Novel Remarks on Point Mass Sources, Firewalls, Null Singularities and Gravitational Entropy

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Abstract

A continuous family of static spherically symmetric solutions of Einstein's vacuum field equations with a *spatial* singularity at the origin $r = 0$ is found. These solutions are parametrized by a real valued parameter λ (ranging from 0 to ∞) and such that the radial horizon's location is *displaced* continuously towards the singularity ($r = 0$) as λ increases. In the limit $\lambda \rightarrow \infty$, the location of the singularity and horizon *merges* leading to a *null* singularity. In this extreme case, any infalling observer hits the null singularity at the very moment he/she crosses the horizon. This fact may have important consequences for the resolution of the fire wall problem and the complementarity controversy in black holes. Another salient feature of these solutions is that it leads to a modification of the Newtonian potential consistent with the effects of the generalized uncertainty principle (GUP) associated to a minimal length. The field equations due to a delta-function point-mass source at $r = 0$ are solved and the Euclidean gravitational action corresponding to those solutions is evaluated explicitly. It is found that the Euclidean action is precisely equal to the black hole entropy (in Planck area units). This result holds in any dimensions $D \geq 3$. The study of the Nonperturbative Renormalization Group flow of the metric $g_{\mu v}[k]$ in terms of the momentum scale k and its relationship to these family of metrics parametrized by λ deserves further investigation.

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1 Family of Static Spherically Symmetric Solutions

There are static spherically symmetric (SSS) *vacuum* solutions of Einstein's equations [1], beyond the Hilbert [4] and Schwarzschild [2] solutions, which are given by a family of metrics parametrized by a family of *area* radial functions $\rho_\lambda(r)$ (in $c = 1$ units), in terms of a real parameter $0 \leq \lambda \leq \infty$, as follows

$$(ds)^2_{(\lambda)} \;=\; (1 - \frac{2GM}{\rho_\lambda(r)})\;(dt)^2 \;-\; (1 - \frac{2GM}{\rho_\lambda(r)})^{-1}\;(d\rho_\lambda)^2 \;-\; \rho^2_\lambda(r)\;(d\Omega)^2. \qquad (1.1)$$

where $(d\rho_\lambda)^2 = (d\rho_\lambda(r)/dr)^2(dr)^2$ and the solid angle infinitesimal element is $(d\Omega)^2 = (d\phi)^2 + sin^2(\phi)(d\theta)^2$. In Appendix **A** we show explicitly that the metric (1.1) is a solution to Einstein's vacuum field equations. This expression for the family of metrics is given in terms of the *areal* radial functions $\rho_\lambda(r)$ which does *not* violate Birkhoff's theorem since the metric (1.1) expressed in terms of the areal radial functions $\rho_\lambda(r)$ has exactly the same functional form as that required by Birkoff's theorem. It is well known that the *extended* Schwarzschild metric solution for $r < 0$ with $M > 0$, corresponds to a solution in the region $r > 0$ with $M < 0$. Negative masses are associated with repulsive gravity. For this reason, the domain of values of r will be chosen to span the whole real axis $-\infty \leq r \leq \infty$. One may notice that our metric solutions (1.1) are invariant under : $r \to -r; M \to -M$ when the areal radial function is chosen to be antisymmetric $\rho(-r) = -\rho(r)$.

The boundary conditions obeyed by $\rho_\lambda(r)$ must be $\rho_\lambda(r = 0) = 0$, $\rho_\lambda(r = \infty) = \infty$. The Hilbert textbook (black hole) solution [4] when $\rho(r) = r$ obeys the boundary conditions but the Abrams-Brillouin [3] choice $\rho(r) = r + 2GM$ does *not*. The *original* solution of 1916 found by Schwarzschild for $\rho(r)$ did not obey the boundary condition $\rho(r = 0) = 0$ as well. The condition $\rho(r = 0) = 2GM$ has a serious $flaw$ and is : how is it *possible* for a point-mass at $r = 0$ to have a non-zero area $4\pi(2GM)^2$ and a *zero* volume *simultaneously* ?; so it seems that one is forced to choose the Hilbert areal radial function $\rho(r) = r$. It is known that fractals have unusual properties related to their lengths, areas, volumes, dimensions but we are not focusing on fractal spacetimes at the moment. For instance, one could have a fractal surface of infinite area but zero volume (space-filling fractal surface). The finite area of $4\pi(2GM)^2$ could then be seen as a regularized value of the infinite area of a "fractal horizon".

The Hilbert choice for the areal radial function $\rho(r) = r$ is ultimately linked to the actual form of the Newtonian potential $V_N = -(Gm_1m_2/r)$. In the last few decades corrections to Newton's law of gravitation and constraints on them have become the subject of considerable study, see the monograph [5]. Yukawa-type corrections to Newton's gravitational law from two recent measurements of the Casimir interaction between metallic surfaces was studied by [6]. A Yukawa-like correction to the Newtonian potential could be chosen to be

$$V(r) \;=\; -\,\frac{Gm_1m_2}{r}\;(1 - \alpha e^{-r/\lambda}), \quad \alpha > 0,\; \lambda > 0 \qquad (1.2a)$$

where α and λ are the strength and interaction range of the Yukawa-type correction. One may notice that the potential (1.2a) can be rewritten in terms of an areal-radial function

$\rho(r)$ as

$$V(r) = -\frac{Gm_1m_2}{\rho(r)}, \quad \rho(r) = \frac{r}{1-\alpha e^{-r/\lambda}}, \quad \alpha \neq 1 \tag{1.2b}$$

One has the correct boundary conditions for the areal radial function when $\alpha \neq 1$

$$\rho(r=0) = 0; \quad \rho(r=r_h) = 2GM; \quad 0 \leq r_h \leq 2GM \tag{1.2c}$$

so that the location of the horizon radius r_h has been *shifted* towards the singularity. In the asymptotic regime one has as expected

$$\rho(r \to \infty) \to r \tag{1.2d}$$

so that the areal-radial function tends to r (as in the Hilbert choice) and the expression for the potential is asymptotic to the Newtonian one.

There are other ways to bypass the Hilbert solution obtaining different modifications to the Newtonian potential and shifting the horizon location from the known $2GM$ value. Instead of the Yukawa-type areal radial function (1.2b), we will propose a one parameter family of areal-radial functions $\rho_\lambda(r)$ [1] such that

$$\rho_\lambda(r=0) = 0; \quad \rho_\lambda(r=r_h^{(\lambda)}) = 2GM; \quad 0 \leq r_h^{(\lambda)} \leq 2GM \tag{1.3a}$$

so that the location of the horizon radius $r_h^{(\lambda)}$ is being *shifted* continuously towards the singularity as λ increases. In the asymptotic regime we shall impose the conditions for $\lambda \neq 0$

$$\rho_\lambda(r \to \infty) \to r + 2G|M| \to r \tag{1.3b}$$

Meaning that the areal radial functions are increasing functions of r and have for *asymptote* (when $r > 0$) the line $f(r) = r + 2G|M|$. We must also set the conditions on the parameter λ as follows

$$\rho_{\lambda=0}(r) = r; \quad \rho_{\lambda=\infty}(r) = r + 2G|M|sgn(r) \tag{1.4}$$

the $\lambda = 0$ case is just the Hilbert radial function choice, and the extreme limiting case $\lambda = \infty$ involves $sgn(r)$ which is the sign function defined as

$$sgn(r=0) = 0, \ sign(r>0) = 1, \ sgn(r<0) = -1, \ sgn(-r) = -sgn(r) \tag{1.5}$$

The sign function is related to the Heaviside step function $\Theta_H(r)$ as follows $2\Theta_H(r) - 1 = sgn(r)$, after setting $\Theta_H(r=0) = \frac{1}{2}$. The sign function vanishes at $r = 0$ because it is an odd function of r. The sign and Heaviside function are not differentiable in the ordinary sense at $r = 0$, but they *are* differentiable under the generalized notion of differentiation in distribution theory. In particular, $d\Theta_H(r)/dr = \delta(r)$ and $dsgn(r)/dr = 2\delta(r)$. For further details we refer to [16].

