Some Algebraic Identities Involving four Square

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ABSTRACT. We have developed some algebraic identities related to power three as: $4(a^2c^2+a^2d^2+b^2c^2+b^2d^2)^3=[(a+b)(a^2-4ab+b^2)(c+d)(c^2-4cd+d^2)]^2+[(a+b)(a^2-4ab+b^2)(c-d)(c^2+4cd+d^2)]^2+[(a-b)(a^2+4ab+b^2)(c+d)(c^2-4cd+d^2)]^2+[(a-b)(a^2+4ab+b^2)(c-d)(c^2+4cd+d^2)]^2$.

I. Identities

Lemma 1. *For* $x,y\in\mathbb{R}$, *then*

(1) $$(x+iy)^3+(x-iy)^3=2x(x^2-3y^2).$$

Proof. We expand the left-hand side of (1)

$$(x+iy)^3+(x-iy)^3=x^3+3ix^2y-3xy^2-iy^3+x^3-3ix^2y-3xy^2+iy^3$$

$$=x^3-3xy^2+x^3-3xy^2$$

$$=2x^3-6xy^2$$

$$=2x(x^2-3y^2).\square$$

Lemma 2. *For* $x,y\in\mathbb{R}$, *then*

(2) $$(y+ix)^3+(y-ix)^3=2y(y^2-3x^2).$$

Proof. We expand the left-hand side of (2)

$$(y+ix)^3+(y-ix)^3=-ix^3-3x^2y+3ixy^2+y^3+ix^3-3x^2y-3ixy^2+y^3$$

$$=-3x^2y+y^3-3x^2y+y^3$$

$$=-6x^2y+2y^3$$

$$=2y(y^2-3x^2).\square$$

Lemma 3. *For* $x,y\in\mathbb{R}$, *then*

(3) $$(x+iy)^3+(x-iy)^3+(y+ix)^3+(y-ix)^3=2(x+y)(x^2-4xy+y^2).$$

Proof. We expand the left-hand side of (3)

$$(x+iy)^3+(x-iy)^3+(y+ix)^3+(y-ix)^3=2x(x^2-3y^2)+2y(y^2-3x^2)\text{ (by Lemma 1 and 2)}$$

$$=2[x(x^2-3y^2)+y(y^2-3x^2)]$$

$$= 2(x^3 - 3x^2y - 3xy^2 + y^3)$$

$$= 2[x(x^2 - 3xy) + y(-3xy + y^2)]$$

$$= 2[x(x^2 - 4xy + xy) + y(xy - 4xy + y^2)]$$

$$= 2[x(x^2 - 4xy + y^2 - y^2 + xy) + y(xy - x^2 + x^2 - 4xy + y^2)]$$

$$= 2[x(x^2 - 4xy + y^2) + x^2y - xy^2 - x^2y + xy^2 + y(x^2 - 4xy + y^2)]$$

$$= 2[x(x^2 - 4xy + y^2) + y(x^2 - 4xy + y^2)]$$

$$= 2(x + y)(x^2 - 4xy + y^2). \square$$

Lemma 4. *For* $x, y \in \mathbb{R}$, *then*

(4) $$(x + iy)^3 - (x - iy)^3 = 2iy(3x^2 - y^2).$$

Proof. We expand the left-hand side of (4)

$$(x + iy)^3 - (x - iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3 - x^3 + 3ix^2y + 3xy^2 - iy^3$$

$$= 3ix^2y - iy^3 + 3ix^2y - iy^3$$

$$= 6ix^2y - 2iy^3$$

$$= 2iy(3x^2 - y^2). \square$$

Lemma 5. *For* $x, y \in \mathbb{R}$, *then*

(5) $$(y - ix)^3 - (y + ix)^3 = 2ix(x^2 - 3y^2).$$

Proof. We expand the left-hand side of (5)

$$(y - ix)^3 - (y + ix)^3 = ix^3 - 3x^2y - 3ixy^2 + y^3 + ix^3 + 3x^2y - 3ixy^2 - y^3$$

$$= ix^3 - 3ixy^2 + ix^3 - 3ixy^2$$

$$= 2ix^3 - 6ixy^2$$

$$= 2ix(x^2 - 3y^2). \square$$

Lemma 6. *For* $x, y \in \mathbb{R}$, *then*

(6) $$(x + iy)^3 - (x - iy)^3 + (y - ix)^3 - (y + ix)^3 = 2i(x - y)(x^2 + 4xy + y^2)$$

