BORELRESUMMATION AND THE SOLUTION OF INTEGRAL EQUATIONS

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ABSTRACT: In this paper we study the methods of Borel resummation applied to the solution of integral equation with symmetric Kernels K(xs) and to the study of the Riesz criterion , which is important to the Riemann Hypothesis

- *Keywords: Integral equation, Borel resummation, divergent series*

1. INTRODUCTION

Divergent series are widely known and appear in many context involving Physics or Mathematics , for example if we integrate by parts the error function :

$$erfc(x)=\int_x^\infty dte^{-t^2} \to \frac{e^{-x^2}}{\sqrt{\pi}}\left(1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{n!(2x)^{2n}}\right) \qquad (1.1)$$

Or if we apply a 'Saddle point mehtod' to evaluate n! For big 'n'

$$\Gamma(x+1)\to\sqrt{2\pi}n^{(n+1)/2}\left(1+\frac{1}{12n}+\frac{1}{288n^2}+.....\right) \qquad \int_0^\infty dte^{-t}t^{x-1}=\Gamma(x) \quad (1.2)$$

But (1.1) and (1.2) are only convergent in the limit $x\to\infty$
for small values of x both serees diverge.

Another example with ODE's is the following

$$x^2\frac{dy}{dx}+y=x \quad ,y(x)=x\int_0^\infty dt\frac{e^{-t}}{1+xt} \qquad \text{(exact solution)} \ (1.3)$$

For (1.3) Euler gave the series solution: $y(x)=x-(1!)x^2+(2!)x^3-(3!)x^4-...$ (1.4)

Which converges only for x=0 !!! , A similar thing happens with the series:

$$a_0+a_1g+a_2g^2+a_3g^3+....... \qquad g<<1 \quad (1.5)$$

That apear in QFT and Quantum Mechanics , here g is the 'coupling constant' in general series of the form (1.5) although divergent are used to calculate the 'mass' or 'charge' , for a given physical theory.

Also as a last example let be the next Taylor series around x=0 :

$$x+x^2+x^3+x^4+.........=\frac{x}{1-x} \quad , \quad x+2x^2+3x^3+4x^4+.........=\frac{x}{(1-x)^2} \qquad (1.6)$$

Convergent for |x| <1 However taking the limit $x \to -1^-$ (-1 by the left) we find the amazing results -1/2 and -1/4 .

Of course this sections pretends to be only a kind of introduction to the subject for further references I strongly recommend 'Divergent series' by G.H Hardy or 'Zeta regularization methods' by E.Elizalde and others for historical examples involving divergent series and integrals.

2. BOREL RESUMMATION FOR SERIES AND INTEGRALS

Let be the divergent (Numerical) series:

$$S=a_0+a_1+a_2+a_3+......... \qquad (2.1)$$

Borel gave a very ingenious method to calculate it, first we multiply and divide each term by n!

$$\sum_{n=0}^{\infty}\frac{a_n}{n!}\int_0^{\infty}dt t^n e^{-t} \qquad \sum_{n=0}^{\infty}\frac{a_n}{n!}t^n=f(t) \qquad (2.2)$$

Then we use (2.1) and (2.2) and supposing that $f(t)=O(e^{bt})$ for a real positive number b then we can writte the 'sum' of the series in (2.1)

$$\int_0^{\infty}dt f(t)e^{-t}=\mathrm{B(S)} \to \sum_{n=0}^{\infty}a_n \quad \text{or} \quad s\int_0^{\infty}dt f(t)e^{-st} \to \sum_{n=0}^{\infty}a_n s^{-n} \qquad s>0 \ (2.3)$$