[1] We thank Matej Pavsic for a discussion on the choices for the radial functions

The reason one chooses an *antisymmetric* radial function $\rho_\lambda(-r) = -\rho_\lambda(r)$, and one uses the *absolute* sign $|M|$ in eq-(1.4), is because the *extended* metric solutions in eq-(1.1) for $r < 0$ with $M > 0$, correspond to a solution in region $r > 0$ with $M < 0$. Namely, the change in sign due to $\rho(r < 0) = -\rho(r > 0)$ in eq-(1.1) is tantamount to changing the sign of the mass M. Hence the metric solutions (1.1) are *invariant* under the transformations $r \to -r; M \to -M$ when $\rho_\lambda(-r) = -\rho_\lambda(r)$.

For a recent analysis of the properties of the maximal extensions (in regions $r < 0$) of the Kerr and Kerr-Newman spacetimes with *negative* mass, see [12]. Negative mass (or regions of negative mass density) imply violations of one or another variant of the positive energy condition of Einstein's general theory of relativity; however, the positive energy condition is not a required condition for the mathematical consistency of the theory as shown by [13]. The authors [14] pointed out that the quantum mechanics of the Casimir effect can be used to produce a locally mass-negative region of space-time. In this article, and subsequent work by others, they showed that negative matter could be used to stabilize a wormhole. Therefore, the solutions described above make sense once one takes into consideration the negative and positive domain of values of $r = \pm\sqrt{x^2 + y^2 + z^2}$, consistent with the $\pm$ signs under the square root, and that the sign and Heaviside functions *are* differentiable under the generalized notion of differentiation in distribution theory [16] as indicated earlier.

The family of interpolating functions $\rho_\lambda(r)$ are all bounded, for all values of λ , as follows

$$r \leq \ \rho_\lambda(r) \ \leq r + 2G|M|, \quad for\ r > 0 \tag{1.6a}$$

and

$$r \geq \ \rho_\lambda(r) \ \geq r - 2G|M|, \quad for\ r < 0 \tag{1.6b}$$

For convenience purposes we may drop the absolute sign in $|M|$ but it should be kept in mind in order to account for the invariance of the metric (1.1) under the transformations $r \to -r; M \to -M$ and which imply that $\rho_\lambda(-r) = -\rho_\lambda(r)$.

We must *stress* that the radial component of the Hilbert-Schwarzschild solution $-(1 - 2GM/r)^{-1}$ is *not* given by a radial reparametrization of the radial component of the metric (1.1) $g_{rr} = -(1 - 2GM/\rho(r))^{-1}(\frac{d\rho(r)}{dr})^2 = g_{\rho\rho}(\frac{d\rho(r)}{dr})^2$. This means that $(1 - 2GM/r)^{-1}(\frac{dr}{d\rho})^2 \neq (1 - 2GM/\rho)^{-1}$. Rigorously speaking, one should interpret the areal radial function $\rho(r) = R(r)$ as if it were an arbitrary *integrating function* which appears in the solutions of Appendix **A** and (1.1). The particular choice $\rho(r) = r$ yields the standard Hilbert-Schwarzschild solution, whereas the infinite family of metric solutions (1.1) are associated to an infinite number of *modifications* to the gravitational Newtonian potential, in the weak field limit. Experiments will tell what type of corrections to the Newtonian potential at shorter distances occur and which will fix the functional form of the areal radial function $\rho(r) = R(r)$.

The Riemann and Weyl Curvature Tensors can be computed from eqs-(A.3) in the Appendix, after equating the areal radial function $R(r)$ with $\rho(r)$. The curvature tensor components are given explicitly by

$$\mathcal{R}_{2121} = \mathcal{R}_{rtrt} \;=\; - \;\frac{2GM}{\rho(r)^3}\;(\frac{d\rho(r)}{dr})^2; \quad \mathcal{R}_{1212} = \mathcal{R}_{trtr} \;=\; - \;\frac{2GM}{\rho(r)^3}\;(\frac{d\rho(r)}{dr})^2 \qquad (1.7a)$$

$$\mathcal{R}_{r\theta r\theta} \;=\; -\frac{1}{2}\;\frac{2GM}{\rho(r)}\;(1-\frac{2GM}{\rho(r)})^{-1}\;(\frac{d\rho(r)}{dr})^2\; sin^2\phi \qquad (1.7b)$$

$$\mathcal{R}_{r\phi r\phi} \;=\; -\frac{1}{2}\;\frac{2GM}{\rho(r)}\;(1-\frac{2GM}{\rho(r)})^{-1}\;(\frac{d\rho(r)}{dr})^2 \qquad (1.7c)$$

One can see that these curvature components *cannot* be obtained from the ordinary Schwarzschild solutions by the simple replacement $r \rightarrow \rho(r)$ due to the fact that $(\frac{d\rho(r)}{dr})^2 \neq 1$, in general.

The other curvature components

$$\mathcal{R}_{titj} \;=\; \frac{1}{2}\;\frac{2GM}{\rho(r)}\;(1-\frac{2GM}{\rho(r)})\;\tilde g_{ij}, \quad \tilde g_{\phi\phi} = 1, \quad \tilde g_{\theta\theta} = sin^2\phi \qquad (1.8)$$

and $\mathcal{R}_{\phi\theta\phi\theta} = \mathcal{R}_{\theta\phi\theta\phi}$ can be obtained from the ordinary Schwarzschild solutions by the simple replacement $r \rightarrow \rho(r)$.

Since the solutions described in Appendix A and eq-(1.1) are Ricci flat $\mathcal{R}_{\mu\nu} = \mathcal{R} = 0$, the Weyl tensor $C_{\mu\nu\rho\sigma}$ coincides with the curvature tensor. Therefore, due to eqs-(1.7) we have that the curvature/Weyl tensor *is not equivalent* to the one obtained in the Schwarzschild solution after the replacement of the radial coordinate $r \rightarrow \rho(r)$.

Furthermore, we should emphasize again that the curvature tensor components in eqs-(1.7) involving the radial direction are *not* obtained by a radial reparametrization $r \rightarrow \rho(r)$ of the Hilbert-Schwarzschild solution. For example,

$$\mathcal{R}^{Schwarzschild}_{rtrt} \;=\; - \;\frac{2GM}{r^3} \;\neq\; \mathcal{R}_{\rho t\rho t}\;(\frac{d\rho(r)}{dr})^2 \;=\; - \;\frac{2GM}{\rho(r)^3}\;(\frac{d\rho(r)}{dr})^2 \;=\; \mathcal{R}_{rtrt} \qquad (1.9)$$

When $\rho(r) = r$, there is an agreement $\mathcal{R}^{Schwarzschild}_{rtrt} = \mathcal{R}_{rtrt}$ but *not* in general.

For Ricci flat solutions, the Weyl tensor $C_{\mu\nu\rho\sigma}$ coincides with the curvature tensor. Type **D** regions in the Petrov [7] algebraic classification of gravitational solutions are associated with the gravitational fields of isolated massive objects. The two double principal null directions define "radially" ingoing and outgoing null congruences near the object which is the source of the field.

The one-parameter family of metrics in eq-(1.1) (and Riemannian curvature tensors) associated with the choice of the areal radial functions $\rho_\lambda(r)$ are *continuous* functions for all values of r. There is a *spatial* singularity at $r = 0$ (for positive masses). It is *only* in the extreme *limiting* case $\lambda \rightarrow \infty$, when the metric component g_{tt} and Riemannian curvature tensor are *discontinuous* at $r = 0$, besides being singular, at the location of the point mass source, due to the discontinuity of the sign function at $r = 0$. Whereas the Ricci tensor and scalar curvature are zero for all values of r, including $r = 0$, and for all values λ, including $\lambda = \infty$, as shown in Appendix A.

If one wishes to avoid these discontinuities due to the use of the sign function in the limiting case $\lambda = \infty$, one could impose a cutoff $\Lambda \neq \infty$ on the upper value of λ, which in turn leads to a *minimal* length value for the location of the radial horizon r_h^Λ and such that $\rho_\Lambda(r_h^\Lambda) = 2GM$. One could set this minimal length r_h^Λ to be of the order of the Planck scale L_P. This is the scale where Quantum Gravity effects play an important role and classical General Relativity is supposed to break down.

