used to identify the system Σ, is greater.

3) Between two different interpolation schemes (but which use the same test inputs) it cannot be decided a priori which is better, as this depends on the particular nature of the system Σ.

4) The approximation method of the system Σ^{-1} is not unique. Instead of the Lagrange type interpolation polynomial, any other approximation formula of a function through intermediate values can be used.

4 Applications

In order to determine the functional characteristics and the performances of an actuator, the classical control method consisted in measuring directly these characteristics and performances with the help of transducers mounted onto the actuator undergoing the test. As it can be easily noticed, this method is disadvantageous for a real time control.

In order to overcome this situation, the novel proposed method of indirect measuring the performance of actuators was given in [1,2,3]. Supposing that the technical performances of the actuators we are about to test are expressed through a set of n standards or functioning specifications, in accordance to the new control method, verifying these parameters is not done directly, but through a reference actuator to which the trial actuators are connected, one at a time. In order to distinguish between the working parameters of the tested actuators and the response values of the functioning parameters of the reference actuator, to which these will be connected, we agree to denote by $p_1,..,p_n$ the working parameters of the trial actuators and by $r_1,..,r_n$ the working parameters of the reference actuator. With these preparations, a mathematical approach for the novel proposed method is to consider it an "input-output system", like (1). It should be noted that, except for the parameter q which is part of the input data $\left(p_1,..,p_n,q\right)$, the values of the other parameters $p_1,..,p_n$ are not known explicitly, as they interact with the system Σ through the trial actuator which they describe implicitly. In this context, the mathematical model we have built has the role of converting the numerical values of the system outputs Σ: $\left(r_1(q),..,r_n(q)\right)$, into the numerical values $\left(p_1(q),..,p_n(q)\right)$, of the trail actuator parameters.

The presented method has the advantage of ensuring a decrease in the trials' duration and the realization of complex trial regimes.

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