Notes on Real Numbers

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This is first part of eight parts of lecture notes on Real Analysis. This notes is well designed and useful to all Undergraduate, Graduate and postgraduate in their regular study. Apart from this, the problems discussed in exercise will increase the readability of readers and they love Number Theory as well as analysis without any doubts. Also, some problems presented in the exercises of this part as well as coming parts will create motivation towards research and development.

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1. Introduction

This chapter concerns what can be thought of as the rules of the game: the axioms of the real numbers. These axioms of the real numbers and, in sense, any set satisfying them is uniquely determined to be the real numbers. This is simple enough to do. However, some basic consequences of the axioms should also be presented so that you know how some rules you have been taught, which are not axioms, follow from the axioms. For instance, ‘a minus times a minus is a plus’, ‘zero times any number is zero itself’, ‘zero is the illegal divisor’, ‘1 > 0’ are not rules and formulas to be committed to memory for future use; they all follow from the axioms.

We assume that the reader is familiar with the real numbers. We shall select those properties as axioms concerning the real number system from which all the other properties of the real numbers can be verified. These axioms are divided into three categories:

(1) Field axioms (2) Order axioms and (3) Completeness axiom

2. Field axioms

The real number system (reals) is first of all a set {a, b, c,…} on which the operations of addition and multiplication are defined so that each pair of real numbers producesa unique sum and product with the following algebraic properties.

Axiom 2.1 (Closure): For any a, b $\in$**R,**

$a + b \in \mathbf{R}$,

$ab \in \mathbf{R}$.

Axiom 2.2 (Commutative): For any a, b $\in$**R,**

$a + b = b + a \in \mathbf{R}$,

$ab = ba \in \mathbf{R}$.

Axiom 2.3 (Associative): For any a, b, c $\in$**R,**

$(a + b) + c = a + (b + c) \in \mathbf{R}$,

$(ab)c = a(bc) \in \mathbf{R}$.

Axiom 2.4 (Distributive): For any a, b, c $\in$ **R,**

$(a + b)c = ac + bc \in \mathbf{R}$.

Axiom 2.5 (Identity): For any a $\in$ **R,** there exists 0, 1 $\in$ **R** such that

$a + 0 = 0 + a = a$ Additive Identity

$a1 = 1a = a$ Multiplicative Identity

Axiom 2.6 (Inverse): For any a $\in$ **R,** there exists b, c $\in$ **R** such that

$a + b = b + a = 0$ Additive Inverse

$ac = ca = 1$ Multiplicative Inverse

Axiom 2.7 (Nontrivial Field): $0 \neq 1$

and for any a $\in$ **R**, such that $a \neq 0$, there exists a real number b such that $ab = 1 = ba$.

Although these axioms seem to contain most properties of the real numbers we normally use, they don' characterize the real numbers; they just give the rules for arithmetic. There are other fields besides the real numbers and can be found in abstract algebra courses.

Example 2.8: As we know that set of rational numbers $Q = \{p/q : p \in Z \wedge q \in N\}$ form a field, implies that Q does not contain all the real numbers as $\sqrt{2} \notin Q$.

Example 2.9: Let the field $F = \{0,1,2\}$ with addition and multiplication can be done by modulo 3. It is easy to verify that the field axioms are satisfied, and it is denoted by Z_3.

Theorem 2.10 (Uniqueness of Identity): The additive and multiplicative identities of a field **F** are unique.

Proof: Let $\bar{0}$ is another additive identity. Then, $0 = 0 + \bar{0}$

$= \bar{0}$ by identity axiom on $\bar{0}$

Similarly, $0 = 0\bar{0}$

$= \bar{0}$ by multiplicative identity on $\bar{0}$.

Theorem 2.11 (Uniqueness of Inverse): Let **F** be a field, if a, b $\in$ **F** with $b \neq 0$, then $-a$ and b^{-1} are unique.

Or

The additive and multiplicative inverses are unique.

Proof:Let a and b are additive inverses of c. Then, a = 0 + a by identity axiom

= (b + c) + a by inverse axiom

= b + (c + a) by associative axiom

= b + 0 by inverse axiom

= b by identity axiom

This shows the additive inverse s unique. The proof is essentially the same for the multiplicative inverse. Because of the uniqueness of inverse, we will denote –a as the additive inverse of a, and a^{-1} as the multiplicative inverse of a. This notation allows us to define subtraction and division as followed.

Definition 2.12 (Subtraction): The difference between two real numbers a and b is defined by $a+(-b)$, and it is denoted by a – b.

Definition 2.13 (Division): The quotient of a real number a by b (≠0) is defined by $a.b^{-1}$, and is denoted by $\frac{a}{b}$ or $a(b^{-1})$.

Remarks: (a) in general a – b ≠ b – a and $\frac{a}{b} \neq \frac{b}{a}$

(b) Division by 0 is not allowed.

(c) Though $\frac{a}{b}$ has meaning, $\frac{b}{a}$ may not be defined.

Theorem 2.14: For all a∈**R,** a0 = 0a = 0.

Proof: a + 0a = 1a + 0a by multiplicative identity

= (1 + 0) a by distributive axiom

= 1a by additive identity

= a by multiplicative identity

$\Rightarrow a+0a=a$

$= a+0a+(-a)=a+(-a)$...(by subtracting a on both sides)

$\Rightarrow 0a=0$

$= a0=0$...(by commutative axiom)

i.e., a0 = 0a = 0.

Theorem 2.15: For all a∈**R,** $-a=-1(a)$.

Proof: $a+(-1)a=(1)a+(-1)a$ by identity axiom

$= (1 - 1)\,a$ by distributive axiom

$= 0a$ by additive inverse

$= 0$ by the previous **theorem2.14**

$\Rightarrow a + (-1)a = 0$

$= a + (-1)a + (-a) = 0 + (-a)$...(by subtracting a on both sides)

i.e., $(-1)a = -a$.