Besides *shifting* the radius horizon location from $r = 2GM$ to $r_h^{(\lambda)} < 2GM$, i.e. towards the singularity, which may be relevant to the resolution of the fire wall problem in black holes [22], another physical motivation in choosing the metric solutions (1.1) is because it leads to a modification of the Newtonian potential which also results from the effects of the generalized uncertainty principle (GUP) associated to a minimal length [24] . The GUP is related to some approaches in quantum gravity such as string theory, black hole physics and doubly special relativity theories (DSR). This leads to a $\sqrt{Area}$-type correction to the area law of entropy which implies that the number of bits N is modified. Therefore, based on Verlinde's enthropic force proposal [20], the authors [24] obtained a modified Newtonian law of gravitation which may have observable consequences at length scales much larger than the Planck scale.

From the asymptotic behavior of the areal radial functions displayed by eq-(1.3) one can infer the corrections to the Newtonian potential obtained in the weak field limit : $g_{tt} \sim (1 + 2V)$. Hence,

$$V = - \frac{GM}{r + 2GM} = - \frac{GM}{r} \left(1 - \frac{2GM}{r} + ... \right) \tag{1.10a}$$

and the modified Newtonian force felt by a test particle of mass m is

$$F = - \frac{GMm}{r^2} \left(1 - \frac{4GM}{r} + ... \right) \tag{1.10b}$$

One has a leading *repulsive* contribution/correction to the modified Newtonian force. At this stage is too early to speculate if this leading repulsive correction has any connection to dark energy. Nevertheless it is worth exploring this possibility. The first two terms of (1.10b) have the same functional form (although with different numerical coefficients) as the modified Newton's law of gravitation found by [24] based on the generalized uncertainty principle (GUP)

$$F = - \frac{GMm}{r^2} \left(1 - \frac{\alpha\sqrt{\mu}}{3r} \right); \quad \alpha = \alpha_o L_P, \quad \alpha_o = constant, \quad \mu = \left(\frac{2.82}{\pi}\right)^2 \tag{1.11}$$

We proceed with a discussion on the possibility of having null singularities in the extreme limiting case $\lambda = \infty$. It is rigorously shown in Appendix B that when $\lambda \rightarrow \infty$, the limiting metric interval $ds^2_{(\infty)}$ in eq-(1.1) is *null* at $r = 0$, instead of being *spacelike*. Hence, it is possible in this limiting $\lambda \rightarrow \infty$ case to have null naked singularities associated to point mass sources. In this *limiting* case the singularity *merges* with the horizon which might have important implications for the resolution of the fire-wall problem in black holes [22], [23].

The physical insignificance of null naked singularities within the context of Penrose's cosmic censorship conjecture was analyzed by [17] in the study of gravitational collapse of general forms matter in the most general of spacetimes. It was shown that the energy is completely trapped inside the null singularity and therefore these null singularities cannot be experimentally observed and cannot cause a breakdown of predictability. This conclusion strongly supports and preserves the essence of the cosmic censorship hypothesis. A timelike singularity is in principle likely to be visible to an outside observer as the redshift is always finite for the light rays emerging from it. For the null singularity surface, the redshift basically diverges as the proper time goes to zero on the null surface. It was argued by [17] that despite that the null singularity is geometrically naked (null geodesics can come out of it) essentially it is not physically visible (naked) as no energy can come out of it due to the infinite redshift. Because one cannot get any information from the null naked singularity it will not have any undesirable physical effect to an outside observer.

The Penrose diagrams associated with the solutions described in (1.1) are the same as the diagrams corresponding to the extended Schwarzchild solutions with the only difference that we must replace the radial variable r for ρ. The horizons at the radial locations $r_h^{(\lambda)}$ all correspond to the *unique* value of the areal radial function $\rho(r_h^{(\lambda)}) = 2GM$ and $t = \pm\infty$. The spatial singularity is located at $\rho_\lambda(r = 0) = 0$. The Fronsdal-Kruskal-Szekeres change of coordinates that permit an analytical extension into the interior region of the black hole has the same functional form as before after replacing r for ρ. In the exterior region $\rho(r) > 2GM$ one has

$$U = (\frac{\rho(r)}{2GM} - 1)^{\frac{1}{2}}\ e^{\rho(r)/4GM}\ cosh\ (\frac{t}{4GM}),\ \ V = (\frac{\rho(r)}{2GM} - 1)^{\frac{1}{2}}\ e^{\rho(r)/4GM}\ sinh\ (\frac{t}{4GM});\ \rho(r) > 2GM \tag{1.12a}$$

and the change of coordinates in the interior region $\rho(r) < 2GM$ is

$$U = (1 - \frac{\rho(r)}{2GM})^{\frac{1}{2}}\ e^{\rho(r)/4GM}\ sinh\ (\frac{t}{4GM}),\ \ V = (1 - \frac{\rho(r)}{2GM})^{\frac{1}{2}}\ e^{\rho(r)/4GM}\ cosh\ (\frac{t}{4GM});\ \rho(r) < 2GM \tag{1.12b}$$

In the overlap $\rho(r) = 2GM$ region one has $U = \pm V$ when $t = \pm\infty$, and $U = V = 0$ for *finite* t.

The coordinate transformations lead to a well behaved metric (except at $\rho(r = 0) = 0$)

$$ds^2 \ = \ \frac{4(2GM)^3}{\rho(U,V)}\ e^{-\rho(U,V)/2GM}\ (dV^2\ - dU^2\ )\ -\ \rho(U,V)^2\ (d\Omega)^2. \tag{1.12c}$$

When $\rho(r = r_{horizon}) = 2GM$ and $d\Omega = 0$, the above interval displacement $ds^2 = 0$ is *null* along the lines $U = \pm V \Rightarrow dU = \pm dV$. It is singular at $\rho(r = 0) = 0$: along the (spacelike) lines $V^2 - U^2 = 1 \Rightarrow dV \neq \pm dU$. The *range* of the radial coordinate r for the family of interpolating functions $\rho_\lambda(r)$, for all values of λ was given by eqs-(1.6).

In the extreme *limiting* case $\lambda = \infty \Rightarrow \rho_{\lambda=\infty}(r) = r + 2G|M|sgn(r)$ the Penrose diagrams can be obtained from the diagrams corresponding to the extended Schwarzchild solutions by simply *removing* the interior regions; i.e. by removing the upper and lower regions (quadrants) of the Rindler wedge, leaving only the left and right exterior (causal diamond-like) regions which are connected to the asymptotically flat portions of spacetime. The horizons at $r = 0^+, t = \pm\infty, \rho_{(\infty)}(r = 0^+) = 2G|M|$ are *causal* boundaries of these left and right diamond-like regions, in addition to the future and past null infinity boundary regions. There is a null-line singularity at $r = 0$ and a null-surface at $r = 0^+$. This may sound quite paradoxically but it is a consequence of the metric *discontinuity* at $r = 0$, the location of the point mass (singularity). Although the spacetime manifold is continuous everywhere, what is *discontinuous* at $r = 0$ is the metric due to the discontinuity of the areal-radial function $\rho_{(\infty)}(r)$ at $r = 0$ since $\rho_{(\infty)}(r = 0) = 0, \rho_{(\infty)}(r = 0^+) = 2G|M|$. In this extreme limiting case, any infalling observer hits the null singularity at the very moment he/she crosses the horizon. This fact may have important consequences for the resolution of the fire wall problem and the complementarity controversy in black holes [22], [23].

