Connections between the three prime factors of 3-Carmichael numbers

Marius Coman
Bucuresti, Romania
email: mariuscoman13@gmail.com

Abstract. It was always obvious to me that, beside Korselt's criterion, that gives a relation between any prime factor of a Carmichael number and the number itself, there must be a relation between the prime factors themselves; here I present a conjecture on the Carmichael numbers with three prime factors expressing the larger two prime factors as a function of the smallest one and few particular cases of connections between all three prime factors.

Introduction:

In the sequence A213812 that I posted in OEIS I showed a formula, derived from Korselt's criterion, to express a Carmichael number as a function of any of its prime factors and an integer. In the sequence A215672 that I posted in OEIS I extended this formula for a Poulet number with three or more prime factors, expressing such a number as a function of at least one of its prime factors and an integer. This formula relates a Fermat pseudoprime to one (in the case of Poulet numbers) or to any (in the case of Carmichael numbers) of its prime factors, but says nothing about the relation between the prime factors themselves.

In the sequence A215672 I showed that most of Fermat pseudoprimes to base 2 with three prime factors (so, implicitly, most of Carmichael numbers with three prime factors) can be written in one of the following two ways:

(1) p*((n + 1)*p - n)*((m + 1)*p - m);
(2) p*((n*p - (n + 1))*(m*p - (m + 1)),

where p is the smallest of the three prime factors and n, m are natural numbers.

Exempli gratia for Poulet numbers from first category:
10585 = 5*29*73 = 5*(5*7 - 6)*(5*18 - 17).
Exempli gratia for Poulet numbers from second category:
6601 = 7*23*41 = 7*(7*4 - 5)*(7*7 - 8).

From the first 37 Poulet numbers with three prime factors, just three (30889, 88561 and 91001) can’t be written in one of this two ways.

Conjecture: For any Carmichael numbers with three prime factors, $C = d_1*d_2*d_3$, where $d_1 < d_2 < d_3$, is true one of the following two statements:

(1) d_2 can be written as $d_1*(n + 1) - n$ and d_3 can be written as $d_1*(m + 1) - m$;

(2) d_2 can be written as $d_1*n - (n + 1)$ and d_3 can be written as $d_1*m - (m + 1)$,

where m and n are natural numbers.

As I showed, this conjecture holds for the first 13 Carmichael numbers with three prime factors checked. In this article I present few connections that express not the larger two prime factors as a function of the smallest one, as above, but connects all the three prime factors.

Observation: For most of the Carmichael numbers with three prime factors, $C = d_1*d_2*d_3$, where $d_1 < d_2 < d_3$, is true one of the following seventh statements:

(1) d_3 can be written as $d_1*(m + 1) - n$ and as well as $d_2*(n + 1) - m$;

(2) d_3 can be written as $d_1*(m - 1) + n$ and as well as $d_2*(n - 1) + m$;

(3) d_3 can be written as $d_1 + (m + 1)*n$ and as well as $d_2 + m*n$;

(4) d_3 can be written as $d_1*m - 2*n$ and as well as $d_2*n + 2*m$;

(5) d_3 can be written as $d_1*m + 2*n$ and as well as $d_2*n - 2*m$;

(6) d_3 can be written as $d_1*m - 2*n$ and as well as $d_2*n + m$;

(7) d_3 can be written as $d_1*m + n$ and as well as $d_2*n - 2*m$,

where m and n are natural numbers.

Carmichael numbers which verify the first statement:

For C = 561 = 3*11*17 we have [m, n] = [5, 1]:
Indeed, 3*(5 + 1) - 1 = 17 and 11*(1 + 1) - 5 = 17.

For C = 162401 = 17*41*233 we have [m, n] = [13, 5]:
Indeed, 17*(13 + 1) - 5 = 233 and 41*(5 + 1) - 13 = 233.

For C = 314821 = 13*61*397 we have [m, n] = [30, 6]:
Indeed, 13*(30 + 1) - 6 = 397 and 61*(6 + 1) - 30 = 397.

Carmichael numbers which verify the second statement:

For C = 1105 = 5*13*17 we have [m, n] = [4, 2]:
Indeed, 5*(4 - 1) + 2 = 17 and 13*(2 - 1) + 4 = 17.

For C = 2821 = 7*13*31 we have [m, n] = [5, 3]:
Indeed, 7*(5 - 1) + 3 = 31 and 13*(3 - 1) + 5 = 31.

For C = 8911 = 7*19*67 we have [m, n] = [10, 4]:
Indeed, 7*(10 - 1) + 4 = 67 and 19*(4 - 1) + 10 = 67.

For C = 10585 = 5*29*73 we have [m, n] = [15, 3]:
Indeed, 5*(15 - 1) + 3 = 73 and 29*(3 - 1) + 15 = 73.

For C = 15841 = 7*31*73 we have [m, n] = [11, 3]:
Indeed, 7*(11 - 1) + 3 = 73 and 31*(3 - 1) + 11 = 73.

