Four conjectures regarding Fermat pseudoprimes and few known types of pairs of primes

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Abstract. There are already known some relations between Fermat pseudoprimes and the pairs of primes [p, 2p – 1]. We will here show few relations between Fermat pseudoprimes and the pairs of primes of the type [p, 2p – 1], [p, 2p + 1], [p, sqrt(2p – 1)], respectivelly [p, k*p – k + 1].

Introduction

Due to mathematician Farideh Firoozbakht, we have in OEIS few interesting observations about the relation between Fermat pseudoprimes and the pairs of primes [p, 2p – 1]. We will list only few of them (see the sequences A005935-A005937):

: if p and 2p – 1 are both primes, and p > 3, then p(2p – 1) is pseudoprime to base 3;

: if p and 2p – 1 are both primes, then p(2p – 1) is pseudoprime to base 5 iff p is of the form 10k + 1;

: if p and 2p – 1 are both primes, then p(2p – 1) is pseudoprime to base 6 iff p is of the form 12k + 1.

Now that the relation between Fermat pseudoprimes and the pairs of primes [p, 2p – 1] appears to be clear, we will make four conjectures regarding the relation between Fermat pseudoprimes and the pairs of primes of the type [p, 2p + 1], [p, 2p - 1], [p, sqrt(2p – 1)], respectivelly [p, k*p – k + 1].

CONJECTURE 1: If p and 2p + 1 are both primes, then the number n = p(2p + 1) – 2*k*p is Fermat pseudoprime to base p + 1 for at least one natural value of k.

Verifying the conjecture:

(for the first 8 such pairs of primes)

For [p, 2p + 1] = [3, 7] we have, for k = 1, n = 15, which is, indeed, pseudoprime to base p + 1 = 4.
For [p, 2p + 1] = [5, 11] we have, for k = 2, n = 35, which is, indeed, pseudoprime to base p + 1 = 6.
For [p, 2p + 1] = [11, 23] we have, for k = 5, n = 143, which is, indeed, pseudoprime to base p + 1 = 12.
For [p, 2p + 1] = [23, 47] we have, for k = 6, n = 805, which is, indeed, pseudoprime to base p + 1 = 24.

For [p, 2p + 1] = [29, 59] we have, for k = 3, n = 1537, which is, indeed, pseudoprime to base p + 1 = 30.
For [p, 2p + 1] = [41, 83] we have, for k = 9, n = 2665, which is, indeed, pseudoprime to base p + 1 = 42.
For [p, 2p + 1] = [53, 107] we have, for k = 4, n = 5247, which is, indeed, pseudoprime to base p + 1 = 54.
For [p, 2p + 1] = [83, 167] we have, for k = 24, n = 9877, which is, indeed, pseudoprime to base p + 1 = 84.

Note: For the list of Sophie Germain primes, see the sequence A005384 in OEIS.

CONJECTURE 2: If p and 2p - 1 are both primes, p > 3, then the number n = p(2p - 1) – 2*k*p is Fermat pseudoprime to base p - 1 for at least one natural value of k.

Verifying the conjecture:
(for the first 6 such pairs of primes)

For [p, 2p - 1] = [7, 13] we have, for k = 4, n = 21, which is, indeed, pseudoprime to base p - 1 = 6.
For [p, 2p - 1] = [19, 37] we have, for k = 10, n = 323, which is, indeed, pseudoprime to base p - 1 = 18.
For [p, 2p - 1] = [31, 61] we have, for k = 5, n = 1581, which is, indeed, pseudoprime to base p - 1 = 30.
For [p, 2p - 1] = [37, 73] we have, for k = 2, n = 2553, which is, indeed, pseudoprime to base p - 1 = 36.
For [p, 2p - 1] = [79, 157] we have, for k = 7, n = 11297, which is, indeed, pseudoprime to base p - 1 = 78.
For [p, 2p - 1] = [97, 193] we have, for k = 8, n = 17169, which is, indeed, pseudoprime to base p - 1 = 96.

Note: For the list of primes p for which 2p – 1 is also prime, see the sequence A005382 in OEIS.

CONJECTURE 3: If p and q are primes, where q = sqrt(2*p – 1), then the number p*q is Fermat pseudoprime to base p + 1.

Verifying the conjecture:
(for the first 8 such pairs of primes)

For [p, q] = [13, 5] we have p*q = 65 which is, indeed, pseudoprime to base 14.
For [p, q] = [61, 11] we have p*q = 671 which is, indeed, pseudoprime to base 62.
For [p, q] = [181, 19] we have p*q = 3439 which is, indeed, pseudoprime to base 182.
For [p, q] = [421, 29] we have p*q = 12209 which is, indeed, pseudoprime to base 422.

For [p, q] = [1741, 59] we have p*q = 102719 which is, indeed, pseudoprime to base 1742.
For [p, q] = [1861, 61] we have p*q = 113521 which is, indeed, pseudoprime to base 1862.
For [p, q] = [2521, 71] we have p*q = 178991 which is, indeed, pseudoprime to base 2522.
For [p, q] = [3121, 79] we have p*q = 246559 which is, indeed, pseudoprime to base 3122.

Note: For the list of primes p for wich sqrt(2p – 1) is also prime, see the sequence A067756 in OEIS.

CONJECTURE 4: If p is prime, p > 3, and k integer, k > 1, then the number n = p*(k*p – k + 1) is Fermat pseudoprime to base k*p - k and to base k*p – k + 2.

Verifying the conjecture:

For the first 4 such pairs of primes, when p = 5:

For [p, 2p - 1] = [5, 9] we have p(2p - 1) = 45 which is, indeed, pseudoprime to bases 8 and 10.
For [p, 3p - 2] = [5, 13] we have p(3p - 2) = 65 which is, indeed, pseudoprime to bases 12 and 14.
For [p, 4p - 3] = [5, 17] we have p(4p - 3) = 85 which is, indeed, pseudoprime to bases 16 and 18.
For [p, 5p - 4] = [5, 21] we have p(5p - 4) = 105 which is, indeed, pseudoprime to bases 20 and 22.

For the first 4 such pairs of primes, when p = 7:

For [p, 2p - 1] = [7, 13] we have p(2p - 1) = 91 which is, indeed, pseudoprime to bases 12 and 14.
For [p, 3p - 2] = [7, 19] we have p(3p - 2) = 133 which is, indeed, pseudoprime to bases 18 and 20.
For [p, 4p - 3] = [7, 25] we have p(4p - 3) = 175 which is, indeed, pseudoprime to bases 26 and 28.
For [p, 5p - 4] = [7, 31] we have p(5p - 4) = 217 which is, indeed, pseudoprime to bases 30 and 32.

For the next 4 such pairs of primes, when k = 3:

For [p, 3p - 2] = [11, 31] we have p(3p - 2) = 341 which is, indeed, pseudoprime to bases 30 and 32.
For [p, 3p - 2] = [13, 37] we have p(3p - 2) = 481 which is, indeed, pseudoprime to bases 36 and 38.
For [p, 3p - 2] = [23, 67] we have p(3p - 2) = 1541 which is, indeed, pseudoprime to bases 66 and 68.
For [p, 3p - 2] = [37, 109] we have p(3p - 2) = 4033 which is, indeed, pseudoprime to bases 108 and 110.

Note: The formula p*(k*p - k + 1), where p is prime and k integer, seems to appear often related to Fermat pseudoprimes (see the sequence A217835 that I submitted to OEIS).