Because a point mass is an infinitely compact source there is nothing *wrong* with the possibility of having a *discontinuity* of the metric at the location of the singularity $r = 0$ when the radial function is chosen $\rho_{\lambda=\infty}(r) = r + 2G|M|sgn(r)$. This discontinuity may appear to be *unappealing* but one cannot disregard such possibility. Similarly, despite the unappealing nature of the black hole singularity at $r = 0$ this was no reason to dismiss those solutions. The study of Einstein equations and the joining of discontinuous metrics when these are discontinuous across the joining (hyper) surface was studied by [9] in the static spherically symmetric case. These discontinuous metrics obey Einstein equations with an energy-momentum tensor which has a delta function type of singularity on the (hyper) surface of discontinuity. It was found that a surface tension is always associated to the cases where the metrics are discontinuous. The kind of metric discontinuity which follows by our choice of the areal radial function $\rho_{(\infty)}(r)$ above is of a $different$ type than the ones studied by [9]. In section **2** we shall study explicitly the case when it is a delta function type of singularity for the energy-momentum tensor (mass density and pressure) associated with the point mass which is the source of a curvature discontinuity, *and* singularity, at $r = 0$.

Finally, as stated earlier, if one wishes to avoid these discontinuities due to the use of the radial function $r + 2G|M|sgn(r)$, when $\lambda = \infty$, one could impose a cutoff $\Lambda \neq \infty$ on the upper value of λ, which in turn leads to a *minimal* length value for the location of the radial horizon r_h^Λ and that could be set to be of the order of the Planck scale L_P. This possibility warrants further investigation, in particular because the imposition of a minimal radius-horizon length is linked directly to the avoidance of metric discontinuities at the location of point mass sources.

Despite that r_h^Λ may be of the order of the Planck length, the value of the areal radial function is still $\rho_\Lambda(r_h^\Lambda) = 2G|M|$ and which is *not* small for macroscopic masses. This interplay between a microscopic length $r_h^\Lambda \sim L_{Planck}$ (where Quantum Gravity effects are relevant) and a much larger scale $2G|M|$ (where classical gravity is valid) corresponding to the horizon radius of the standard black hole solutions (when $\rho(r) = r$) might be relevant to the Hawking radiation phenomenon and viewing a black hole as a macroscopic quantum system due to a Bose Einstein condensate of a very large number of soft gravitons [26].

2 Point Mass Sources and Euclidean Gravitational Action as Entropy

A rigorous correct treatment of point mass distributions in General Relativity has been provided based on Colombeau's [10] theory of nonlinear distributions, generalized functions and nonlinear calculus. This permits the proper multiplication of distributions since the old Schwarz theory of linear distributions is invalid in nonlinear theories like General Relativity. Colombeau's nonlinear distributional geometry supersedes the no-go results of Geroch and Traschen [18] stating that there is no proper framework to study distributions of matter of co-dimensions higher than two (neither points nor strings in $D = 4$) in General Relativity. Colombeau's theory of Nonlinear Distributions (and Nonstandard Analysis) is the proper way to deal with point-mass sources in nonlinear theories like Gravity and where one may rigorously solve the problem without having to introduce a boundary of spacetime at $r = 0$.

Nevertheless one may still arrive at some interesting physical results by recurring to the ordinary Dirac delta functions. In order to generate $\delta(r)$ terms in the right hand side of Einstein's equations in the presence of a point-mass source, it was argued in [11] that one must replace everywhere $r \rightarrow |r|$ as required when point-mass sources are inserted. The Newtonian gravitational potential (in three dimensions) due to a point-mass source at $r = 0$ is given by $-GM/|r|$. It is consistent with Poisson's law which states that the non-zero Laplacian of the Newtonian potential $\nabla^2(-GM/|r|) = 4\pi G\rho$ is proportional to the mass density distribution $\rho = (M/4\pi r^2)\delta(r)$. However, the Laplacian in spherical coordinates of $(1/r)$ is identically *zero*.

For this reason, there is a *fundamental* difference in dealing with expressions involving absolute values $|r|$ like $1/|r|$ from those which depend on r like $1/r$. This is a direct consequence of the *discontinuity* of the *derivatives* of the function $|r|$ at $r = 0$. However, despite this discontinuity in the derivatives we shall be working next with a metric that is *continuous* at $r = 0$.

Let us begin now with the temporal and radial components of a *continuous* metric at $r = 0$ and whose signature is chosen to be $(-, +, +, +)$

$$g_{tt} = -\,(1 - \frac{2GM}{|r|}) = -\,(1 - \frac{2GM}{r}\,\frac{r}{|r|}) = -\,(1 - \frac{2GM}{r}\,f(r)); \quad f(r) \equiv \frac{r}{|r|}. \tag{2.1}$$

the function $f(r) = \frac{r}{|r|}$ is by definition the same as the sign function $sgn(r) = f(r)$. Because it is an odd function of r it vanishes at $r = 0$, and is ± 1 when $r > 0, r < 0$, respectively. The mass is chosen to be positive $M > 0$. We must emphasize that the expression (2.1) for the metric component does make *sense*, as we shall prove below, when we study the metric associated with a mass distribution given by a smeared delta function [19] and take the limit when the mass density distribution becomes a delta function. The other metric components are

$$g_{rr} = -\,\frac{1}{g_{tt}}, \quad g_{\phi\phi} = r^2, \quad g_{\theta\theta} = r^2 sin^2(\phi) \tag{2.2}$$

In section **1** we stressed the fact that the values of r may span the region $-\infty \leq r \leq \infty$ and despite that the sign and Heaviside functions are not differentiable in the ordinary sense at $r = 0$, they *are* differentiable under the generalized notion of differentiation in distribution theory [16]. In particular, the derivative of the sign function is [16]

$$f(r) \equiv sgn(r) \;\Rightarrow\; f'(r) \;=\; \frac{df(r)}{dr} \;=\; 2\,\delta(r); \quad f''(r) \;=\; \frac{d^2 f(r)}{dr^2} \;=\; 2\,\delta'(r). \tag{2.3}$$

and one learns that the curvature scalar $\mathcal{R}$ corresponding to the metric (2.1, 2.2) is now *nonvanishing* at $r = 0$

$$\mathcal{R} \;=\; -\,2GM\;[\;\frac{f''(r)}{r} \;+\; 2\;\frac{f'(r)}{r^2}\;] \;=$$

$$-\;4GM\;[\;\frac{\delta'(r)}{r} \;+\; 2\;\frac{\delta(r)}{r^2}\;] \;=\; -\,8\pi G\;T \tag{2.4}$$

where T in eq-(2.4) is the trace of the stress energy tensor $g^{\mu\nu}T_{\mu\nu}$ in the Einstein's field equations due to the presence of matter. We should emphasize that after using the distributional relation $r\delta'(r) = -\delta(r)$ allows to rewrite the scalar curvature (2.4) solely in terms of $\delta(r)$ as $\mathcal{R} = -\frac{4GM\delta(r)}{r^2}$. This latter expression for the scalar curvature is precisely the one obtained after taking the trace of eqs-(A.4-A.6) and using the metric expression (2.1-2.2) solely in terms of the absolute values $|r|$ and not in terms of the sign function $sgn(r)$. The metric (2.1-2.2) and scalar curvature $\mathcal{R} = -\frac{4GM\delta(r)}{r^2}$ are invariant under $r \rightarrow -r; M \rightarrow M$. There is *no* sign change in M as it occurred in the solutions (1.1).

The Ricci tensor associated to the metric (2.1, 2.2) can be obtained by using eqs-(A.4-A.6) in the Appendix

$$\mathcal{R}_{tt} \;=\; \frac{2GM}{r^2}\;(1 - \frac{2GM}{|r|})\;\delta(r), \quad \mathcal{R}_{rr} \;=\; -\;\frac{2GM}{r^2}\;(1 - \frac{2GM}{|r|})^{-1}\;\delta(r) \tag{2.5a}$$

$$\mathcal{R}_{\theta\theta} \;=\; 0, \quad \mathcal{R}_{\phi\phi} \;=\; 0 \tag{2.5b}$$

Hence, after taking the trace of eqs-(2.5) yields $\mathcal{R} = g^{tt}\mathcal{R}_{tt} + g^{rr}\mathcal{R}_{rr} = -\frac{4GM\delta(r)}{r^2}$ as announced.