Theorem 2.16: If $ab = 0$, then $a = 0$ or $b = 0$.

Proof: Let us prove $\Rightarrow$ by considering the cases $a \neq 0$, $b \neq 0$, or $a, b = 0$.

(a) Case $a \neq 0$

$(a.b = 0) \wedge (a \neq 0) \Rightarrow (a.b = 0) \wedge \{(\exists a^{-1})(a^{-1}a = 1)\}$

$\Rightarrow a^{-1}ab = a^{-1}0 \quad (\because ab = 0)$

$\Rightarrow (a^{-1}a)b = a^{-1}0$

$\Rightarrow 1b = a^{-1}0$

$\Rightarrow b = 0.$

(b) Case $b \neq 0$

$(b.a = 0) \wedge (b \neq 0) \Rightarrow (b.a = 0) \wedge \{(\exists b^{-1})(b^{-1}b = 1)\}$

$\Rightarrow ab^{-1}b = b^{-1}0 \quad (\because ab = 0)$

$\Rightarrow (b^{-1}b)a = b^{-1}0$

$\Rightarrow 1a = b^{-1}0$

$\Rightarrow a = 0.$

(c) Case $a, b = 0$

$\Rightarrow (a = 0) \vee (b = 0)$, independent of the fact that $ab = 0$.

Proof by contradiction: Let $a \neq 0$ and $b \neq 0$. Then, $ab = 0$

$\Rightarrow a^{-1}ab = a^{-1}0$

$\Rightarrow (a^{-1}a)b = 0$

$\Rightarrow 1b = 0$

$\Rightarrow b = 0$

Oops! We assume that $b \neq 0$. Thus, our assumption is wrong. Therefore, we can realize that, both a and b cannot be non-zero or at least one of them is zero.

Corollary 2.17: For any $a_1, a_2, \ldots, a_n \in \mathbf{R}$, $a_1a_2a_3\ldots a_n = 0 \Leftrightarrow (a_1 = 0) \vee (a_2 = 0) \vee \ldots \vee (a_n = 0)$.

Proof: $a_1a_2a_3\ldots a_n = 0 \Leftrightarrow a_1(a_2a_3\ldots a_n) = 0$

$$\Leftrightarrow (a_1 = 0) \vee (a_2a_3\ldots a_n) = 0$$

$$\Leftrightarrow (a_1 = 0) \vee (a_2 = 0) \vee (a_3\, a_4\ldots a_n = 0)$$

$$\vdots$$

$$\Leftrightarrow (a_1 = 0) \vee (a_2 = 0) \vee\ldots\vee (a_n = 0)$$

3. The Order Relation

The axiom of this part gives the order and metric properties of the real numbers. There is a subset P of **R**, called the set of positive numbers, satisfying the following.

(i) If a, b $\in$ P, the a + b and ab$\in$ P

(ii) For all a $\in$**R**, either a $\in$ P or a = 0 or –a $\in$ P **(TRICHOTOMY)**

Any field **R** satisfying the axioms so far listed is generally called an ordered field. The axioms (i) and (ii) indicates that 1 is a positive number. Indeed, since $1 \neq 0$, the axiom (i) indicates that either 1 or -1 is positive. As 1 = 1. 1 = (-1) (-1), the axiom (i) implies that 1 is positive. Also, by (ii), we see that **R** is divided into three pairwise disjoint sets. Namely, P, {0} and {-a, a $\in$ P}. The notation a < b means that b – a $\in$ P. More precisely, look the following odder properties of real numbers.

(a) For any real number a, b, exactly one of the following holds: a = b, a < b or a > b.

(b) For all real numbers a, b, c, if a < b and b < c $\Rightarrow$ a < c.

(c) For all real numbers a, b, c, if a < b $\Rightarrow$ a + c<b + c.

(d) For all real numbers a, b, c with c > 0, a < b $\Rightarrow$ ac <bc.

Theorem 3.1: If $a \le b$ and $b \le a$, then a = b.

Proof: If $a \le b \Rightarrow b - a \in P$ or a = b. Also, for $b \le a \Rightarrow a - b \in P$ or b = a.

But, we know that a – b = -1 (b –a) and by the Trichotomy axiom (see 3(ii)) only b – a or $-1(b-a)$ can be in P. Thus, b – a $\in$ P $\Rightarrow$ a – b $\in$ P. The only other situation a = b hold.

Theorem 3.2: If **R** is an ordered field and a, b, c $\in$**R**, then the following hold:

(1) $a < b \Leftrightarrow a + c < b + c$

(2) $(a < b) \wedge (b < c) \Rightarrow a < c$

(3) $(a < b) \wedge (c > 0) \Rightarrow ac < bc$

(4) $(a < b) \wedge (c < 0) \Rightarrow ac > bc$.

Proof: (1) if $a < b \Leftrightarrow b - a \in P \Leftrightarrow (b + c) - (a + c) \in P \Leftrightarrow a + c < b + c$

(2) Let us consider b – a and c – b are in P. As P is closed under addition, we see that $(b-a)+(c-b) = c-a \in P \Rightarrow c > a.$

(3) As $b - a \in P$ and $c \in P$ with P is closed under multiplication, $c(b-a) = cb - ca \in P$ $\Rightarrow ac < bc$.

(4) By considering b – a and –c are in P, then from (3), we can see ac >bc.

Theorem 3.3: If **F** be an ordered field and $a \in$**F**. If $a > 0$, then $a^{-1} > 0$.

Proof: We know that $a = 0 \Leftrightarrow a^2 = 0$ for $a^2 \geq 0$.

Therefore, $a^{-1}a^{-1} > 0 \Rightarrow aa^{-1}a^{-1} > 0$

$\Rightarrow (aa^{-1})a^{-1} > 0 \Rightarrow 1a^{-1} > 0 \Rightarrow a^{-1} > 0$.