As a 'toy model' of our Borel resummation method we have:

$$1-1+1-1+1-1+1-1+.......\to 1/2 \quad f(t)=\exp(-t) \qquad (2.4)$$

Unfortunately we can't always know an exact expression for f(t),
To give an approximate evaluation of our Borel transform, we can use the 'Euler-Abel' transform applied ot our divergent series

$$\sum_{n=0}^{\infty}a(n)x^n \approx \sum_{k=0}^{p}\frac{(-1)^k x^k \Delta^k[(-1)^n a_n]_{n=0}}{(1+x)^{k+1}} + \left(\frac{x}{1+x}\right)^{p+1}\sum_{n=0}^{\infty}(-1)^{n+p+1}\Delta^{p+1}[(-1)^n a_n]_{n=0}\, x$$

$$\Delta^n a(0)=\sum_{n=0}^{n}(-1)^m \frac{n!}{n!(n-m)!}a_{n-m} \qquad (2.5)$$

Also we need another well-known property of the 'Laplace transform'

$$\int_0^\infty dt \frac{t^p}{(t+c)^{p+1}}e^{-st}=\frac{1}{p!}\frac{\partial^p}{\partial s^p}\frac{\partial^p}{\partial c^p}E_1(cs)e^{cs} \qquad E_1(x)=\int_x^\infty dt\frac{e^{-t}}{t}=-E_i(-x) \quad (2.6)$$

The first expression in (2.5) is an approximate evaluation for f(x) ,let x=t the B(S) 'Borel sum' for our divergent series (3.1) is:

$$B(S)\approx \frac{1}{s}\sum_{k=0}^{p}(-1)^k \frac{1}{k!}\frac{\partial^k}{\partial s^k}\frac{\partial^k}{\partial c^k}\Big|_{c=1,s=1} \Delta^k[(-1)^n b_n]_{n=0}E_1(s)e^{cs} \qquad (2.7)$$

With $a_n=b_n n!$, only in case that the coefficients of our initial series (2.1) were of the form $a_n=(-1)^n P(n)$ with P(n) a Polynomial (2.7) is exact.

The error term is given by the expression:

$$E=O\left(\frac{1}{s}\frac{1}{p!}\frac{\partial^{p+1}}{\partial s^{p+1}}\frac{\partial^p}{\partial c^p}E_1(cs)e^{cs}\Big|_{c=1,s=1}\right) \qquad (2.8)$$

In case (2.1) were convergent, then its ' Borel sum' is equivalent to the term-by-term Laplace transform at s=1.

The formalism of Borel resummation for integrals is inmediatly acomplished if we define the Riemann sum multiplying and dividing each term by a Gamma function we have:

$$\sum_{n=0}^{\infty}\frac{f(a+n\Delta x)s^{\Delta x+a}}{\Gamma(a+n\Delta x+1+\alpha)}\int_0^\infty dtt^{a+n\Delta x+}e^{-t}t^\alpha \Delta x \qquad (2.9)$$

Now we take the limit , $\Delta x \to 0$,the sums becomes the double-integral :

$$s\int_0^\infty dt\left(\int_0^\infty dx\frac{f(x)t^x}{\Gamma(x+1+\alpha)}\right)t^\alpha e^{-st} \qquad (2.10)$$

Of course in general, unless $\int_0^\infty dxf(x)$ is convergent 'Fubini's theorem' does not hold for (2.9) and (2.10) so:

$$\int_0^\infty dt\int_0^\infty dx\sigma(x,t)\neq \int_0^\infty dx\int_0^\infty dt\sigma(x,t) \qquad \sigma(x,t)=\frac{f(x)}{\Gamma(x+1+\alpha)}t^{x+\alpha}e^{-t}$$

(2.11)

Now if we define the integral transform

$$H_{\alpha}(t)=\int_{0}^{\infty}dx\frac{f(x)}{\Gamma(x+1+\alpha)}t^{x} \qquad \text{With } H_{\alpha}(t)=O(e^{b_2 t}) \quad (2.12)$$

If the 2 conditions inside (2.14) holds then the 'Borel integral' is just the Laplace transform of $H_{\alpha}(t)t^{\alpha}$, $\alpha>0$

But. Can a 'Borel sum' be the real sum of the series?, let's take :

$$\mathrm{E_i}(x)=-\int_{-x}^{\infty}dt\frac{e^{-t}}{t}\approx\frac{e^{-x}}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{x^n}n! \qquad (2.13)$$

The alternating series has the Borel transform:

$$x\int_{0}^{\infty}dte^{-xt}\frac{1}{t+1}\to\sum_{n=0}^{\infty}\frac{(-1)^n}{x^n}n! \qquad (2.14)$$

Using the result for the Laplace transform of 1/(t+1) ,we find:

$$L\left\{\frac{1}{t+1}\right\}=e^{s}E_i(s) \qquad (2.15)$$

Setting s=1 we find that the 'asymptotic' expansion (2.14) can be 'summed' even for high values of x.