For C = 115921 = 13*37*241 we have [m, n] = [19, 7]:
Indeed, 13*(19 - 1) + 7 = 241 and 37*(7 - 1) + 19 = 241.

For C = 314821 = 13*61*397 we have [m, n] = [31, 7]:
Indeed, 13*(31 - 1) + 7 = 397 and 61*(7 - 1) + 31 = 397.

For C = 334153 = 19*43*409 we have [m, n] = [22, 10]:
Indeed, 19*(22 - 1) + 10 = 409 and 43*(10 - 1) + 22 = 409.

Carmichael numbers which verify the third statement:

For C = 1729 = 7*13*19 we have [m, n] = [1, 6]:
Indeed, 7 + 2*6 = 19 and 13 + 6 = 19.

For C = 2465 = 5*17*29 we have [m, n] = [1, 12]:
Indeed, 5 + 2*12 = 29 and 17 + 12 = 29.

For C = 29341 = 13*37*61 we have [m, n] = [1, 24]:
Indeed, 13 + 2*24 = 61 and 37 + 24 = 61.

For C = 252601 = 41*61*101 we have [m, n] = [2, 32]:
Indeed, 41 + 3*20 = 101 and 61 + 2*20 = 101.

For C = 294409 = 37*73*109 we have [m, n] = [1, 36]:
Indeed, 37 + 2*36 = 109 and 73 + 36 = 109.

For C = 399001 = 31*61*211 we have [m, n] = [5, 36]:
Indeed, 31 + 6*30 = 211 and 61 + 5*30 = 211.

For C = 410041 = 41*73*137 we have [m, n] = [2, 32]:
Indeed, 41 + 3*32 = 137 and 73 + 2*32 = 137.

For C = 488881 = 37*73*181 we have [m, n] = [3, 36]:
Indeed, 37 + 4*36 = 181 and 73 + 3*36 = 181.

For C = 512461 = 31*61*271 we have [m, n] = [7, 30]:
Indeed, 31 + 8*30 = 271 and 61 + 7*30 = 271.

For C = 1152271 = 43*127*211 we have [m, n] = [1, 84]:
Indeed, 43 + 2*84 = 211 and 127 + 84 = 211.

For C = 1152271 = 43*127*211 we have [m, n] = [1, 84]:
Indeed, 43 + 2*84 = 211 and 127 + 84 = 211.

For C = 1857241 = 31*181*331 we have [m, n] = [1, 150]:
Indeed, 31 + 2*150 = 331 and 181 + 150 = 331.

Carmichael numbers which verify the fourth statement:

For C = 52633 = 7*73*103 we have [m, n] = [15, 1]:
Indeed, 7*15 – 2*1 = 103 and 73*1 + 2*15 = 103.

For C = 1461241 = 37*73*541 we have [m, n] = [15, 7]:
Indeed, 37*15 – 2*7 = 541 and 73*7 + 2*15 = 541.

Carmichael numbers which verify the fifth statement:

For C = 46657 = 13*37*97 we have [m, n] = [7, 3]:
Indeed, 13*7 + 2*3 = 97 and 37*3 - 2*7 = 97.

Carmichael numbers which verify the sixth statement:

For C = 1193221 = 31*61*631 we have [m, n] = [21, 10]:
Indeed, 31*21 - 2*10 = 631 and 61*10 + 21 = 631.

Carmichael numbers which verify the seventh statement:

For C = 530881 = 13*97*421 we have [m, n] = [32, 5]:
Indeed, 13*32 + 5 = 421 and 97*5 - 2*32 = 421.

Note: From the first 31 Carmichael numbers with three prime factors checked, only four of them (6601 = 7*23*41, 1024651 = 19*199*271, 1615681 = 23*199*353 and 1909001 = 41*101*461) don't satisfy any of the seventh statements.

Note: Obviously the prime factors of Chernick's Carmichael numbers satisfy the third statement.

Note: There are Carmichael numbers, like 314821 = 13*61*397, that satisfy both the first and the second statement. The triplets of primes like $[p_1, p_2, p_3]$ = [13, 61, 397], for which $p_3 = p_1*(m + 1) - n = p_2*(n + 1) - m = p_1*m + n + 1 = p_2*n + m + 1$, deserve further study, also the question if and when the products $p_1*p_2*p_3$ are Carmichael numbers.

Note: The Carmichael number 252601 = 41*61*101 can be written as p*(p*n – m)*(p*(n + 1) – (m + 1)), where p is prime and m, n natural numbers (because 61 = 41*2 – 21 and 101 = 41*3 – 22). Also the triplets of primes of the form [p, p*n – m, p*(n + 1) – (m + 1)] deserve further study as well as the question if and when the products of the primes that form such a triplet are Carmichael numbers.

Note: For Carmichael numbers with three prime factors, see the sequence A087788 in OEIS.