Corollary 3.4: If $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$.

Proof: By the previous theorem, $a^{-1} > 0$ and $b^{-1} > 0$. For $a < b$: $\Rightarrow a(a^{-1}b^{-1}) < b(a^{-1}b^{-1})$

$\Rightarrow (aa^{-1})b^{-1} < a^{-1}(bb^{-1}) \Rightarrow 1b^{-1} < a^{-1}1 \Rightarrow \frac{1}{b} < \frac{1}{a}$.

3.1 Metric Properties

The order axiom on a field **F** allows us to introduce the idea of a distance between points in **F**. To study this, we begin with the following definition.

Definition 3.1.1: Let **F** be an ordered field. The absolute value function on **F** is a function $|.| : F \rightarrow F$ defined as $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$

The definition of $|x|$ shows that: $|-x| = |x| \geq 0$ for all x $\in$**F**.

It is also useful to observe that $|x|$ is the larger of x and –x. When we think of $|a-b|$ as measuring the difference between a and b, the needy property of the $|.|$ is contained in the following theorem.

Theorem 3.1.1(The Transitive Property): If a and b are any two real numbers then $|a+b| \leq |a| + |b|$.

Proof: To complete the proof, the following four properties are required.

(a) If a ≥ 0 and b ≥ 0 , then a+ b ≥ 0 $\Rightarrow |a+b| = a+b = |a|+|b|$.

(b) If a ≤ 0 and b ≤ 0, then a + b ≤ 0 $\Rightarrow |a+b| = -a+(-b) = |a|+|b|$.

(c) If a ≥ 0 and b ≤ 0, then a + b = |a| - |b|.

(d) If a ≤ 0 and b ≥ 0, then a + b = -|a|+ |b|.

Precisely, |a+ b| ≤ |a| + |b| hold as:

$$|a+b| = \begin{cases} |a| - |b| \text{ when} |a| \geq |b| \\ |b| - |a| \text{ when} |b| \geq |a| \end{cases}.$$

We are used to thinking of |a – b| as the distance between the numbers a and b. This notation of a distance between two points of a set can be generalized.

Definition 3.1.2: Let X be a set. A function $d : X \times X \to [0,\infty)$ is called a metric if

(a) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if x = y,

(b) $d(x, y) = d(y, x)$,

(c) $d(x, z) \leq d(x, y) + d(y, z)$ (the triangular inequality)

We call (X, d) a metric space.

A metric is a function, which defines the distance between any two points of a set.

Example 3.1.3:(a) Take **X = R** and define $d(x, y) = |x - y|$

(b) Take **X = R^2** and define $d((x_1, y_1),(x_2, y_2)) = | y_1 - x_1 | + | y_2 - x_2 |$.

(c) Take **X = R^2** and define $d((x_1, y_1),(x_2, y_2)) = \sqrt{| y_1 - x_1 |^2 + | y_2 - x_2 |^2}$.

(d) Any subset of **R** with the same metric.

4. The Completeness Axiom

All the axioms given so far are so common from pre-algebra, and, on the surface, it's not obvious they haven't captured all the properties of the real numbers. As **Q** satisfies all of them, the following theorem shows that we're not yet discussed.

Theorem 4.1: There is no Ω∈**Q** such that $\Omega^2 = 2$.

Proof: let us assume that, the contrary, there is Ω in **Q**$\Rightarrow \Omega^2 = 2$. Obviously, there exist p, q in **N** such that, $\Omega = \frac{p}{q}$ with (p, q) = 1 $\Rightarrow \left(\frac{p}{q}\right)^2 = 2 \Rightarrow p^2 = 2q^2$, which shows p^2 is even. As the square of an odd number is odd, p should be even or p = 2k for some k ∈ **N**.

i.e., $4k^2 = 2q^2 \Rightarrow 2k^2 = q^2$ (since $p^2 = 2q^2$ and p = 2k)

The some argument as above establishes that q is also even. This contradicts our assumption that p and q are relatively primes or (p, q) = 1.

Therefore, there is no such Ω exists.

Since we suspect √2 is a perfectly fine number, there's still something missing from our earlier discussion of axioms. The completeness axiom is somewhat more difficult than the previous axioms, and several definitions are required to complete it.

4.2 Bounded Sets

Definition 4.2.1: The subset $S \subset R$ is said to be bounded above if there is a real number $M \in R$ such that $x \le M$ for all $x \in S$ and M is called an upper bound of S. Note that, if M is an upper bound for S then any bigger number is also an upper bound. Not all sets have an upper bounded.

Example4.2.2: The set N is bounded below that it is not bounded above. Hence N is not bounded.

The set $S = \left\{\frac{1}{n} \mid n \in N\right.$ is bounded because every element of S is less than 1 and greater than 0.

$\Rightarrow S \subset [0,1]$.

Definition 4.2.3: The supremum (least upper bound) of a set $S \subseteq R$ which is bounded above is an upper bound. For $b \in R$ of S such that $b \le u$ for any upper bound u of S. We usually denoted by b = sup S for supremums. A number b is said to be the supremum (least upper bound) of the set S if:

(a) b is an upper bound: ∀x ∈ S satisfies x ≤ b, and

(b) b is the smallest upper bound.in other words, any smallest number is not an upper bound.

If u < b then there exist x ∈ S with u < x.

i.e., $b = \sup S = \underset{x \in S}{SUP} - x$, upper bounds of S may, or may not belong to S.

For example, (-2, 3) is bounded above by 100, 85, 5, 4, 3.55, 3. In fact 3 is its least upper bound. In the case of (-2, 3] also has 3 as its least upper bound. Note that, the supremum of S is a number that belongs to S then it is also called the maximum of S.