Also, if the integral is convergent then using the property of Laplace transform with s=1 $L\{t^x\}_{s=1}=\Gamma(x+1)$ then the definition of 'integral' (2.12) is the same as the usual definition for the integral in terms of convergent Riemann sums.

The relationship of this 'Borel resummation' for integrals can be written as this, using the next property for Laplace transforms:

$$L\left\{\int_{0}^{\infty}dxt^{x}\frac{f(x)}{\Gamma(x+1}\right\}=\frac{F(\ln s)}{s\ln s} \qquad \text{and} \qquad L\{t^n f(t)\}=(-1)^n D^n F(s) \quad (2.16)$$

Then we can write (2.14) in terms of Laplace transforms:

$$\int_{0}^{\infty}dxf(x)s^{-x}\to s(-1)^{\alpha}\frac{d^{\alpha}}{ds^{\alpha}}\left(\frac{F_{\alpha}(\ln s)}{s\ln s}\right) \qquad (2.17)$$

Where $F_{\alpha}(s)$ is the Laplace transform of $g(x)=\dfrac{f(x)}{(x+1)(x+2).....(x+\alpha)}$

Valid for $\alpha > 0$ and integer

For $\alpha \notin Z$,we must apply the analytic prolongation of the Gamma function $\Gamma(z)$ and use the definition of the differintegral $D_x^{\alpha} f$.

For the case of Fourier sums $g(x) = \sum_{n=0}^{\infty} a_n \cos(nx)$ the Borel resummation method can be applied, if we use the Real part of the identity $\sum_{n=0}^{\infty} t^n e^{inx} =_{reg} \frac{1}{1-te^{ix}}$ and use the Borel resummation formula

$$\sum_{n=0}^{\infty} a_n \cos(nx) = p.v\int_0^{\infty} dt \frac{g(t) - g(t)\cos(x)}{1-2t\cos(x)+t^2} \qquad a_n = \int_0^{\infty} dt g(t) t^n \qquad (2.18)$$

This last integral in (2.18) will only exist in Cauchy's principal value sense due to the singularities of the integrand when $1-2t\cos(x)+t^2 = 0$

3. **BOREL RESUMMATION AND INTEGRAL EQUATIONS**

We could write a generalization to (2.3) as the integral expressions

$$B(a_n) = \int_0^{\infty} dt f(t) h(t) \qquad f(t) = \sum_{n=0}^{\infty} \frac{a_n}{M(n+1)} t^n \qquad (3.1)$$

With $M(n+1) = \int_0^{\infty} dt h(t) t^n$, in case $h(t) = e^{-t}$ and M(n+1)=n! ,

Expression (4.1) is just the Borel transform of the sequence $\{a_n\}$, with the advantage that now f(t) can grow faster than e^{bt} so $f(t) \neq O(e^{bt})$,

We will study the applications of this Borel resummtion to solve integral equations and to study the Riesz criterion for Riemann Hypothesis

In order to apply the Borel generalized resummation to integral equations, let be the Fredholm equation of first kind :

$$g(s) = s\int_0^{\infty} dt K(st) y(t) \qquad \hat{K}(n+1) = \int_0^{\infty} dt K(t) t^n \qquad (3.2)$$

Here K(st) is the Kernel of the integral equation , and g(s) has the form of a Z-transform involving inverse powers of 's' :

$$g(s) = c_0 + \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s^3} + ... \qquad c_n = \frac{1}{2\pi i}\int_{\gamma} dz g(z) z^{n-1} \qquad (3.3)$$