Proof. We expand the left-hand side of (6)

$$(x + iy)^3 - (x - iy)^3 + (y - ix)^3 - (y + ix)^3 = 2iy(3x^2 - y^2) + 2ix(x^2 - 3y^2) \text{ (by Lemma 4 and 5)}$$

$$= 2i[y(3x^2 - y^2) + x(x^2 - 3y^2)]$$

$$= 2i(3x^2y - y^3 + x^3 - 3xy^2)$$

$$= 2i[x(x^2 + 3xy) - y(3xy + y^2)]$$

$$= 2i[x(x^2 + 4xy - xy) - y(-xy + 4xy + y^2)]$$

$$= 2i[x(x^2 + 4xy + y^2 - y^2 - xy) - y(-x^2 - xy + x^2 + 4xy + y^2)]$$

$$= 2i[x(x^2 + 4xy + y^2) - xy^2 - x^2y + x^2y + xy^2 - y(x^2 + 4xy + y^2)]$$

$$= 2i[x(x^2 + 4xy + y^2) - y(x^2 + 4xy + y^2)]$$

$$= 2i(x - y)(x^2 + 4xy + y^2). \square$$

Theorem 1. *For* $x, y \in \mathbb{R}$, *then*

(7) $[(x + iy)^3 + (x - iy)^3 + (y + ix)^3 + (y - ix)^3]^2 - [(x + iy)^3 - (x - iy)^3 + (y - ix)^3 - (y + ix)^3]^2 =$

$$= [2(x + y)(x^2 - 4xy + y^2)]^2 + [2(x - y)(x^2 + 4xy + y^2)]^2 = [2(x^2 + y^2)]^3.$$

Proof. We expand the left-hand side of (7)

$[(x + iy)^3 + (x - iy)^3 + (y + ix)^3 + (y - ix)^3]^2 - [(x + iy)^3 - (x - iy)^3 + (y - ix)^3 - (y + ix)^3]^2 =$

$$= [2(x + y)(x^2 - 4xy + y^2)]^2 - [2i(x - y)(x^2 + 4xy + y^2)]^2$$

$$= [2(x + y)(x^2 - 4xy + y^2)]^2 + [2(x - y)(x^2 + 4xy + y^2)]^2$$

$$= 4\{[(x + y)(x^2 - 4xy + y^2)]^2 + [(x - y)(x^2 + 4xy + y^2)]^2\}$$

$$= 4(x^6 - 6x^5y + 3x^4y^2 + 20x^3y^3 + 3x^2y^4 - 6xy^5 + y^6$$

$$+x^6 + 6x^5y + 3x^4y^2 - 20x^3y^3 + 3x^2y^4 + 6xy^5 + y^6)$$

$$= 4(2x^6 + 6x^4y^2 + 6x^2y^4 + 2y^6)$$

$$= 8(x^6 + 3x^4y^2 + 3x^2y^4 + y^6)$$

$$= [2(x^2 + y^2)]^3. \square$$

Theorem 2. *For* $x, y \in \mathbb{R}$, *then*

(8) $\{(x - y)^3 + (x + y)^3 - i[(y - x)^3 + (y + x)^3]\}^2 - \{(x - y)^3 - (x + y)^3 - i[(y - x)^3 - (y + x)^3]\}^2 =$

$$= [2(x + iy)(x^2 - 4ixy - y^2)]^2 + [2(x - iy)(x^2 + 4ixy - y^2)]^2 = [2(x^2 - y^2)]^3.$$