For example, the interval (-2, 3) has supremum equal to 3 and no maximum; (-2, 3] has supremum, and maximum, equal to 3.

4.2.4 Bounded sets do have a least upper bound.

This is a fundamental property of real numbers, as it allows us to discuss about limits. Before that, let us have a look at some following needy theorems on supremums.

Theorem 4.2.5: The set S has unique least upper bound (l.u.b).

Proof:let $S \subseteq R$ is bounded above and that a, b $\in$**R** are supremums of S. Note that, both a and b are upper bounds of S. As a is a least upper bound of S and b is an upper bound of S, a ≤ b. Similarly, b is a least upper bound and a is a upper bound of S, b ≤ a. Thus a = b, sowing that the supremum of a set S is unique.

Intuitively, we can state the definition of supremum is in another way that, no number smaller than the supremum can be upper-bound of the given set. For better understanding, look at the following:

Theorem 4.2.6: An upper bound b of a set $S \subseteq R$ is the supremum of S if and only if for any $\varepsilon > 0$ there exists s $\in$S such that $b-\varepsilon < S$.

Proof:Let us take the small piece of the theorem that, 'there exists s $\in$ S such that $b-\varepsilon < s'$ says that $b-\varepsilon$ is not an upper bound of S. Or there is some other upper bound u in **R** of S. For s $\in$ S, s ≤ u. clearly, $b-\varepsilon$ varies over all real numbers smaller than b as ε varies. Therefore, an upper bound b of S, b = sup S if and only if no number smaller than b is an upper bound of S.

Theorem 4.2.7: Any nonempty set of real numbers which is bounded above has a supremum.

Proof:The proof of existence of supremums that I saw relied on the completeness of the real numbers, and not much else. Basically, we constructed a Cauchy (and thus convergent) sequence of real numbers that was always an upper bound of the set, and got infinitely close to the set.

Here's an outline:

Let $U \subset R$ *be a bounded set of real numbers. Then there exists* $s \in R$ *such that for all* $u \in U, s \geq u$, *and for any* $\varepsilon > 0, s-\varepsilon$ *is not an upper bound of* U.

Let $M > 0$ be the upper bound for U (for example, if $U = (0,1)$, we could sa $M = 100000000($

and it would be fine). Take $T_1 := M$, and B_1 to be some number that is *not* an upper bound for U.

Now, we given T_i and B_i, we will define T_{i+1} and B_{i+1} as follows:

We take the midpoint of T_i and B_i, to be called m_i, and see if m_i is an upper bound for U. If it is, then we'll take $T_{i+1} := m_i$ and $B_{i+1} := B_{i.}$ Let us define a_i to be m_i. If m_i is not an upper bound for U, then we define $a_i := a_i - 1$, and $T_{i+1} := T_i$, and $B_{i+1} := B_i$.

The distance between T_i and B_i halves for each iteration, so since a_i, is contained in the interval, it is squeezed into convergence. Its limit a must be an upper bound, and the claim (I will leave for you to play with) is that a is the least upper bound of the set U.

Definition 4.2.8: The subset S ⊂ R is said to be bounded below if there is a real number m ∈ R such that m ≤ x for all x ∈ S, and m is called a lower bound of S. Note that, if the set S is bounded above as well as below, then S is said to be bounded. i.e. S is bounded if and only if $S \subset \{x \mid m \le x \le M = [m, M]$.

Definition 4.2.9: The infimum (the greatest lower bound) of the set if (a) b is a lower bound: any x ∈ S satisfies x ≥ b, and (b) b is the greatest lower bound. In other words, any greater number is not a lower bound.

If b < u then there is x ∈ S with x < u.

$\Rightarrow b = \inf S = \inf_{x \in S} x,$ Greatest lower bounds of S may not belong to S.

For example, (-2, 3) is bounded below by -100, -15,-4, -2. In fact, -2 is its infimum. In the interval [-2,3) also has -2 as its infimum. i.e., if the infimum of S belongs to S then it is said to be minimum of S.

Theorem 4.2.10: Every non-empty bounded subset of the real numbers has an infimum.

Proof: Let E is a non-empty set bounded below. Construct E_1 = -E = {-x | x ∈ E}. We expect that β = inf E = - sup E_1 = - Ω. Now we should establish the existence ofΩ.

To do so we show E_1 is bounded above. As E is bounded from below, there exists a k such that x ≥ k for all x ∈ E. Or –x ≤ -k, and every element of E_1 are bounded by –k. Clearly, E_1 is non-empty set and bounded above. By the Completeness axiom it has a supremum, Ω (say). Now we expect β = -Ω is an infimum for E. Let us verify that β is lower bound and it is the greatest lower bound. To do so, we see

(a) As Ω is an upper bound of $E_1 \Rightarrow -x \leq \Omega$ for all x in E. i.e., $x \geq -\Omega = \beta$ for all x in E, and β is a lower bound for E.

(b) In case $\beta < y \Rightarrow -\Omega < y$ or $-y < \Omega$. By infimum statement, there exists a $t \in E_1$ with $-y < t$ or $y > -t$. As -t in $E \Rightarrow$ y cannot be a lower bound. Thus β is the greatest lower bound.

Example 4.2.11: Show that S is bounded and find sup S and inf S, where $S = \left\{ \frac{2+n}{n}, n \in N. \right.$

We know that for any n $\in$N, $\frac{2+n}{n} = \frac{2}{n} + 1 > 1,$ and $\frac{2}{n} + 1 \leq 2 + 1 = 3.$

i.e., Max S = 3 as 3 = (2/1) + 1 $\in$ A for a least value of n = 1, as n $\in$ N.

Now we show that inf S = 1. We have from above, 1 is a lower bound for S. Let $\varepsilon > 0$ and consider $1 + \varepsilon$.