Since the Mellin transform for Kernel K(u) exists, we will apply Borel resummation to solve the integral equation given in (3.2)

$$g(s)=\sum_{n=0}^{\infty}c_n s^{-n}=\int_0^{\infty}dt\left(\sum_{n=0}^{\infty}\frac{c_n}{\hat{K}(n+1)}\cdot\frac{t^n}{s^n}\right)K(t)=s\int_0^{\infty}dtK(st)y(t) \qquad (3.4)$$

Here 'γ' is a certain closed path on the complex plane
From expression (3.4) We have proven that a infinite power series in the form $\sum_{n=0}^{\infty}\frac{c_n}{\hat{K}(n+1)}t^n=y(t)$ can be used to solve an integral equation of the form (4.2) with the Kernel $K(xs)$.

As an example of our method let be the integral equation for the Prime counting function $\pi(x)=\sum_{p\le x}1$ [1]

$$\log\zeta(s)=s\int_0^{\infty}dt\frac{\pi\left(e^t\right)}{e^{st}-1} \qquad (3.5)$$

The Kernel of (3.5) is $K(t)=\frac{1}{e^t-1}$ and its Mellin transform $\Gamma(s+1)\zeta(s+1)$, then the solution to (3.5) is given by the Gram series $\pi(x)\approx\sum_{n=1}^{\infty}\frac{1}{n}\frac{\left(\log x\right)^n}{n!\zeta(n+1)}$, from the Prime number theorem $\lim_{x\to\infty}\frac{\pi(x)\log x}{x}=1$, the coefficients inside the Gramm series must be equal to $c_n\approx\frac{1}{n}$, so in the limit $x\to\infty$, the Gram series is just the Taylor expansion for the logarithmic integral

$$Li(x)=\int_2^{\infty}\frac{dt}{\log t}\approx\sum_{n=1}^{\infty}\frac{1}{n}\frac{\left(\log x\right)^n}{n!}$$

This method of the Borel transform can be extended to include integral equation with non-constant limit of integration or Volterra equations in the form

$$sg\left(\frac{1}{s}\right)=\int_s^{\infty}dtK\left(\frac{t}{s}\right)f(t) \qquad f(t)=\sum_{n=0}^{\infty}\frac{c_n}{\hat{K}(n+1)_1}t^n \qquad (3.6)$$

With $\hat{K}(n+1)_1=\int_1^{\infty}dtK(t)t^{n-1}$, the Mellin transform with lower limit '1' using the same reasoning as we did inside (3.2 -3.4) and with the change of variable $s \to \frac{1}{s}$, we can inmediatly check that $f(t)=\sum_{n=0}^{\infty}\frac{c_n}{\hat{K}(n+1)_1}t^n$ solves the integral equation in (3.6)

In these cases , to solve integral equations of the form $g(s)=s\int_0^{\infty}dtK(st)f(t)$, we have assumed that the Mellin transform of the Kernel K(t) will exists for every positive integer 'n' and for n =0 , but what happens for example if $\int_0^{\infty}dt\frac{t^{s-1}}{1+t}=\frac{\pi}{\sin(\pi s)}$? , in this case the Mellin transform is singular and has poles at the integers.

If the Mellin transform of the Kernel $M[K(t)](s)=\int_0^{\infty}dtK(t)t^{s-1}$ has poles at the integers, we introduce a real or complex number α , so the Mellin trasform defined by analytic continuation $M[K(t)](s+\alpha)=\int_0^{\infty}dtK(t)t^{s+\alpha-1}$ has no poles $\forall n\in Z$. IN this case we re-write the integral equation in the form

$$g(s)=s\int_0^{\infty}dtK(st)y(t)\to s^{\alpha}g(s)=s\int_0^{\infty}dt\left(st\right)^{\alpha}K(st)\phi(t) \quad (3.7)$$

Here $\phi(t)=\frac{f(t)}{t^{\alpha}}$, in this case and due to the extra term inside the Kernel (3.7) , the solution may have fractional powers of 't' , if the function g(s) admits the expansion $g(s)=\sum_{n=0}^{\infty}c_n s^{-n}$, then the solution to (3.7) is given by the power series $f(t)=\sum_{n=0}^{\infty}\frac{c_n}{\hat{K}(n+1+\alpha)}t^{n+\alpha}$ and $\hat{K}(s)$ is the Mellin transform of the Kernel K(t).