Proof. We let $y \to iy$ in both side of Theorem 1. $\square$

Examples

$10^3 = [2(3^2 - 2^2)]^3 = [2(63 - 62i)]^2 + [2(63 + 62i)]^2;$

$14^3 = [2(4^2 - 3^2)]^3 = [2(172 - 171i)]^2 + [2(172 + 171i)]^2.$

Theorem 3. *For* $x, y \in \mathbb{R}$, *then*

(9) $$\left\{\left(\sqrt{\frac{x}{2}} - \sqrt{\frac{y}{2}}\right)^3 + \left(\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}}\right)^3 - i\left[\left(\sqrt{\frac{y}{2}} - \sqrt{\frac{x}{2}}\right)^3 + \left(\sqrt{\frac{y}{2}} + \sqrt{\frac{x}{2}}\right)^3\right]\right\}^2$$

$$-\left\{\left(\sqrt{\frac{x}{2}}-\sqrt{\frac{y}{2}}\right)^3-\left(\sqrt{\frac{x}{2}}+\sqrt{\frac{y}{2}}\right)^3-i\left[\left(\sqrt{\frac{y}{2}}-\sqrt{\frac{x}{2}}\right)^3-\left(\sqrt{\frac{y}{2}}+\sqrt{\frac{x}{2}}\right)^3\right]\right\}^2=$$

$$=\frac{1}{2}\left\{\left[(\sqrt{x}+i\sqrt{y})(x-y-4i\sqrt{xy})\right]^2+\left[(\sqrt{x}-i\sqrt{y})(x-y+4i\sqrt{xy})\right]^2\right\}=(x-y)^3.$$

Proof. We let $x\to\sqrt{\frac{x}{2}}, y=\sqrt{\frac{y}{2}}$, in both side of (8), this completes the proof. □

Examples

$$2^3=(5-3)^3=\frac{1}{2}\left[\left(14\sqrt{5}-18i\sqrt{3}\right)^2+\left(14\sqrt{5}+18i\sqrt{3}\right)^2\right];$$

$$3^3=(5-2)^3=\frac{1}{2}\left[\left(11\sqrt{5}-17i\sqrt{2}\right)^2+\left(11\sqrt{5}+17i\sqrt{2}\right)^2\right];$$

$$4^3=(7-3)^3=\frac{1}{2}\left[\left(16\sqrt{7}-24i\sqrt{3}\right)^2+\left(16\sqrt{7}+24i\sqrt{3}\right)^2\right].$$

Theorem 4. *For* $x,y\in\mathbb{R}$, *then*

$$(10)\left\{\left(\sqrt{\frac{x}{2}}-i\sqrt{\frac{y}{2}}\right)^3+\left(\sqrt{\frac{x}{2}}+i\sqrt{\frac{y}{2}}\right)^3-i\left[\left(-\sqrt{\frac{x}{2}}+i\sqrt{\frac{y}{2}}\right)^3+\left(\sqrt{\frac{x}{2}}+i\sqrt{\frac{y}{2}}\right)^3\right]\right\}^2$$

$$-\left\{\left(\sqrt{\frac{x}{2}}-i\sqrt{\frac{y}{2}}\right)^3-\left(\sqrt{\frac{x}{2}}+i\sqrt{\frac{y}{2}}\right)^3-i\left[\left(-\sqrt{\frac{x}{2}}+i\sqrt{\frac{y}{2}}\right)^3-\left(\sqrt{\frac{x}{2}}+i\sqrt{\frac{y}{2}}\right)^3\right]\right\}^2=$$

$$=\frac{1}{2}\left\{\left[(\sqrt{x}-\sqrt{y})(x+y+4\sqrt{xy})\right]^2+\left[(\sqrt{x}+\sqrt{y})(x+y-4\sqrt{xy})\right]^2\right\}=(x+y)^3.$$