The scalar curvature (2.4) is $\mathcal{R} = 0$ for $r > 0$ and it is singular at $r = 0$. Whereas the scalar curvature $\mathcal{R}$ and Ricci tensor $\mathcal{R}_{\mu\nu}$ associated with the standard Schwarzschild (Hilbert) solutions, involving r instead of $|r|$, are identically *zero* for *all* values of r, including $r = 0$. What is not zero, and singular at $r = 0$, is the Riemannian curvature tensor. [2]

The Einstein tensor corresponding to eqs-(2.1, 2.2, 2.5) is

[2] One may notice that by choosing $f(r) = \kappa/r$ in eq-(2.4) for κ= constant, it yields $\mathcal{R} = 0$ which implies a zero trace for the stress energy tensor $T = 0$, as it happens in Electromagnetism due to the conformal invariance of Maxwell equations in $D = 4$. The Reisner-Nordstrom solutions (in the massless case) have for temporal metric component $g_{tt} = 1 - e^2/r^2$, which has the same functional form as $g_{tt} = 1 - (2GM/r)f(r) = 1 - 2GM\kappa/r^2 \leftrightarrow 1 - e^2/r^2$.

$$\mathcal{R}_{tt} - \frac{1}{2}\, g_{tt}\, \mathcal{R} \;=\; \frac{2GM}{r^2}\,(1 - \frac{2GM}{|r|})\, \delta(r) \,-\frac{1}{2}\,(-(1-\frac{2GM}{|r|}))\,(-\frac{4GM\delta(r)}{r^2}) \;=\; 0 \quad (2.6a)$$

$$\mathcal{R}_{rr} - \frac{1}{2}\, g_{rr}\, \mathcal{R} \;=\; -\,\frac{2GM}{r^2}\,(1 - \frac{2GM}{|r|})^{-1}\, \delta(r) \,-\frac{1}{2}\,(1-\frac{2GM}{|r|})^{-1}\,(-\frac{4GM\delta(r)}{r^2}) \;=\; 0 \quad (2.6b)$$

$$\mathcal{R}_{\theta\theta} - \frac{1}{2}\, g_{\theta\theta}\, \mathcal{R} \;=\; g_{\theta\theta}\,\frac{2GM\delta(r)}{r^2} \;=\; 8\pi G\; g_{\theta\theta}\; p_\theta \;\Rightarrow\; p_\theta \;=\; \frac{2M\delta(r)}{8\pi r^2} \qquad (2.6c)$$

$$\mathcal{R}_{\phi\phi} - \frac{1}{2}\, g_{\phi\phi}\, \mathcal{R} \;=\; g_{\phi\phi}\,\frac{2GM\delta(r)}{r^2} \;=\; 8\pi G\; g_{\phi\phi}\; p_\phi \;\Rightarrow\; p_\phi \;=\; \frac{2M\delta(r)}{8\pi r^2} \qquad (2.6d)$$

Therefore we have found that the Einstein tensor associated with the metric (2.1, 2.2) and Ricci tensors (2.5) correspond to a matter distribution such that

$$\rho \;=\; p_r \;=\; 0,\;\; p_\theta \;=\; p_\phi \;=\; \frac{2M\delta(r)}{8\pi r^2} \qquad (2.7)$$

The weak energy conditions $\rho \geq 0$, $\rho + p_i \geq 0$, for $i = 1, 2, 3$, are satisfied by (2.7). Also the strong energy conditions $\rho + \sum_i p_i \geq 0$ are satisfied. While on the other hand, the dominant energy conditions $\rho \geq |p_i|$ are not satisfied.

The non-trivial Einstein-Hilbert action is

$$S = \;-\,\frac{1}{16\pi G}\int\!\!\int \mathcal{R}\; 4\pi r^2\; dr\; dt = \frac{1}{16\pi G}\int\!\!\int \frac{4GM\;\delta(r)}{r^2}\; 4\pi r^2\; dr\; dt \;=$$

$$\frac{1}{16\pi G}\int\!\!\int\; 16\pi GM\;\delta(r)\; dr\; dt \;=\; \int\!\!\int\; M\;\delta(r)\; dr\; dt \qquad (2.8)$$

Because the radial integral (2.8) is symmetric in r due to $\delta(-r) = \delta(r)$, the radial integral from $r = 0$ to $r = \infty$ can be rewritten as one half the integral from $r = -\infty$ to $r = \infty$

$$S \;=\; \int_{r=0}^{r=\infty}\int\; M\;\delta(r)\; dr\; dt \;=\; \frac{1}{2}\int_{r=-\infty}^{r=\infty}\delta(r)\; dr\int\; M\; dt \;=\; \frac{1}{2}\int\; M\; dt \qquad (2.9)$$

The metric in eqs-(2.1, 2.2) has a well defined notion of surface gravity at $r = 2GM$, which is the location of the standard horizon, and the radial derivatives of $GM/|r|$ are well defined and finite at $r = 2GM$. Therefore, the concepts of entropy and Hawking temperature [15] are meaningful in this case.

The Euclideanized Einstein-Hilbert action (2.8, 2.9) associated with the non-trivial scalar curvature is obtained after a compactification of the temporal direction along a circle S^1 whose net Euclidean time integration interval is $2\pi t_E$. The latter interval can be defined in terms of the Hawking temperature T_H and the Boltzman constant k_B as $2\pi t_E = (1/k_B T_H) = 8\pi GM$. The temperature T_H also agrees with the Unruh-Rindler temperature $\frac{\hbar}{2\pi}|a|$ (in units $\hbar = c = 1$), where $|a| = \frac{1}{4GM}$ is the magnitude of the surface gravity (acceleration) at the standard horizon location $r = 2GM$. Integrating with respect to the Euclidean temporal coordinate, the Euclidean gravitational action becomes then

$$S_E = (\frac{M}{2})\ (2\pi t_E)\ =\ 4\pi\ G\ M^2 = \ \frac{1}{4}\ \frac{4\pi(2GM)^2}{G}\ =\ \frac{Area}{4\ L_P^2}. \tag{2.10}$$

which is precisely the black bole Entropy in Planck area units $G = L_P^2$ ($\hbar = c = 1$).

This result that the Euclideanized gravitational action (associated with a non-trivial scalar curvature involving delta functions due to point-mass sources) is the same as the black hole entropy can be generalized to higher dimensions. In the Reissner-Nordsrom (massive-charged) and Kerr-Newman black hole case (massive-rotating-charged) [11] has shown that the Euclidean action in a bulk domain bounded by the inner and outer horizons is the same as the black hole entropy. These findings should be compared to Verlinde's enthropic gravity proposal [20] based on the holographic principle.

Another approach [11] to tackle delta function sources is by smoothing the point-mass distribution by a *smeared* delta function as proposed by [19]

$$\rho(r)\ =\ M\ \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}}\ \Rightarrow\ lim_{\sigma\rightarrow 0}\ M\frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}}\ \rightarrow 2M\frac{\delta(r)}{4\pi r^2} \equiv\ M\delta^3(r,\phi,\theta) \tag{2.11a}$$

The integral of the mass density is

$$\int_{r=0}^{r=\infty}\ M\ \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}}\ 4\pi r^2\ dr\ =\ M \tag{2.11b}$$

after using the result

$$\int\ r^2\ e^{-r^2/a^2}\ dr\ =\ \frac{a^2}{4}\ (\ \sqrt{\pi}\ a\ erf(\frac{r}{a})\ -\ 2r\ e^{-r^2/a^2}\ ),\quad erf(r=0)=0,\quad erf(r=\infty)=1 \tag{2.11c}$$

where $erf(x)$ is the error function. The result in (2.11b) is also consistent with the integral of the mass density in the $\sigma \rightarrow 0$ limit given by eq-(2.11a), after making use of the symmetry property $\delta(-r) = \delta(r)$ such that the radial integral from $r = 0$ to $r = \infty$ can be rewritten as one half the integral from $r = -\infty$ to $r = \infty$

$$\int_{r=0}^{r=\infty}\ 2M\frac{\delta(r)}{4\pi r^2}\ 4\pi r^2\ dr\ =\ \frac{1}{2}\ \int_{r=-\infty}^{r=\infty}\ 2M\frac{\delta(r)}{4\pi r^2}\ 4\pi r^2\ dr\ =\ M\ \int_{r=-\infty}^{r=\infty}\ \delta(r)\ dr\ =\ M \tag{2.11d}$$

Therefore one arrives at the expected result in (2.11d).