We can extend our method to include also negative powers of 't' if the function g(s) admits a Z-tranform

$$\sum_{m=-\infty}^{\infty}\frac{c_m}{s^m}=g(s) \qquad f(t)=\sum_{m=-\infty}^{\infty}\frac{c_n}{\hat{K}(m+1+\alpha)}t^{m+\alpha} \quad (3.8)$$

The number α is chosen so the Mellin transform $\hat{K}(n+1+\alpha)$ has no zeros nor poles for integer 'n'

In the special case of the integral equation $g\left(\frac{1}{s}\right)=s\int_0^{\infty}dtK(st)f(t)$ with a function g(x) which can be expanded into a Taylor-Riemann series involving integer or fractional derivatives with coefficients $\frac{D^{n+\alpha}g(0)}{\Gamma(n+1+\alpha)}$ the solution is a bit easier and it can be written in the form

$$f(t)=\sum_{m=-\infty}^{\infty}\frac{D^{n+\alpha}g(0)}{\hat{K}(m+1+\alpha)\Gamma(n+\alpha+1)}t^{m+\alpha} \qquad (3.9)$$

For the special case $\alpha=0$, there would be only normal derivatives and positive powers of 'x'.

Although we have only applied our method to integral equation of first kind , we can also apply this method of power series to the integral equation of second kind

$$g(s)+f\left(\frac{1}{s}\right)=s\int_{0}^{\infty}dtK(st)f(t) \qquad (3.10)$$

In this case we can use the trick $s\int_{0}^{\infty}dt\delta(st-1)f(t)=f\left(\frac{1}{s}\right)$, so we have the new kernel $R(st)=K(st)-\delta(st-1)$, and the solution to equation (3.10) is the following

$$f(t)=\sum_{m=-\infty}^{\infty}\frac{a_n}{\hat{K}(m+1+\alpha)-1}t^{m+\alpha} \qquad g(s)=\sum_{n=-\infty}^{\infty}\frac{a_n}{s^n} \qquad (3.11)$$

Depending on the value of α it may happen that $\hat{K}(n+\alpha+1)=1$, for some value of 'n' and the equation may have no solution

This method of power series may seem useless since for example for an integral equation with Kernel $K(st)$ the solution may be obtained with the aid of the Mellin transform

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{ds}{x^s}\frac{G(1-s)}{K(1-s)} \qquad g(s)=\int_{0}^{\infty}dtK(st)f(t) \quad (3.12)$$

Where $G(s)=\int_{0}^{\infty}dtg(t)t^{s-1}$ and $K(s)=\int_{0}^{\infty}dtk(t)t^{s-1}$ are the Mellin transforms of the Kernelt and of the function g(t) , however it may happen that the function g(t) has no well-defined Mellin transform, for example in the case of the Gram series , also the function $K(1-s)$ may have an infinite number of zeros , so by residue theorem the evaluation of the complex integral inside (3.12) is also an infinite series.

4. RIESZ CRITERION AND THE BOREL TRANSFORM:

The Riesz function, introduced by Marcel Riesz , ref [6] has the 2 equivalent formulations

$$Riesz(x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{(k-1)!\zeta(2n)} \qquad \frac{Riesz(x)}{x} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \exp\left(-\frac{x}{n^2}\right) \quad (4.1)$$

Here $\mu(n) = \begin{cases} 0 \text{ if n is not square-free.} \\ (-1)^k \text{if n is square-free with k- distinct prime factor} \end{cases}$ and $\mu(1) = 1$ is the Möbius function

We will use the Borel transform method to give an integral equation for the Riesz(x) function using the Borel resummation method

$$1 - e^{-x} = -\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n!} = \int_0^{\infty} \frac{dt}{t} \left\{ \frac{1}{\sqrt{t}} \right\} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{\zeta(2n)} \frac{(xt)^n}{(n-1)!} \right) \quad (4.2)$$

