On A Property of Pascal's Triangle II

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Abstract

Based on the paper viXra:1303.0163 (vixra.org/abs/1303.0163), we show a few more properties of Pascal's Triangle.

1 Lemmas 1, 2, 3 and 4

Lemma 1. *If a is a positive odd integer, then* $\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots + \binom{a}{a} = 1$.

Proof. See viXra:1303.0163 (vixra.org/abs/1303.0163). □

Lemma 2. *If a is a positive odd integer, then* $-\binom{a}{1} + \binom{a}{2} - \binom{a}{3} + \binom{a}{4} - \cdots - \binom{a}{a} = -1$.

Proof. According to Lemma 1, if a is a positive odd integer then

$$\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots + \binom{a}{a} \tag{1}$$

equals 1. Now, what happens if in the expression (1) we 'switch signs'? In other words, what happens if in the expression (1) we replace positive signs with negative signs and vice versa? Switching signs in (1) is the same as multiplying (or dividing) that expression by -1. Therefore, we have

$$\binom{a}{1} - \binom{a}{2} + \binom{a}{3} - \binom{a}{4} + \cdots + \binom{a}{a} = 1$$

$$\left(\binom{a}{1}-\binom{a}{2}+\binom{a}{3}-\binom{a}{4}+\cdots+\binom{a}{a}\right)(-1)=1(-1).$$

This means that

$$-\binom{a}{1}+\binom{a}{2}-\binom{a}{3}+\binom{a}{4}-\cdots-\binom{a}{a}=-1,$$

which proves the lemma. □

Lemma 3. *If a is a positive odd integer, then* $-\binom{a}{0}+\binom{a}{1}-\binom{a}{2}+\binom{a}{3}-\binom{a}{4}+\cdots+\binom{a}{a}=0$.

Proof. This can be easily proved if we take into account that $\binom{a}{1}-\binom{a}{2}+\binom{a}{3}-\binom{a}{4}+\cdots+\binom{a}{a}=1$ (see Lemma 1) and $-\binom{a}{0}=-1$. □

Lemma 4. *If a is a positive odd integer, then* $\binom{a}{0}-\binom{a}{1}+\binom{a}{2}-\binom{a}{3}+\binom{a}{4}-\cdots-\binom{a}{a}=0$.

Proof. This can be easily proved if we consider that $-\binom{a}{1}+\binom{a}{2}-\binom{a}{3}+\binom{a}{4}-\cdots-\binom{a}{a}=-1$ (see Lemma 2) and $\binom{a}{0}=1$. □

2 Lemmas 5, 6, 7 and 8

Lemma 5. *If b is a positive even integer, then* $\binom{b}{1}-\binom{b}{2}+\binom{b}{3}-\binom{b}{4}+\cdots+\binom{b}{b-1}-\binom{b}{b}=1$.

Proof. See viXra:1303.0163 (`vixra.org/abs/1303.0163`). □

Lemma 6. *If b is a positive even integer, then* $-\binom{b}{1}+\binom{b}{2}-\binom{b}{3}+\binom{b}{4}-\cdots-\binom{b}{b-1}+\binom{b}{b}=-1$.

Proof. According to Lemma 5, if b is a positive even integer, then

$$\binom{b}{1}-\binom{b}{2}+\binom{b}{3}-\binom{b}{4}+\cdots+\binom{b}{b-1}-\binom{b}{b}=1.$$

This means that

$$\left(\binom{b}{1}-\binom{b}{2}+\binom{b}{3}-\binom{b}{4}+\cdots+\binom{b}{b-1}-\binom{b}{b}\right)(-1)=1(-1)$$

$$-\binom{b}{1}+\binom{b}{2}-\binom{b}{3}+\binom{b}{4}-\cdots-\binom{b}{b-1}+\binom{b}{b}=-1,$$

which proves the lemma. □

Lemma 7. *If b is a positive even integer, then* $-\binom{b}{0}+\binom{b}{1}-\binom{b}{2}+\binom{b}{3}-\binom{b}{4}+\cdots+\binom{b}{b-1}-\binom{b}{b}=0$.

Proof. It is easy to prove that this is true if we take into account that $\binom{b}{1}-\binom{b}{2}+\binom{b}{3}-\binom{b}{4}+\cdots+\binom{b}{b-1}-\binom{b}{b}=1$ (see Lemma 5) and $-\binom{b}{0}=-1$. □

Lemma 8. *If b is a positive even integer, then* $\binom{b}{0}-\binom{b}{1}+\binom{b}{2}-\binom{b}{3}+\binom{b}{4}-\cdots-\binom{b}{b-1}+\binom{b}{b}=0$.

Proof. It is easy to prove that this is true if we consider that $-\binom{b}{1}+\binom{b}{2}-\binom{b}{3}+\binom{b}{4}-\cdots-\binom{b}{b-1}+\binom{b}{b}=-1$ (see Lemma 6) and $\binom{b}{0}=1$. □

3 Conclusion

According to Lemmas 1, 2, 3 and 4, if a is any positive odd integer, then

$$\sum_{i=1}^{\frac{a+1}{2}}\binom{a}{2i-1}-\sum_{j=1}^{\frac{a-1}{2}}\binom{a}{2j}=1,$$

$$\sum_{j=1}^{\frac{a-1}{2}}\binom{a}{2j}-\sum_{i=1}^{\frac{a+1}{2}}\binom{a}{2i-1}=-1,$$

$$\sum_{i=1}^{\frac{a+1}{2}}\binom{a}{2i-1}-\sum_{j=0}^{\frac{a-1}{2}}\binom{a}{2j}=0, \text{ and}$$

$$\sum_{j=0}^{\frac{a-1}{2}}\binom{a}{2j}-\sum_{i=1}^{\frac{a+1}{2}}\binom{a}{2i-1}=0.$$

According to Lemmas 5, 6, 7 and 8, if b is any positive even integer, then

$$\sum_{i=1}^{\frac{b}{2}}\binom{b}{2i-1}-\sum_{j=1}^{\frac{b}{2}}\binom{b}{2j}=1,$$

$$\sum_{j=1}^{\frac{b}{2}}\binom{b}{2j}-\sum_{i=1}^{\frac{b}{2}}\binom{b}{2i-1}=-1,$$

$$\sum_{i=1}^{\frac{b}{2}} \binom{b}{2i-1} - \sum_{j=0}^{\frac{b}{2}} \binom{b}{2j} = 0, \text{ and}$$

$$\sum_{j=0}^{\frac{b}{2}} \binom{b}{2j} - \sum_{i=1}^{\frac{b}{2}} \binom{b}{2i-1} = 0.$$

A Appendix

The following is a more visual representation of Lemmas 1 and 5:

$$\binom{1}{1} = 1$$

$$\binom{2}{1} - \binom{2}{2} = 1$$

$$\binom{3}{1} - \binom{3}{2} + \binom{3}{3} = 1$$

$$\binom{4}{1} - \binom{4}{2} + \binom{4}{3} - \binom{4}{4} = 1$$

$$\binom{5}{1} - \binom{5}{2} + \binom{5}{3} - \binom{5}{4} + \binom{5}{5} = 1$$

$$\binom{6}{1} - \binom{6}{2} + \binom{6}{3} - \binom{6}{4} + \binom{6}{5} - \binom{6}{6} = 1$$

$$\binom{7}{1} - \binom{7}{2} + \binom{7}{3} - \binom{7}{4} + \binom{7}{5} - \binom{7}{6} + \binom{7}{7} = 1$$

$$\dots$$

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