$\Rightarrow$ For $a \in S$: $a < 1 + \varepsilon$.

But a should be in $\frac{2+n}{n} = 1 + \frac{2}{n}$ for some n $\in$ N. So we have $1 + \frac{2}{n} < 1 + \varepsilon \Leftrightarrow \frac{2}{n} < \varepsilon \Leftrightarrow \frac{2}{q} < n.$

Now such existence, define $n = \left[\frac{2}{q} \right] + 1,$ where [x] is the greatest integer of x.

Thus, we proved that $1 + \varepsilon$ cannot be a lower bound for any $\varepsilon > 0$, inf S = 1.

4.2.12 Some Consequences of Completeness:

The property of completeness is what divides analytic from geometry and algebra. It required the use if approximation, infinity and more dynamic visualizations.

Theorem 4.2.13 (The Archimedean Property): Let *a* be any real number and *b* any positive real. Then there exist a positive integer n such that nb > a.

Proof: Let us consider the theorem is false and a is an upper bound of the set $S = \{x \mid x = nb, n \text{ is an intger}\} \Rightarrow S$ has a supremumβ (say), by Completeness property. Therefore, $nb \leq \beta$ for any integer n (*). As n + 1 is an integer, when n is (*)

$\Rightarrow (n + 1) b \leq \beta$

$\Rightarrow nb \leq \beta - b$ for some integer n.

Thus, $\beta - b$ is an upper bound of S. Since, $\beta - b < \beta$, our assumption is wrong.

Therefore, the above theorem is true.

Theorem 4.2.14: For any real number a, there exists an integer n such that a < n.

Proof: Let us consider the theorem is false and a > n for all integers n. i.e., the set of integers N is bounded above, and by the Completeness axiom it has a least upper bound M (say). As M is the least upper bound, M – 1 cannot be upper bound. Obviously, there exists an integer n such that M – 1 < n. i.e., M < n + 1 where n + 1 is an integer and greater than the upper bound, which is not possible. Thus, the theorem is true.

Theorem 4.2.15: For any real number b > 0, there exists an integer n such that $0 < \frac{1}{n} < b.$

Proof: By **theorem 4.2.14**, there is an integer n such that $0 < \frac{1}{b} < n.$ Also by **corollary 3.4**, we have $0 < \frac{1}{n} < b.$

Density of the Rational and Irrationals.

Definition 4.2.16: A set D is dense in the real's if every open interval (a, b) contains a member of D.

Theorem 4.2.17: The set of rational numbers are dense in the intervals OR if a and b are real's numbers with a < b, there is a rational number $\frac{p}{q}$ such that $\frac{p}{q} \in (a,b).$

Proof:Let a and b are any two distinct real numbers and let a < b ⇒ b – a > 0. By the Archimedean Property of real's, there exist a positive integer q such that

q(b – a) > 1 or qb > qa + 1….(*)

Also there exist a unique integer p such that

p – 1 ≤ qa < p

⇒qa + 1 ≥ p > qa….(**)

By combining (*) and (**), we get;

qb > qa + 1 ≥ p > qa

⇒ qa < p < qb

$\Rightarrow a < \frac{p}{q} < b$

Clearly, p, q are integers with q ≠ 0 $\Rightarrow \frac{p}{q}$ is a rational. Let us consider $\frac{p}{q} = k$ ⇒ a < k < b.

Thus, there exist a rational number k, in between a and b. Also, by repeating the above process for a and k, and k and b, we get new rationales k_1 and k_2 such that;

$a < k_1 < k$ and $k < k_2 < b$

$\Rightarrow a < k_1 < k < k_2 < b$

Again by continuing the same process, we find infinitely many rationales between any two distinct reals.

Remark: The rational number system is not complete (see **theorem 4.1**).

Theorem 4.2.18: The set of irrational numbers is dense in the reals.

Proof: By **theorem 4.2.17**, there are rational numbers k_1 and k_2 such that $a < k_1 < k_2 < b$.

Let $p = k_1 + \frac{1}{\sqrt{2}}(k_2 - k_1)$

$\Rightarrow$p is irrational and $k_1 < q < k_2$

$\Rightarrow a < p < b$ as $a < k_1 < k_2 < b$.

Theorem 4.2.19: Between any two distinct real numbers, there exist an infinite number of reals.

Proof: The above **theorems 4.2.17** and **4.2.18** will complete the proof.

4.4.20 The extended real number system

It is often convenient to extend the system of the real numbers by the addition of two elements ∞ and $-\infty$. The arithmetic relationships among ∞, $-\infty$, and the real numbers are defined as follows:

$\mathbf{R}^* = \mathbf{R} \cup \{-\infty, \infty\}$

Where ∞ and $-\infty$ are largest and smallest element of the real line **R.** We extend the order relation to **R** by $-\infty < x < \infty$ for all $x \in \mathbf{R}$.

Also we define addition on $\mathbf{R}^*$ as

$x + \infty = \infty = \infty + x$ for all $x \in \mathbf{R}^*$ with $x > -\infty$

$y + (-\infty) = -\infty = (-\infty) + y$ for all $y \in \mathbf{R}^*$ with $y < \infty$.

Also for $x > 0$, $x\infty = \infty x = \infty$,

$x(-\infty) = (-\infty)x = -\infty$.

for $x < 0$, $x\infty = \infty x = -\infty$,

$x(-\infty) = (-\infty)x = \infty$.

We also define $\infty + \infty = \infty\infty = (-\infty)(-\infty) = \infty$

$-\infty - \infty = \infty(-\infty) = (-\infty)\infty = -\infty$

$|-\infty| = |\infty| = \infty$

Note that, $\infty + (-\infty)$, $0.\infty$, $\infty.0$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are not defined.