The last expression inside the integral in (4.2) is precisely the Riesz function, so for the Riesz function using the Borel resummation method we have obtained the integral equation for the Riesz function

$$e^{-x} - 1 = \int_0^{\infty} \frac{dt}{t} \left\{ \sqrt{\frac{x}{t}} \right\} Riesz(t) \qquad \{t\} = t - [t] = \frac{1}{2} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi kt)}{k} \quad (4.3)$$

Inside (4.2) and (4.3) we have used the definition of the Riemann Zeta function as the Mellin transform of the fractional part $-\frac{\zeta(2s)}{s} = \int_0^{\infty} \frac{dt}{t} \left\{ \sqrt{\frac{1}{t}} \right\} t^s$, before applying Borel resummation

Truncation of the Fourier series inside (4.3) may be needed if we want to use numerical method to solve this integral equation).

We can check that (4.3) is correct , if we apply the Mellin transform to both sites and use the Mellin convolution theorem

$$\int_0^{\infty} (f * g)(x) x^{s-1} dx = \int_0^{\infty} dt \int_0^{\infty} dx f(xt) g(t) \frac{dt}{t} x^{s-1} = \hat{F}(s) \int_0^{\infty} dt g(t) t^{-s-1} \quad (4.4)$$

Together with the change of variable $z = xt$, if we apply this to (4.3)

$$-\Gamma(s) = -\frac{\zeta(-2s)}{s} \int_0^{\infty} dt Riesz(t) t^{s-1} \qquad \int_0^{\infty} dt Riesz(t) t^{s-1} = \frac{\Gamma(s+1)}{\zeta(-2s)} \quad (4.5)$$

Inside (4.5) is the Mellin transform of the Riesz function, so our integral equation (4.3) is correct.

Also if we take the derivative with respect to ‘x’ on both sides of (4.3) and use the fact that

$$\frac{d}{dx}\{x\} = 1 - \frac{d[x]}{dx} = 1 - \sum_{m=-\infty}^{\infty} \delta(x-m) \quad (4.6)$$

And the identity relating a sum and an integral

$$\sum_{n=1}^{\infty} f(n) = -\int_0^{\infty} dx[x]f'(x) \qquad \lim_{x\to\infty}[x]f(x) = 0 \quad (4.7)$$

Where f(x) is a suitable function with continous first derivative, then, we recover the well known functional equation for the Riesz function $xe^{-x} = \sum_{n=1}^{\infty} Riesz\left(\frac{x}{n^2}\right)$. To prove this equation we have used a change of variable of the form $\frac{x}{t} = u^2$ inside (4.3) , also we have used formulae (4.6) and (4.7) and the differential identity (assuming both derivatives with respect to ‘u’ and ‘x’ exist)

$$-u\frac{\partial f}{\partial u} = 2x\frac{\partial f}{\partial x} \qquad f = f\left(\frac{x}{u^2}\right) \quad (4.8)$$

An alternative to equation (4.3) is to use the representation of the Riemann zeta function $\frac{\zeta(s)}{s} = \int_0^{\infty} \frac{dt}{t}\left[\sqrt{\frac{1}{t}}\right]t^s$, in this case we have on the left side of (4.3) the expression $1-e^{-x}$, here $[x]$ is the floor function, the kernel of the integral eqution would be now $\left[\sqrt{\frac{x}{t}}\right]$.

In any case both Kernel can give two equivalent kernel and two equivalent integral equations, if we replace the function $\sqrt{\frac{x}{t}}$ by $\left(\frac{x}{t}\right)^{\frac{1}{2}+\varepsilon}$ for any positive epsilon , then we find the regularized integral

$$\lim_{\varepsilon\to 0}\int_0^{\infty} \frac{dx}{x^{\frac{3}{2}+\varepsilon}} R(x) = 0 \qquad \text{since} \quad \zeta(1+2\varepsilon) = \frac{1}{2\varepsilon} + \gamma + O(\varepsilon) \qquad \varepsilon \to 0 \quad (4.9)$$

This regularization is admissible since the Mellin transform of the Riesz function is equal to $\frac{\Gamma(s+1)}{\zeta(-2s)}$ for every ‘s’ so $\mathrm{Re}(s) < -\frac{1}{2}$.