The field equations

$$\mathcal{R}_{00}\ -\frac{1}{2}\ g_{00}\ \mathcal{R}\ =\ 8\pi G\ T_{00},\quad \mathcal{R}_{ij}\ -\frac{1}{2}\ g_{ij}\ \mathcal{R}\ =\ 8\pi G\ T_{ij} \tag{2.12}$$

were solved by [19] when the stress energy tensor T_{00}, T_{ij} elements are comprised of a density $\rho(r)$, a radial and tangential pressures $p_r(r), p_\theta(r), p_\phi(r)$ associated to a self-gravitating anisotropic fluid and were given in terms of a smeared delta function (a Gaussian of width σ) as follows [19]

$$\rho(r) \ = \ M \ \frac{e^{-r^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}}, \quad p_r \ = \ - \rho(r), \quad p_{tan} \ = \ p_\theta \ = \ p_\phi \ = -\rho(r) - \frac{r}{2}\ \frac{d\rho}{dr}. \quad (2.13)$$

The metric solutions to the field equations (2.12) corresponding to the stress energy tensor associated with the density and pressure in (2.13) have a similar form as the Hilbert-Schwarzschild solution after *replacing* the mass parameter M for a radial-dependent mass function given by $M(r) = \int_0^r \rho(r') 4\pi r'^2 dr'$ [19]. In the $\sigma \rightarrow 0$ limit, the elements of eq-(2.13) become

$$\rho(r) \ = \ - p_r(r) \ = \ 2M \frac{\delta(r)}{4\pi r^2} \ \equiv \ M\delta^3(r,\phi,\theta), \quad p_\theta(r) \ = \ p_\phi(r) \ = \ - 2M \frac{\delta'(r)}{8\pi r} \ =$$

$$-2M \frac{r\delta'(r)}{8\pi r^2} \ = \ 2M \frac{\delta(r)}{8\pi r^2} \quad (2.14)$$

where in the last terms of eq-(2.14) we have used the important distributional relation $r\delta'(r) = -\delta(r)$ [3] and which enables to write everything in eq-(2.14) directly in terms of the delta function $\delta(r)$.

The scalar curvature in the $\sigma \rightarrow 0$ limit can be expressed in terms of the trace of the energy stress tensor of eq-(2.14) and it yields $-8\pi G T^\mu_\mu = \mathcal{R} = -4GM\delta(r)/r^2$ which turns out to be equal to the scalar curvature obtained from taking the trace of eqs-(2.5a,2.5b). Therefore, despite that the point-mass stress energy source distribution in eq-(2.14) *differs* from the point-mass stress energy source in eq-(2.7), both have the same trace and consequently generate they same scalar curvature $\mathcal{R} = -4GM\delta(r)/r^2$, which in turn, leads to the same value of the Euclideanized gravitational action and which coincides with the black-hole entropy (2.10) $S = Area/4L_P^2$. To conclude, the metric solutions (2.1, 2,2) and eq-(2.3) do *make sense*, from the mathematical and physical point of view.

An important remark is in order before presenting our conclusions. The Gaussian width limit $\sigma \rightarrow 0$ must be taken afterwards one inserts the metric given by [19] into the field equations. In one takes this limit *before* evaluating the Einstein tensor, it leads directly to the standard Hilbert-Schwarzschild metric which furnishes a *zero* Einstein tensor and corresponds to the textbook static spherically symmetric Ricci flat vacuum solution $\mathcal{R}_{\mu\nu} = 0$ (a zero stress energy tensor in the right hand side of Einstein's equations).

It is instructive to study the energy conditions for the given density and pressure configurations of eq-(2.14) when $M > 0$. After simple algebra one learns that the weak energy conditions $\rho \geq 0$ and $\rho + p_i \geq 0$, for $i = 1,2,3$, are satisfied due to $d\rho/dr \leq 0$ when $r \geq 0$. The strong energy conditions $\rho + \sum_i p_i \geq 0$ are only satisfied in the region $r \geq 2\sigma$ but *not* in the core region $r < 2\sigma$. While on the other hand, the dominant energy conditions $\rho \geq |p_i|$ are satisfied for the values of $r \leq 2\sqrt{2}\sigma$ but are violated for the values of $r > 2\sqrt{2}\sigma$. Thus, the matter field configuration studied in eq-(2.14) obeys only the weak energy conditions for all values of r such that $r \geq 0$.

We finalize by adding some remarks [11] about how a *fuzzy* point mass may admit a short distance cut-off of the Brillouin form $\rho(r = 0) = 2GM$ (instead of *zero*) if

[3] In general, when n derivatives are involved one has the relation $r^n \delta^{(n)}(r) = (-1)^n n! \delta(r)$ [16]

one has a Noncommutative spacetime coordinates algebra $[x^\mu, x^\nu] = i\Sigma^{\mu\nu}$, $[p^\mu, p^\nu] = 0$, $[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}$ where $\Sigma^{\mu\nu}$ are c-numbers of $(Planck\ length)^2$ magnitude. A change of coordinates in phase space $x'^\mu = x^\mu + \frac{1}{2}\Sigma^{\mu\nu}\ p_\nu$ leads to commuting coordinates x'^μ and allows to define $r'(r) = \sqrt{(x^i + \frac{1}{2}\ \Sigma^{i\rho}\ p_\rho)\ (x_i + \frac{1}{2}\ \Sigma_{i\tau}\ p^\tau)}$. One can select $\Sigma^{\mu\nu}$ such that $r'(x^i = 0) = r'(r = 0) = 2GM$, after using the on-shell condition $p_\mu p^\mu = M^2$. Therefore one recovers the cut-off corresponding to the Brillouin area radial function $\rho(r) = r + 2GM \Rightarrow \rho(r = 0) = 2GM$. Thus a fuzzy point mass has non-zero area and volume.

Another Planck scale cut-off can be derived in terms of noncommutative Moyal star products $f(x) * g(x)$ simply by replacing $r \to r_* = \sqrt{r * r} = \sqrt{r^2 + \Sigma^{ij} x_i x_j / r^2 + ...}$ so $r_*(x^i = 0) \neq 0$, and it receives Planck scale corrections. A point is fuzzy and delocalized, henceforth it has a non-zero fuzzy area and fuzzy volume. An open problem is to verify whether or not Schwarzschild deformed metrics of the form

$$g_{tt}(r_*) = 1 - \frac{2GM}{r_*}, \quad g_{rr} = -[g_{tt}^{-1}]_*, \quad r_* = \sqrt{r * r} = \sqrt{r^2 + \Sigma^{ij} x_i x_j / r^2 + ...}\ . \quad (2.15)$$

solve the Noncommutative Gravity field equations to *all* orders in the noncommutative parameter $\Sigma^{\mu\nu}$. The angular part is given by $r_*\ *\ r_*\ (d\Omega)^2$, and the star inverse $[g_{tt}^{-1}]_*$ is defined in terms of a Taylor series involving star products . This is a very difficult problem. To conclude, one has to wait for a theory of Quantum Gravity to fully address these issues of the avoidance of singularities due to the noncommutativity of spacetime coordinates. Another relevant topic is to explore the Nonperturbative Renormalization Group flow [25] of the metric $g_{\mu v}[k]$ in terms of the momentum scale k and its relationship (if any) to the family of metrics (1.1) parametrized by λ.