Remark: In general, one may get doubt on ∞ (number/ quantity or not). (#)

Infinity can be a number if you want it to be, as Mathematicians can define any sort of number system. What is important is if it's useful and interesting. We see that infinity is not considered a number in the set of real numbers**(R)**. However, in Calculus and other subjects, it helps to informally (sometimes formally) consider infinity a number with a special properties in order to evaluate limits. This number system is called ***the extendedreal number system***. To say yes or no of our (#),and it is **not**. The answer depends entirely on what we are working with. In some number systems, infinity is defined. In other number systems, infinity is not defined. Regarding, modulus symbol, usually, when we are working with the extended real number system, we take continues that are defined on the reals (or some subset) and extended them continuity to have +∞ or -∞ whenever possible. In the case of absolute values, we define $|+\infty|=+\infty$ and $|-\infty|=+\infty$ for this reason:i.e. because

$$\underset{x\to+\infty}{Lt} |x| = +\infty \text{ and } \underset{x\to-\infty}{Lt} |x| = +\infty$$

Whether you want to call +∞ and -∞ numbers or not is not really relevant to this situation. I believe that it is very useful to consider them numbers... but maybe that should only be done once you are comfortable doing arithmetic with them.

Exercise 1.1

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1. Show that the set $S = \{x: x^2 < 1 - x\}$ is bounded above. Also find the least upper bound of S.
2. Show that $\sup\left\{1-\frac{1}{n} \mid n \in N\right\} = 1.$
3. Justify: $a > b \Rightarrow (\exists \varepsilon > 0 : a \geq b + \varepsilon).$
4. Find the least upper bound of $S = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}$ and justify your answer.
5. Find lower and upper bounds of y = f(x), where $f(x) = -x^4 + 2x^2 + x$ and $x \in [-1, 1.5]$.
6. Prove that **Z** is unbound both above and below.

7. Estimate the size of $f(x) = x^2 - 1$ in $x \in (0, 2)$.
8. Prove that $\sqrt{p}$ is irrational, where p is prime.
9. For $S \subseteq R$ be a bounded set with $-S = \{-x : x \in S\}$, prove that inf S = - sup (-S).
10. For $S \subseteq R, T \subseteq R, S \neq \phi, T \neq \phi$ with $S \subset T$, then prove that inf T ≤infS≤ sup S ≤ sup T.
11. Prove that **Q** is not complete.
12. Given x ∈**R**, prove that there is a unique (∃!) n ∈**Z** such that x ∈ [n -1, n)
13. Find the max, min, sup and inf, and justify your answer by proof for $S = \left\{ \frac{2n+1}{n+1} \middle| n \in N \right\}$.
14. Find the bound of $f(x) = \frac{x^3 - 3x + 1}{x^2 + 1}$ for $x \in [-2, 2]$.
15. Find the max, min, sup and inf of $S = \left\{ 1 + \frac{1 + (-1)^n}{n} \middle| n \in N \right\}$.
16. Establish formulas for sup K and inf K in terms of Sup S and inf S for $K = \{al + b | l \in N\}$, where a and b are fixed real numbers with S is bounded non-empty set.
17. Prove that, the infimum of a set, if it exits, is unique.
18. If M and N are non-empty subsets of **R**, then prove that Sup (M – N) =Sup (M) – Inf (N).
19. Let S be an ordered field and x, y ∈ S. Then prove the following:
    a) $|x| \geq 0$ and $|x| = 0 \Leftrightarrow x = 0$.
    b) $|x| = |-x|$
    c) $-|x| \leq x \leq |x|$
    d) $|x| \leq y \Leftrightarrow -y \leq x \leq y$
20. Prove that for any x ∈**R**, x < x + 1 and for x ≠ 0, x.x> 0.
21. Prove that for any real numbers a and b, $ab \leq \left(\frac{a+b}{2} \right)^2$.
22. Let a ∈**Q** with a ≠ 0, and b is some irrational. Prove that ab also an irrational.
23. Let S = {.1, .12, .123, .1234, … , .12345678910, …}, then find g.l.b.(S) and how would you write l.u.b.(S)?

24. Prove that $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc.$

25. Prove that a^{-1} is positive, when a is positive.

26. If $a_1, a_2, \ldots, a_n$ are real numbers, then prove that

    (i) $|a_1 + a_2 + \ldots + a_n| \le |a_1| + |a_2| + \ldots + |a_n|$.

    (ii) $|a_1.a_2 \ldots a_n| = |a_1|. |a_2| \ldots |a_n|$.

27. Let f, g be real-valued functions defined on non-empty set D, and such that $R_f = f(D)$ and $R_g = g(D)$ are bounded subsets of R. Then, Prove that

    $\sup\{f(x)+g(x): x \in D\} \le \sup\{f(x): x \in D\} + \sup\{g(x): x \in D\}$,

    $\Rightarrow \sup(f+g)(D) \le \sup f(D) + \sup g(D)$. Note that, R_f means *Range of f.*

28. Prove that the set of negative real numbers is not bounded below.

29. Show that, for a and b are fixed real number, $\frac{|a+b|}{1+|a+b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$ is true.

30. Justify the statement " $a > b \Rightarrow (\exists \varepsilon > 0: a \ge b + \varepsilon)$"

31. If $S \subset \mathbf{R}$ has maximum with a supremum. Prove that, Sup S = Max S.

32. Find the least and greatest upper bound of $S = \left\{ \frac{n-1}{n+1} \cos \frac{2n\pi}{3} \middle| n \in N \right\}$.

33. If x and y are real numbers, then show that max(x, y) $= \frac{x+y+|x-y|}{2}$ and min(x, y) $= \frac{x+y-|x-y|}{2}$.

5. Mathematical Induction

Aresemblance of the principal of mathematical induction is the game of dominoes. Let the dominoes are lined up properly, so that when one falls, the next one will also fall and so on. Therefore, the basic principal of mathematical induction is as follows. To prove that a statement holds for all positive integers n, we first verify that it holds for n = 1, and then we prove that if it holds for k (a certain natural number), and then it holds for k + 1.