The integral equation (4.3) can be written also in the form

$$xe^{-\frac{1}{x}} - x = x\int_0^\infty \frac{dt}{t}\left\{\sqrt{\frac{1}{xt}}\right\}Riesz(t) \qquad e^{-\frac{1}{x}} - 1 = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!x^n} \qquad (4.10)$$

If we insert this series inside (3.9) for the function $e^{-\frac{1}{x}} - 1$, and use the fact that $-\frac{\zeta(2s)}{s} = \int_0^\infty \frac{dt}{t}\left\{\sqrt{\frac{1}{t}}\right\}t^s$, with $\alpha = 0$ then we get the solution to (4.10) in terms of the series $-\sum_{n=1}^{\infty}\frac{(-x)^n}{(k-1)!\zeta(2n)}$.

APPENDIX A: RAMANUJAN'S MASTER THEOREM AND BOREL RESUMMATION

Let be the function $f(x) = \sum_{n=0}^{\infty}(-1)^n a(n)x^n$, which can be expanded into a Taylor series in a neighborhood of x=0 , Ramanujan's master theorem states

$$\int_0^\infty dx f(x)x^{s-1} = \frac{\pi}{\sin(\pi s)}a(-s) \qquad (A.1)$$

We can prove (A.1) with the Borel generalized transform

$$f(x) = \sum_{n=0}^{\infty}(-1)^n a(n)x^n = \int_0^\infty dt\left(\sum_{n=0}^{\infty}(-1)^n(xt)^n\right)g(t) \qquad a(n) = \int_0^\infty dt t^n g(t) \quad (A.2)$$

The sum $\sum_{n=0}^{\infty}(-xt)^n = \frac{1}{1+xt}$ can be evaluated without any problem, then we apply the Mellin transform to both sides of (A.2) , and use the Mellin covonlutio theorem for the expression $\int_0^\infty dt\frac{g(t)}{1+xt}$

$$\int_0^\infty f(x)x^{s-1}dx = \int_0^\infty x^{s-1}dx\int_0^\infty dt\frac{g(t)}{1+xt} = \frac{\pi}{\sin(\pi s)}\int_0^\infty dt g(t)t^{-s} = \frac{\pi}{\sin(\pi s)}a(-s) \quad (A.3)$$

The last expression (A.3) is precisely Ramanujan's master theorem, here we have used the identity $\int_0^\infty dt\frac{t^{s-1}}{1+t} = \frac{\pi}{\sin(\pi s)}$, and the definiton of a(n) in terms of the function g(t)

APPENDIX B: RIESZ FUNCTION AND A SUM OVER RIEMANN ZEROS

From the formula (4.1) $\frac{Riesz(x)}{x}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}\exp\left(-\frac{x}{n^2}\right)$ a question is could we express a sum involving the Möbius function using the Riemann Zeros ??.

Titchmarsh [8] used the Residue theorem and assumed that all the Riemann zeros were simple to obtain the following formula for the Mertens function

$$M(x)=\sum_{n\le x}\mu(n)=\sum_{\rho}\frac{x^{\rho}}{\rho\zeta'(\rho)}-2+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)!n\zeta(2n+1)}\left(\frac{2\pi}{x}\right)^{2n} \quad \text{(B.1)}$$

Valid for x >1 , if we set $x=e^u$ and differentiate respect to 'x' , since the Mertens function is just an step function its derivative must be a delta comb

$$\int_0^{\infty}due^{-u/2}\frac{dM_0(e^u)}{du}g(u)=\sum_{n=1}^{\infty}\frac{\mu(n)}{\sqrt{n}}g(\log n) \quad \text{(B.2)}$$

Now if we write the Riemann zeros as $\rho=1/2+i\gamma$, then after some trivial manipulations we get from (B.2) the formula