APPENDIX A : Schwarzschild-like solutions in $D > 3$

In this Appendix we follow closely the calculations of the static spherically symmetric *vacuum* solutions to Einstein's equations in any dimension $D > 3$. Let us start with the line element with signature $(-, +, +, +, ..., +)$

$$ds^2 = -e^{\mu(r)}(dt)^2 + e^{\nu(r)}(dr)^2 + R^2(r)\tilde{g}_{ij} d\xi^i d\xi^j. \quad (A.1)$$

where the areal radial function $\rho(r)$ is now denoted by $R(r)$ and which must *not* be *confused* with the scalar curvature $\mathcal{R}$. Here, the metric $\tilde{g}_{ij}$ corresponds to a homogeneous space and $i, j = 3, 4, ..., D - 2$ and the temporal and radial indices are denoted by $1, 2$ respectively. In our text we denoted the temporal index by 0. The only non-vanishing Christoffel symbols are given in terms of the following partial derivatives with respect to the r variable and denoted with a prime

$$\begin{array}{lll} \Gamma^1_{21} = \frac{1}{2}\mu', & \Gamma^2_{22} = \frac{1}{2}\nu', & \Gamma^2_{11} = \frac{1}{2}\mu' e^{\mu-\nu}, \\ \Gamma^2_{ij} = -e^{-\nu} R R' \tilde{g}_{ij}, & \Gamma^i_{2j} = \frac{R'}{R}\delta^i_j, & \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}, \end{array} \quad (A.2)$$

and the only nonvanishing Riemann tensor are

$$\mathcal{R}^1_{212} = -\tfrac{1}{2}\mu'' - \tfrac{1}{4}\mu'^2 + \tfrac{1}{4}\nu'\mu', \qquad \mathcal{R}^1_{i1j} = -\tfrac{1}{2}\mu' e^{-\nu} RR'\tilde{g}_{ij},$$

$$\mathcal{R}^2_{121} = e^{\mu-\nu}(\tfrac{1}{2}\mu'' + \tfrac{1}{4}\mu'^2 - \tfrac{1}{4}\nu'\mu'), \qquad \mathcal{R}^2_{i2j} = e^{-\nu}(\tfrac{1}{2}\nu' RR' - RR'')\tilde{g}_{ij}, \tag{A.3}$$

$$\mathcal{R}^i_{jkl} = \tilde{R}^i_{jkl} - R'^2 e^{-\nu}(\delta^i_k \tilde{g}_{jl} - \delta^i_l \tilde{g}_{jk}).$$

The vacuum field equations are

$$\mathcal{R}_{11} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\mu'\nu' + \frac{(D-2)}{2}\mu'\frac{R'}{R}) = 0, \tag{A.4}$$

$$\mathcal{R}_{22} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu' + (D-2)(\frac{1}{2}\nu'\frac{R'}{R} - \frac{R''}{R}) = 0, \tag{A.5}$$

and

$$\mathcal{R}_{ij} = \frac{e^{-\nu}}{R^2}(\frac{1}{2}(\nu' - \mu')RR' - RR'' - (D-3)R'^2)\tilde{g}_{ij} + \frac{k}{R^2}(D-3)\tilde{g}_{ij} = 0, \tag{A.6}$$

where $k = \pm 1$, depending if $\tilde{g}_{ij}$ refers to positive or negative curvature. From the combination $e^{-\mu+\nu}\mathcal{R}_{11} + \mathcal{R}_{22} = 0$ we get

$$\mu' + \nu' = \frac{2R''}{R'}. \tag{A.7}$$

The solution of this equation is

$$\mu + \nu \ = \ \ln R'^2 \ + \ C, \tag{A.8}$$

where C is an integration constant that one sets to *zero* if one wishes to recover the flat Minkowski spacetime metric in spherical coordinates in the asymptotic region $r \to \infty$.

Substituting (A.7) into the equation (A.6) we find

$$e^{-\nu} \ (\ \nu' RR' - 2RR'' - (D-3)R'^2 \ ) \ = \ - k(D-3) \tag{A.9}$$

or

$$\gamma' RR' + 2\gamma RR'' + (D-3)\gamma R'^2 = k(D-3), \tag{A.10}$$

where

$$\gamma = e^{-\nu}. \tag{A.11}$$

The solution of (A.10) for an ordinary D-dim spacetime (one temporal dimension) corresponding to a $D-2$-dim sphere for the homogeneous space can be written as

$$\gamma = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} R^{D-3}}) \ (\frac{dR}{dr})^{-2} \Rightarrow$$

$$g_{rr} = e^{\nu} = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} R^{D-3}})^{-1} \ (\frac{dR}{dr})^2. \qquad (A.12)$$

where Ω_{D-2} is the appropriate solid angle in $D-2$-dim and G_D is the D-dim gravitational constant whose units are $(length)^{D-2}$. Thus $G_D M$ has units of $(length)^{D-3}$ as it should. When $D = 4$ as a result that the 2-dim solid angle is $\Omega_2 = 4\pi$ one recovers from eq-(A.12) the 4-dim Schwarzchild solution. The solution in eq-(A.12) is consistent with Gauss law and Poisson's equation in $D-1$ spatial dimensions obtained in the Newtonian limit.

For the most general case of the $D-2$-dim homogeneous space we should write

$$-\nu = \ln(k - \frac{\beta_D G_D M}{R^{D-3}}) - 2\ln R'. \qquad (A.13)$$

β_D is a constant equal to $16\pi/(D-2)\Omega_{D-2}$, where Ω_{D-2} is the solid angle in the $D-2$ transverse dimensions to r, t and is given by $(D-1)\pi^{(D-1)/2}/\Gamma[(D+1)/2]$.

Thus, according to (A.8) we get

$$\mu = \ln(k - \frac{\beta_D G_D M}{R^{D-3}}) \ + \ constant. \qquad (A.14)$$

we can set the constant to zero, and this means the line element (A.1) can be written as

$$ds^2 = -(k - \frac{\beta_D G_D M}{R^{D-3}})(dt)^2 \ + \ \frac{(dR/dr)^2}{(k - \frac{\beta_D G_D M}{R^{D-3}})}(dr)^2 \ + \ R^2(r)\tilde{g}_{ij} d\xi^i d\xi^j \ =$$

$$-(k - \frac{\beta_D G_D M}{R^{D-3}})(dt)^2 \ + \ \frac{1}{(k - \frac{\beta_D G_D M}{R^{D-3}})}(dR)^2 \ + \ R^2(r)\tilde{g}_{ij} d\xi^i d\xi^j \qquad (A.15)$$

One can verify, that the equations (A.4)-(A.6),leading to eqs-(A.9)-(A.10), do *not* determine the form $R(r)$. It is also interesting to observe that the only effect of the homogeneous metric $\tilde{g}_{ij}$ is reflected in the $k = \pm 1$ parameter, associated with a positive (negative) constant scalar curvature of the homogeneous $D-2$-dim space. $k = 0$ corresponds to a spatially flat $D-2$-dim section. The metric solution in eq-(1.1) is associated to a different signature than the one chosen in this Appendix, and corresponds to $D = 4$ and $k = 1$.

We finalize this Appendix by studying what happens when the radial function is given by $R(r) = r + 2G|M| sgn(r)$ in the limiting case $\lambda = \infty$. It is important to emphasize that despite that the derivatives $\frac{dR}{dr} = 1 + 4G|M|\delta(r)$ and $(d^2R/dr^2) = 4G|M|\delta'(r)$ are *singular* at $r = 0$, there is an exact and precise *cancellation* of these singular derivatives (involving delta functions) in the evaluation of the Ricci curvature tensor components yielding a zero Ricci tensor $\mathcal{R}_{\mu\nu} = 0$ and a zero Ricci scalar $\mathcal{R} = 0$. What is *not* zero is the Riemann curvature tensor $\mathcal{R}_{\mu\nu\rho\tau}$. Therefore, the conditions $\mathcal{R}_{\mu\nu} = 0$ and $\mathcal{R} = 0$ are satisfied for *any* area radial function $R(r)$, irrespective if it has singular derivatives at $r = 0$ or not.

Furthermore, despite that $(\frac{dR}{dr})^2 = (1 + 4G|M|\delta(r))^2$ in eq-(A.15) involves the ill defined product of distributions, one should notice that it is well defined at $r > 0 \Rightarrow (\frac{dR}{dr})^2 = 1$, and also the radial component of the metric (A.15) is well defined at $r = 0$ because the product

$$lim_{r\to 0}\ [\ (1-\frac{2GM}{R})^{-1}\ (\frac{dR}{dr})^2\ ]\ \to\ lim_{r\to 0}\ \ [\ -\frac{R(r)}{2GM}\ (1+4G|M|\delta(r))^2\ ]\ \to 0 \quad (A.16)$$

so that $g_{rr}(r=0)=0$. This is a consequence of the fact that $R(r=0)(\delta(r=0))^2 = 0\times(\delta(r=0))^2=0$ because the expression $R(r)(\delta(r))^2$ is an *odd* function of r which must *vanish* at the origin $r=0$.