Theorem 5.1: Let $S_1, S_2,\ldots,S_n, \ldots$ be propositions, one of for each positive integer, such that;

(i) S_1 is true

(ii) $S_n \to S_{n+1}$, for each positive integer n, then $S_n$is true for all positive integers of n.

Proof: Let $K = \{n|n\in N$ and S_n is true$\}$

From (i), $1 \in K$ and from (ii), $n + 1 \in K$ whenever $n \in K \Rightarrow K = N$ □

In general, we use to hear the weak and strong induction. What is the real meaning of the weak and strong? Where can we apply?...

The answer is simply that we use one that works. we don't choose ahead of time which form to use; we use the one that gives you the strength of hypothesis needed to make our proof work.

In general, the hypothesis $P(n)$ simply isn't strong enough to let us derive $P(n+1)$, but we *can* derive $P(n+1)$ if we assume $P(n)$ and $P(n-1)$. Sometimes we have to assume $P(k)$ for **all** $k$such that $n_0 \le k \le n$ in order to be able to infer $P(n+1)$. (Here n_0 is the initial value for the induction.) In practice you might as well simply assume that $P(k)$ holds for $k=n_0,\ldots,n$ when trying to prove $P(n+1)$; if it turns out that you don't actually need that strong a hypothesis, no harm has been done. In other words, when attacking a new proof, always remember that you can use the full strength of strong induction, though in many cases you won't need to do so.

It's unfortunate that so-called strong and weak induction are so often taught as different things, when in fact they are just very slightly different special cases of a considerably more general concept that covers transfinite induction and structural induction as well. Roughly speaking, it's a method that applies whenever the setting is such that it's meaningful to talk about a *minimal counterexample* to the theorem that you're trying to prove. In the case of induction over the integers, a minimal counterexample is simply the smallest n for which $P(n)$ is false. You can think of a proof by induction as a proof that no such minimal counterexample can exist. You suppose that n is a minimal counter example and you getting a contradiction. Sometimes the contradiction can be obtained just from the hypothesis that $P(n-1)$ is true; sometimes you find that you need a bit more – the truth of both $P(n-1)$ and $P(n-2)$, for instance, or even of all $P(k)$ for $n_0 \le k < n$. Since you're assuming that n is a minimal counterexample, however, you **are** assuming that $P(k)$is true for $n_0 \le k < n$, so you can use as much of that assumption as you need in order to get your contradiction.

Example 5.2: Prove that 1 + 2 + 3 +…+ n = $\frac{n(n+1)}{2}$…(*) for any integer n ≥ 1.

For n = 1 (*) is true, since 1 = $\frac{1(1+1)}{2}$.

Let us assume that, * is true for n = k ≥ 1, that is 1 + 2 + 3 +…+ k = $\frac{k(k+1)}{2}$ …(**)

Prove that * is true for n = k +1, that is

$$1 + 2 + 3 + \ldots + k + (k + 1) \stackrel{?}{=} \frac{k(k+1)}{2} + (k+1)$$

As we have, $1 + 2 + 3 + \ldots + k + (k + 1) \stackrel{**}{=} \frac{k(k+1)}{2} + (k+1)$

$$\Rightarrow (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2}.$$

Example 5.2: Prove that 1 + 3 + 5 +…+ (2n - 1) = n^2…(*) for any integer n ≥ 1.

For n = 1 (*) is true, since 1 = 1^2.

Let us assume that, * is true for n = k ≥ 1, that is 1 + 3 + 5 +…+ (2k - 1) = k^2…(**)

Prove that * is true for n = k +1, that is

$$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) \stackrel{?}{=} (k+1)^2.$$

As we have, $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) \stackrel{**}{=} k^2 + (2k+1) = (k+1)^2$.

The following examples deal with problems for which induction is a natural and efficient method of solution.

Example 5.3: Let $a_n = \begin{cases} 1 \text{ for } n = 1 \,\&\, 2 \\ a_{n-2} + a_{n-1} \text{ for } n \ge 2, \text{ where} \end{cases}$

a_n is the formula for n[th] term of the Fibonacci sequence. Prove that by mathematical induction,

$$a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5}.2^n}.$$

For n = 1, a_1 = 1 = $\frac{(1+\sqrt{5}) - (1-\sqrt{5})}{\sqrt{5}.2}$ is true.

For n = 2, a_2 = 1 = $\frac{(1+\sqrt{5})^2-(1-\sqrt{5})^2}{\sqrt{5}.2^2}$ is also true.

Let us assume the truth of the statement for some n – 1 and n, that is

$$a_{n-1}=\frac{(1+\sqrt{5})^{n-1}-(1-\sqrt{5})^{n-1}}{\sqrt{5}.2^{n-1}}\ldots(*)$$

and, $a_n=\frac{(1+\sqrt{5})^{n}-(1-\sqrt{5})^{n}}{\sqrt{5}.2^{n}}\ldots(**)$

By adding (*) and (**), we get;

$$\Rightarrow a_{n+1}=a_{n-1}+a_n$$

$$=\frac{(1+\sqrt{5})^{n-1}-(1-\sqrt{5})^{n-1}}{\sqrt{5}.2^{n-1}}+\frac{(1+\sqrt{5})^{n}-(1-\sqrt{5})^{n}}{\sqrt{5}.2^{n}}$$

$$=\frac{4(1+\sqrt{5})^{n-1}-4(1-\sqrt{5})^{n-1}+2(1+\sqrt{5})^{n}-2(1-\sqrt{5})^{n}}{\sqrt{5}2^{n+1}}$$

$$=\frac{(1+\sqrt{5})^{n-1}[4+2((1+\sqrt{5})]-(1-\sqrt{5})^{n-1}[4+2((1-\sqrt{5})]}{\sqrt{5}.2^{n+1}}$$

$$=\frac{(1+\sqrt{5})^{n-1}[6+2\sqrt{5}]-(1-\sqrt{5})^{n-1}[6-2\sqrt{5}]}{\sqrt{5}.2^{n+1}}$$

$$=\frac{(1+\sqrt{5})^{n-1}[1+\sqrt{5}]^2-(1-\sqrt{5})^{n-1}[1-\sqrt{5}]^2}{\sqrt{5}.2^{n+1}}$$

$$=\frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{\sqrt{5}.2^{n+1}}, \text{which is true for } n+1.$$

i.e., the statement is true for n = 1 and n = 2 and its truth for n – 1 and n implies its truth for n + 1.