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{\sqrt{n}}g(\log n)=\sum_{\gamma}\frac{h(\gamma)}{\zeta'\left(\frac{1}{2}+i\gamma\right)}+\sum_{n=1}^{\infty}\frac{2(2\pi)^{2n}(-1)^n}{(2n)!\zeta(2n+1)}\int_{-\infty}^{\infty}dug(u)e^{-(2n+1/2)u} \quad \text{(B.3)}$$

The right part of (B.3) runs over all the Riemann zeros on the critical strip $0<\mathrm{Re}(s)<1$ and $g(x)=\frac{1}{\pi}\int_0^{\infty}h(u)\cos(ux)du=g(-x)$ is a Fourier transform pair

A straight application to the Riesz function of (B.3) with $g(t,x)=e^{-xe^{-2t}}$ gives for big x>>1

$$\frac{R(x)}{x}\approx\sum_{\gamma}\frac{1}{\zeta'(1/2+i\gamma)}\cdot\frac{\Gamma\left(\frac{3-2i\gamma}{4}\right)}{2x^{\left(\frac{3-2i\gamma}{4}\right)}} \quad x\to\infty \quad \text{(B.4)}$$

We have used the integral representation of the Gamma function $\Gamma(s)=\int_0^{\infty}dte^{-t}t^{s-1}$ inside the equation (B.4)

So if all the imaginary part of the Riemann zeros were REAL , Riemann Hypothesis true then the Riesz function would obey the bound $R(x)=O\left(x^{\frac{1}{4}+\varepsilon}\right)$ for any positive epsilon, at least for big values of 'x'

APPENDIX C: A FUNCTIONAL DIFFERENTIAL EQUATION FOR INFINITE PRODUCTS

Let be the infinite product

$$S(x)=\prod_{n=0}^{\infty}\left(1+\frac{x}{a_n}\right) \qquad S(0)=1 \qquad \text{(C.1)}$$

This product also can be defined (regularization) As

$$-\partial_s Z(0,x)+\partial_s Z(0,0)=\log S(x) \qquad Z(s,x)=\sum_{n=0}^{\infty}\frac{1}{\left(x+a_n\right)^s} \quad \text{(C.2)}$$

We may take the logarithmic derivative inside (C.1), this is equal to

$$\frac{S'(x)}{S(x)}=\frac{d\log S(x)}{dx}=\sum_{n=0}^{\infty}\frac{1}{\left(x+a_n\right)} \quad \text{(C.3)}$$

If we take now the (s-1) derivative inside (C.3) we get

$$\frac{d^{s-1}}{dx^{s-1}}\left(\frac{S'(x)}{S(x)}\right)=-\sum_{n=0}^{\infty}\frac{(-1)^s\,\Gamma(s)}{\left(x+a_n\right)^s} \quad \text{(C.4)}$$

Then, if we combine and , and set $y(x)=\log S(x)$, we have a functional differential equation valid for every infinite product of the form (C.1)

$$\frac{d}{ds}\left(\frac{D_x^{s-1}}{(-1)^s\,\Gamma(s)}\left(\frac{dy}{dx}(x)\right)\right)_{s=0}-\frac{d}{ds}\left(\frac{D_x^{s-1}}{(-1)^s\,\Gamma(s)}\left(\frac{dy}{dx}(0)\right)\right)_{s=0}=y(x) \quad \text{(C.5)}$$

Where $(-1)^s=e^{i\pi s}$ and $D_x^{s-1}=\frac{d^{s-1}}{dx^{s-1}}$ is the (s-1) fractional derivative operator, this operator can be defined by the Grunwald-Letnikov differintegral.

$$D_x^{s-1}y(x)=\lim_{h\to 0}\frac{1}{h^{s-1}}\left(\sum_{m=0}^{\infty}(-1)^m\binom{s-1}{m}f(x-mh)\right) \qquad \text{(C.6)}$$

$\binom{s-1}{m}=\frac{\Gamma(s)}{\Gamma(s-m)m!}$ are the generalized binomial coefficients.

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