APPENDIX B : Null-like singularities in the limiting $\lambda=\infty$ case

As mentioned earlier, in the limiting $\lambda=\infty$ case, the radial function is $R(r) = r+2G|M|sgn(r)$ and there is a discontinuity at $r=0:\ R(r=0)=0; R(r=0^+)=2GM$ (we shall omit the absolute symbol in M for simplicity), and our solutions can be described by focusing on the right and left regions (quadrants) of the Rindler-wedge formed by the straight (null) lines $U=\pm V$, corresponding to $r=0^+,\ t=\pm\infty$, and whose slope is $+45,-45$ degrees respectively. In the standard textbook solution, the Fronsdal-Kruskal-Szekeres change of coordinates [8] in the exterior region $R>2GM$ is given by

$$U=(\frac{R}{2GM}-1)^{\frac{1}{2}}\ e^{R/4GM}\ cosh\,(\frac{t}{4GM}),\quad V=(\frac{R}{2GM}-1)^{\frac{1}{2}}\ e^{R/4GM}\ sinh\,(\frac{t}{4GM});\ R>2GM \quad (B.1)$$

and the change of coordinates in the interior region $R<2GM$ is

$$U=(1-\frac{R}{2GM})^{\frac{1}{2}}\ e^{R/4GM}\ sinh\,(\frac{t}{4GM}),\quad V=(1-\frac{R}{2GM})^{\frac{1}{2}}\ e^{R/4GM}\ cosh\,(\frac{t}{4GM});\ R<2GM \quad (B.2)$$

In the overlap $R=2GM$, one has that $U=\pm V$ and $t=\pm\infty$; and $U=V=0$ for $finite$ t. The coordinate transformations lead to a well behaved metric (except at $R(r=0)=0$)

$$ds^2\ =\ \frac{4(2GM)^3}{R(U,V)}\ e^{-R(U,V)/2GM}\ (dV^2\ -dU^2\ )\ -R(U,V)^2(d\Omega)^2. \quad (B.3)$$

the functional form $R(U,V)$ is defined implicitly by the equation

$$U^2\ -\ V^2\ =\ (\frac{R}{2GM}-1)\ e^{R/2GM}\ \Rightarrow\ \frac{R}{2GM}\ =\ 1\ +\ W(\frac{U^2-V^2}{e}) \quad (B.4)$$

where W is the Lambert function defined implicitly by $z=W(z)e^{W(z)}$. When $R=2GM$ and $d\Omega=0$, the above interval displacement $ds^2=0$ is *null* along the lines $U=\pm V \Rightarrow dU=\pm dV$. It is singular at $R(r=0)=0$ along the (spacelike) lines $V^2-U^2=1\Rightarrow dV\neq\pm dU$.

However in the case of our solutions (1.1) one will still retain the Kruskal-Szekeres change of coordinates in the region $R\geq 2GM$, but one must *replace*, instead, the change of coordinates in the *interior* region $R<2GM$ in eqs-(B.2) for the following one

$$V\ =\ (\frac{R}{2GM})^{\frac{1}{2}}\ cosh(\frac{t}{4GM});\quad U\ =\ (\frac{R}{2GM})^{\frac{1}{2}}\ sinh(\frac{t}{4GM});\quad R<2GM \quad (B.5)$$

leading to $V^2 - U^2 = \frac{R}{2GM}$ and $\frac{U}{V} = tanh(t/4GM)$. In doing so one has that the points $R(r=0) = 0$ and $t = \pm\infty$ are mapped to the straight lines $U = \pm V$ with a ± 45 degree slope, respectively. While $R(r=0) = 0$ is mapped to the origin of coordinates $U = V = 0$ for arbitrary but $finite$ values of t. In this fashion there is geodesic completeness and there are no disconnected points along the geodesics. The incoming radial null geodesics (and future-oriented time like geodesics) all end up in the *null* singularity described now by the straight line $U = V$, instead of the (spacelike) hyperbola $V^2 - U^2 = 1$, and without "*tunneling*" through the interior region $R < 2GM$.

To show that now one has a *null* singularity at $U = \pm V$ one inserts the above change of coordinates (B.5) for the region $R < 2GM$ into the metric (1.1), such that it leads to a $different$ expression for the metric than in eq-(B.3) and given by

$$ds^2 \;=\; g_{UU}\; dU^2 \;+\; g_{VV}\; dV^2 \;+\; 2\; g_{UV}\; dU\; dV \;+\; R^2(U,V)\; d\Omega^2, \quad R < 2GM \qquad (B.6)$$

where

$$g_{UU} \;=\; (1 - \frac{1}{V^2 - U^2})\; (\frac{4GMV}{V^2 - U^2})^2 \;-\; (1 - \frac{1}{V^2 - U^2})^{-1}\; (4GMU)^2 \qquad (B.7a)$$

$$g_{VV} \;=\; (1 - \frac{1}{V^2 - U^2})\; (\frac{4GMU}{V^2 - U^2})^2 \;-\; (1 - \frac{1}{V^2 - U^2})^{-1}\; (4GMV)^2 \qquad (B.7b)$$

$$g_{UV} \;=\; g_{VU} \;=\; -\; (1 - \frac{1}{V^2 - U^2})\; (\frac{4GMV}{V^2 - U^2})\; (\frac{4GMU}{V^2 - U^2}) \;+ (4GM)^2\; (1 - \frac{1}{V^2 - U^2})^{-1}\; U\; V \qquad (B.7c)$$

Despite the different expression for the metric components in eqs-(B.7) from those in eq-(B.3) , one still has a *null* interval displacement $ds^2 = 0$ along the lines $U = \pm V$, and which correspond to the values $R(r=0) = 0$ and $t = \pm\infty$, respectively. Therefore, one has now a *null singularity* along the lines $U = \pm V$ instead of a spacelike singularity along the hyperbola $V^2 - U^2 = 1$. One can verify explicitly that when $U = \pm V, dU = \pm dV$ there is an *exact* cancellation of the singular terms

$$2\; \frac{(4GM)^2 UV}{(V^2 - U^2)^3}\; dU dV \;-\; \frac{(4GM)^2 U^2}{(V^2 - U^2)^3}\; dV^2 \;-\; \frac{(4GM)^2 V^2}{(V^2 - U^2)^3}\; dU^2 \qquad (B.8a)$$

and

$$-2\; \frac{(4GM)^2 UV}{(V^2 - U^2)^2}\; dU dV \;+\; \frac{(4GM)^2 U^2}{(V^2 - U^2)^2}\; dV^2 \;+\; \frac{(4GM)^2 V^2}{(V^2 - U^2)^2}\; dU^2 \qquad (B.8b)$$

in the above infinitesimal interval ds^2 of eqs-(B.7, B.8). Whereas there is also an *exact* cancellation of the non-singular terms when $U = \pm V, dU = \pm dV$. Since $R(r=0) = 0$, one obtains a net zero value for the displacement $ds^2 = 0$ in eq-(B.6) furnishing then a *null* interval. Because the curvature-squared Kretschmann invariant blows up $\mathcal{R}_{\mu\nu\rho\tau}\mathcal{R}^{\mu\nu\rho\tau} \sim$

$(2GM)^2/R(r)^6 \to \infty$ when $R(r) = 0$ at $r = 0$, one has then a *null* singularity at $r = 0$, as opposed to a *spacelike* singularity in the traditional solutions.

In the r, t coordinate picture when one evaluates $g_{tt}[R(r = 0)](dt)^2$ along the *constant* $t = \pm\infty$ lines, whose $dt = 0$, it yields an undetermined product of the form $\infty \times 0$ because $dt = 0$. This undetermined product is resolved when one writes the interval ds^2 in the form provided by eqs-(B.7) leading to a *null* result as we have shown. Future and past infinity $t = \pm\infty(U = \pm V)$ are well defined and the metric (1.1) has a proper causal structure.

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