Example 5.4: Prove that $n! \le n^n$… (*) for any integer n ≥ 1

For n = 1 (*) is true as $1! = 1^1$

Let (*) is true for some n = k ≥ 1, that is $k! \le k^k$ …(**)

Prove that (*) is true for n = k + 1, that is $(k + 1)! \overset{?}{\le} (k+1)^{k+1}$.

We have $(k + 1)! = k!\,(k + 1) \overset{**}{\le} k^k(k+1) < (k+1)^k(k+1) = (k+1)^{k+1}$.

Example 5.5: Prove that $7|n^7 - n$ … (*) for any integer n ≥ 1.

For n = 1 (*) is true, since $7|1^7 - 1$

Let (*) is true for some n = k ≥ 1, which is $7|k^7 - k$… (**)

Prove that (*) is true for n = k + 1, that is $7|(k+1)^7 - (k+1)$.

We have $(k+1)^7 - (k+1) = k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1 - k - 1$

$= (k^7 - k) + (7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k)$

$= (k^7 - k) + 7(k^6 + 3k^5 + 5k^4 + 5k^3 + 3k^2 + k)$

= divisibly by 7. $(\because by - (**))$

Example 5.6: Prove that $n^3 - n$ … (*) is divisible by 3 $\forall n \in N$.

For n = 1 (*) is true, since $3|1^3 - 1$

Let (*) is true for some n, that is $3|n^3 - n$…(**)

$\Rightarrow (n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 = (n^3 - n) + 3(n^2 + n)$ (#)

By (**), (#) is divisible by 3.

Exercise 1.2

Prove the following by induction:

1. The sum of square of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$.
2. $1^2 + 3^2 + \ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$.
3. Show that if a is a real number with a > -1, then $(1+a)^n \geq 1 + na \forall n \in N$.
4. $1^5 + 2^5 + 3^5 + \ldots + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$.
5. $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$.
6. $(n+1)(n+2)\ldots(2n-1)(2n) = 2^n.1.3.5\ldots(2n-1)$.
7. $\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}}_{n-radicals} = 2\cos\frac{\pi}{2^{n+1}}$.
8. $a_{n+1} = 2a_n + 1 (n \in N), \{a_n\}$ is a sequence, show that $a_{n+1} = 2^{n-1}(a_1 + 1)$.
9. $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}$.
10. Given a sequence $a_1, a_2, \ldots, a_n, \ldots$ such that $a_1 = 1$ and $a_n = a_{n-1} + 3$, $(n \geq 2)$. Show that $a_n = 3n - 2$ $\forall n \in N$.

11. $6|n(n^2 + 5)$

12. $21|5^n - 2^n$ for all positive even integers of n.

13. $30|5^n - 3^n - 2^n$ for all positive odd integers of n.

14. If α, β are the roots of $x^2 - 14x + 36 = 0$. Show that $\alpha^n + \beta^n \mid 2^n$, $\forall\ n \in N$.

15. If $n \in Z$ and $n \geq 0$ then $\sum_{i=0}^{n} i.i! = (n+1)! + 1$.

16. $2^n \geq n^2 \forall n \geq 5$ and $n \in N$.

17. For all $n \in N$, there exists distinct integers x, y, z for which $x^2 + y^2 + z^2 = 14^n$.

18. $\binom{n}{k} = \binom{n}{n-k}$

19. For all integers n and k with $1 \leq k \leq n$; $\binom{n}{n-1} + 2\binom{n}{k} + \binom{n}{n+k} = \binom{n+2}{k+1}$.

20. Prove that every positive integer greater than 1 can be written as a product of primes.

21. $\sum_{n=1}^{n} \cos(2k-1)x = \frac{\sin 2nx}{2\sin x}$, for $x \in R$ with $\sin x \neq 0$.

22. $n! > 3^n$ for $n \geq 7$.

23. A set of n elements has 2^n subsets.

24. 'Everything is the same color'. Explain the fallacy by induction.

25. n^{th} derivative of x^n is n!.

26. Ramanujan result: $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\ldots}}}}} = 3$, show that

$$\sqrt{1+n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+\ldots}}}}} = n+1, \forall n \in N.$$

27. $9|5^{2n} + 3n - 1$.

28. $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2$.

29. $|x_{n+1} - x_n| \leq r|x_n - x_{n-1}|, \forall n \geq 1$, where r is fixed positive integer.

30. $\sin x + \sin 3x + \ldots + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x}, n \geq 1.$

31. $\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$, where $a_1, a_2, \ldots, a_n$ are positive integers.

32. Ever integer = $(3+\sqrt{5})^n + (3-\sqrt{5})^n$.

33. $\int_0^{\pi} \sin^n x dx = \frac{n-1}{n} \int_0^{\pi} \sin^{n-2} x dx$; is true for $n \geq 2$. (HINT: integrate by parts)

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The following reference are applicable to all the parts of this notes series. As I said, earlier, the entire notes is well designed in 8 parts. These references are belongs to 8 parts (entire notes).

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