Proof. We let $y\to -y$, in both side of (9), this completes the proof. □

Examples

$$3^3=(2+1)^3=\frac{1}{2}\left[\left(5-\sqrt{2}\right)^2+\left(5+\sqrt{2}\right)^2\right];$$

$$4^3=(3+1)^3=\frac{1}{2}\left\{\left[(\sqrt{3}-1)(4+4\sqrt{3})\right]^2+\left[(\sqrt{3}+1)(4-4\sqrt{3})\right]^2\right\};$$

$$5^3=(3+2)^3=\frac{1}{2}\left[\left(7\sqrt{2}-3\sqrt{3}\right)^2+\left(7\sqrt{2}+3\sqrt{3}\right)^2\right];$$

$$6^3=(5+1)^3=\frac{1}{2}\left[\left(2\sqrt{5}-14\right)^2+\left(2\sqrt{5}+14\right)^2\right];$$

$$7^3=(4+3)^3=\frac{1}{2}\left[\left(9\sqrt{3}-10\right)^2+\left(9\sqrt{3}+10\right)^2\right];$$

$$8^3=(5+3)^3=\frac{1}{2}\left[\left(12\sqrt{3}-4\sqrt{5}\right)^2+\left(12\sqrt{3}+4\sqrt{5}\right)^2\right];$$

$$9^3 = (7+2)^3 = \frac{1}{2}\left[\left(\sqrt{7} - 19\sqrt{2}\right)^2 + \left(\sqrt{7} + 19\sqrt{2}\right)^2\right].$$

Theorem 5. *For* $a, b, c, d \in \mathbb{R}$, *then*

$$\begin{aligned}(11)\quad & [4(a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2)]^3 \\ &= [4(a+b)(a^2 - 4ab + b^2)(c+d)(c^2 - 4cd + d^2)]^2 \\ &+ [4(a+b)(a^2 - 4ab + b^2)(c-d)(c^2 + 4cd + d^2)]^2 \\ &+ [4(a-b)(a^2 + 4ab + b^2)(c+d)(c^2 - 4cd + d^2)]^2 \\ &+ [4(a-b)(a^2 + 4ab + b^2)(c-d)(c^2 + 4cd + d^2)]^2.\end{aligned}$$

Proof. In (7), we have

$$[2(x^2 + y^2)]^3 = [2(x+y)(x^2 - 4xy + y^2)]^2 + [2(x-y)(x^2 + 4xy + y^2)]^2,$$

thereof,

$$(12)\qquad [2(a^2 + b^2)]^3 = [2(a+b)(a^2 - 4ab + b^2)]^2 + [2(a-b)(a^2 + 4ab + b^2)]^2,$$

$$(13)\qquad [2(c^2 + d^2)]^3 = [2(c+d)(c^2 - 4cd + d^2)]^2 + [2(c-d)(c^2 + 4cd + d^2)]^2,$$

multiplying (12) by (13), we obtain

$$\begin{aligned}& [4(a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2)]^3 \\ &= [4(a+b)(a^2 - 4ab + b^2)(c+d)(c^2 - 4cd + d^2)]^2 \\ &+ [4(a+b)(a^2 - 4ab + b^2)(c-d)(c^2 + 4cd + d^2)]^2 \\ &+ [4(a-b)(a^2 + 4ab + b^2)(c+d)(c^2 - 4cd + d^2)]^2 \\ &+ [4(a-b)(a^2 + 4ab + b^2)(c-d)(c^2 + 4cd + d^2)]^2. \square\end{aligned}$$

Theorem 6. *For* $a, b, c, d \in \mathbb{R}$, *then*

$$\begin{aligned}(14)\quad & 4(a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2)^3 \\ &= [(a+b)(a^2 - 4ab + b^2)(c+d)(c^2 - 4cd + d^2)]^2 \\ &+ [(a+b)(a^2 - 4ab + b^2)(c-d)(c^2 + 4cd + d^2)]^2 \\ &+ [(a-b)(a^2 + 4ab + b^2)(c+d)(c^2 - 4cd + d^2)]^2 \\ &+ [(a-b)(a^2 + 4ab + b^2)(c-d)(c^2 + 4cd + d^2)]^2.\end{aligned}$$

Proof. In (11) we set $c \to \frac{c}{2}$ and $d \to \frac{d}{2}$, this completes the proof. □