The Problem of Points on a Parabola

Edigles Guedes

e-mail: edigles.guedes@gmail.com

March 14, 2013

ABSTRACT

By means of geometrical problem of how many points can you find on the (half) parabola, such that the distance between any pair of them is rational, we construct some parametric equations.

1. INTRODUCTION

We explore the problem, see [1]: How many points can you find on the (half) parabola $y = x^2$, for $x > 0$, such that the distance between any pair of them is rational? This is a geometrical problem, which requires for its solution a focus on number theory. That is, to determine whether the distance (r, v) and (s, t) is rational, we need to know when $(r - s)^2 + (v - t)^2$ is the square of a rational number. Thereof, it follows that our goal is to find a parametric equation for the variables r, s, t, v, X and Y from equation:

(1) $$(r - s)^2 + (v - t)^2 = \left(\frac{X}{Y}\right)^2 \Leftrightarrow X^2 = [Y(r - s)]^2 + [Y(v - t)]^2.$$

2. THEOREM AND COROLLARY

THEOREM 1. *Any integers r, s, t, v, X and Y of form*

$$X^2 = (u_1^2 + u_2^2 + u_3^2 + u_4^2)(w_1^2 + w_2^2 + w_3^2 + w_4^2),$$

$$Yr = u_1w_1 + u_1w_4 + u_2w_3 + u_4w_1,$$

$$Ys = u_2w_2 + u_3w_2 + u_3w_3 + u_4w_4,$$

$$Yv = u_1w_2 + u_2w_1 + u_2w_4 + u_3w_4,$$

$$Yt = u_1w_3 + u_3w_1 + u_4w_2 + u_4w_3,$$

satisfy the equation

$$(r - s)^2 + (v - t)^2 = \left(\frac{X}{Y}\right)^2.$$

Proof. The Brahmagupta-Fibonacci identity [2] asseverate that

(2) $$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2.$$

Comparing (1) with (2), we have

(3) $$X^2 = (a^2 + b^2)(c^2 + d^2) = (ac)^2 + (ad)^2 + (bc)^2 + (bd)^2,$$

(4) $$[Y(r-s)]^2=(ac+bd)^2 \Leftrightarrow Yr-Ys=ac+bd,$$

(5) $$[Y(v-t)]^2=(ad-bc)^2 \Leftrightarrow Yv-Yt=ad-bc.$$

On the other hand, Euler's four-square identity [3] says that

(6) $$(u_1^2+u_2^2+u_3^2+u_4^2)(w_1^2+w_2^2+w_3^2+w_4^2)=$$

$$=(u_1w_1-u_2w_2-u_3w_3-u_4w_4)^2+(u_1w_2+u_2w_1+u_3w_4-u_4w_3)^2$$

$$+(u_1w_3-u_2w_4+u_3w_1+u_4w_2)^2+(u_1w_4+u_2w_3-u_3w_2+u_4w_1)^2.$$

Comparing (3) with (6), we find

(7) $$X^2=(u_1^2+u_2^2+u_3^2+u_4^2)(w_1^2+w_2^2+w_3^2+w_4^2),$$

(8) $$ac=u_1w_1-u_2w_2-u_3w_3-u_4w_4,$$

(9) $$ad=u_1w_2+u_2w_1+u_3w_4-u_4w_3,$$

(10) $$bc=u_1w_3-u_2w_4+u_3w_1+u_4w_2,$$

(11) $$bd=u_1w_4+u_2w_3-u_3w_2+u_4w_1.$$

From (4), (5), (8), (9), (10) and (11), we encounter

(12) $$Yr-Ys=u_1w_1+u_1w_4+u_2w_3+u_4w_1-u_2w_2-u_3w_2-u_3w_3-u_4w_4,$$

(13) $$Yv-Yt=u_1w_2+u_2w_1+u_2w_4+u_3w_4-u_1w_3-u_3w_1-u_4w_2-u_4w_3,$$

whence, we conclude that

(14) $$Yr=u_1w_1+u_1w_4+u_2w_3+u_4w_1,$$

(15) $$Ys=u_2w_2+u_3w_2+u_3w_3+u_4w_4,$$

(16) $$Yv=u_1w_2+u_2w_1+u_2w_4+u_3w_4,$$

(17) $$Yt=u_1w_3+u_3w_1+u_4w_2+u_4w_3.\ \square$$

COROLLARY 1. *Any integers k,l,m,n and p of form*

$$X=4p^2(k^2+l^2+m^2+n^2),$$

$$r=k(3l+2m-n)+l(l-m+2n)+n(m+n),$$

$$s=k(l+2m+n)-l(l-m-2n)+n(3m-n),$$

$$v=k(k-l+2m+n)+2l^2+m(3l+m-n),$$

$$t=k(k+l-2m+3n)+m(l+m+n)+2n^2$$

and

$$Y=2p^2,$$

satisfy the equation

$$(r-s)^2+(v-t)^2=\left(\frac{X}{Y}\right)^2.$$

Proof. We assume that

(18) $$u_1^2+u_2^2+u_3^2+u_4^2=w_1^2+w_2^2+w_3^2+w_4^2.$$

On the other hand, in [4], we knew the J. Zehfuss identity:

(19) $$(2a)^2+(2b)^2+(2c)^2+(2d)^2=$$

$$=(-a+b+c+d)^2+(a-b+c+d)^2+(a+b-c+d)^2+(a+b+c-d)^2,$$

thence, it follows that

(20) $$X=(2a)^2+(2b)^2+(2c)^2+(2d)^2,$$

and

(21) $$u_1=2a, u_2=2b, u_3=2c, u_4=2d, w_1=-a+b+c+d, w_2=a-b+c+d,$$

$$w_3=a+b-c+d, w_4=a+b+c-d.$$

Substituting (21) in (14), (15), (16) and (17), we obtain

(22) $$Yr=2a(-a+b+c+d)+2a(a+b+c-d)+2b(a+b-c+d)+2d(-a+b+c+d),$$

(23) $$Ys=2b(a-b+c+d)+2c(a-b+c+d)+2c(a+b-c+d)+2d(a+b+c-d),$$

(24) $$Yv=2a(a-b+c+d)+2b(-a+b+c+d)+2b(a+b+c-d)+2c(a+b+c-d),$$

(25) $$Yt=2a(a+b-c+d)+2c(-a+b+c+d)+2d(a-b+c+d)+2d(a+b-c+d).$$

Simplifying (22), (23), (24) and (25), we find

(26) $$Yr=2[a(3b+2c-d)+b(b-c+2d)+d(c+d)],$$

(27) $$Ys=2[a(b+2c+d)-b(b-c-2d)+d(3c-d)],$$

(28) $$Yv=2[a(a-b+2c+d)+2b^2+c(3b+c-d)],$$

(29) $$Yt=2[a(a+b-2c+3d)+c(b+c+d)+2d^2].$$

Let $a=kp, b=lp, c=mp, d=np$ in (20), (26), (27), (28) and (29)

(30) $$X=(2kp)^2+(2lp)^2+(2mp)^2+(2np)^2,$$

(31) $$Yr=2[kp(3lp+2mp-np)+lp(lp-mp+2np)+np(mp+np)],$$

(32) $$Ys = 2[kp(lp + 2mp + np) - lp(lp - mp - 2np) + np(3mp - np)],$$

(33) $$Yv = 2[kp(kp - lp + 2mp + np) + 2l^2p^2 + mp(3lp + mp - np)],$$

(34) $$Yt = 2[kp(kp + lp - 2mp + 3np) + mp(lp + mp + np) + 2n^2p^2].$$

Again, simplifying (30), (31), (32), (33) and (34), we have

$$X = 4p^2(k^2 + l^2 + m^2 + n^2),$$

$$Yr = 2p^2[k(3l + 2m - n) + l(l - m + 2n) + n(m + n)],$$

$$Ys = 2p^2[k(l + 2m + n) - l(l - m - 2n) + n(3m - n)],$$

$$Yv = 2p^2[k(k - l + 2m + n) + 2l^2 + m(3l + m - n)],$$

$$Yt = 2p^2[k(k + l - 2m + 3n) + m(l + m + n) + 2n^2].$$

We set $Y = 2p^2$ and complete the proof. □

REFERENCES

[1] Geometry/Number Theory Open Problems, http://dimacs.rutgers.edu/~hochberg/undopen/geomnum/geomnum.html.

[2] http://en.wikipedia.org/wiki/Brahmagupta%E2%80%93Fibonacci_identity.

[3] http://en.wikipedia.org/wiki/Euler%27s_four-square_identity.

[4] A Collection of Algebraic Identities, https://sites.google.com/site/tpiezas/005.