GAUSSIAN QUADRATURE OF THE INTEGRALS $\int_{-\infty}^{\infty} f(x)dx/\cosh x$

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Abstract. The manuscript delivers nodes and their weights for Gaussian quadratures with a "non-classical" weight in the integrand defined by a reciprocal hyperbolic cosine. The associated monic orthogonal polynomials are constructed; their coefficients are simple multiples of the coefficients of Hahn polynomials. A final table shows the abscissae-weight pairs for up to 128 nodes.

1. Introduction

We tabulate the abscissae x_i and weights w_i for Gaussian integration with a reciprocal hyperbolic cosine in the integral kernel,

$$\int_{-\infty}^{\infty} f(x)\frac{dx}{\cosh x} = 2\int_{-\infty}^{\infty} f(x)d[\arctan e^x] \approx \sum_{i=1}^{N} w_i f(x_i). \tag{1}$$

The function $1/\cosh x = \operatorname{sech} x$ has the same even parity and apart from a stretch factor $\sqrt{2}$ the same concave quadratic shape for small x as the Gaussian e^{-x^2}, but falls off $\sim e^{-x}$ for large x. In that sense the Gaussian quadrature shares the symmetry properties with the more familiar Gauss-Hermite quadratures, whereas for large x the node's distances, weights and densities resemble those of the Gauss-Laguerre quadratures.

The variable substitutions $x = \operatorname{arcosh} t$ and $t = 1/\cos\phi = 1/y$ define a small family of related rules,

$$\int_0^{\infty} f(x)\frac{dx}{\cosh x} = \int_1^{\infty} f(\operatorname{arcosh} t)\frac{dt}{t\sqrt{t^2-1}} = \int_0^{\pi/2} f(\operatorname{arcosh}\frac{1}{\cos\phi})d\phi = \int_0^1 f(\operatorname{arcosh}\frac{1}{y})\frac{dy}{\sqrt{1-y^2}}. \tag{2}$$

The last format on the right hand side is usually evaluated with Gauss-Chebyshev quadratures [1, (25.4.38)][10, 8, 7].

2. Gaussian Integration

2.1. Moments.

The w_i and x_i in Eq. 1 are computed with the standard theory from roots of a system of orthogonal polynomials p_n with norm [4, 6, 11]

$$\langle f, g\rangle \equiv \int_{-\infty}^{\infty} f(x)g(x)\frac{dx}{\cosh x}. \tag{3}$$

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Moments μ are defined as

$$\mu_n \equiv \int_{-\infty}^{\infty} \frac{x^n}{\cosh x} dx. \tag{4}$$

The even moments are powers of π multiplied by Euler numbers [5, 3.523.4],

$$\int_0^{\infty} \frac{x^{2n}}{\cosh(ax)} dx = [\pi/(2a)]^{2n+1} |E_{2n}|, \quad a > 0. \tag{5}$$

Example 1.

$$\mu_0 = 2\int_0^{\infty} dx/\cosh x = \pi. \tag{6}$$

$$\mu_2 = 2\int_0^{\infty} \frac{x^2}{\cosh x} dx \approx 7.7515691700749550 43869 = \pi^3/4; \tag{7}$$

$$\mu_4 = 2\int_0^{\infty} \frac{x^4}{\cosh x} dx \approx 95.631151495400454144606659 = 5\pi^5/16. \tag{8}$$

The odd moments are zero because $\cosh x$ is an even function.

Remark 1. *The associated semi-infinite integrals start with* [5, 3.521.2]

$$\int_0^{\infty} \frac{x}{\cosh x} dx = 2C, \tag{9}$$

where $C \approx 0.915965$ *is Catalan's constant* [9, A006752]*. Expansion of* $1/\cosh(ax)$ *in a Taylor series of* e^{-ax} *and exchange of summation and integration yields the Mellin transform* [5, 3.523.3]

$$\int_0^{\infty} \frac{x^{\tau-1}}{\cosh(ax)} dx = \frac{2}{a^{\tau}} \Gamma(\tau) \sum_{k=0}^{\infty} (-)^k \frac{1}{(2k+1)^{\tau}} \quad \Re\tau > 0, a > 0. \tag{10}$$

So the semi-infinite integrals are essentially Dirichlet beta-functions with some factorial growth.

Example 2. [9, A006752,A175572][1, Table 23.3]

$$\beta(2) = \sum_{k\ge 0} \frac{(-)^k}{(2k+1)^2} \approx 0.91596559418; \tag{11}$$

$$\beta(4) = \sum_{k\ge 0} \frac{(-)^k}{(2k+1)^4} \approx 0.988944551741. \tag{12}$$

These β*-values turn out to be irrelevant for the assembly of the orthogonal polynomials further down.*

2.2. **Recurrence.** The set of orthogonal (monic) polynomials $p_n(x)$ is bootstrapped from

$$p_0(x) = 1; \quad p_1(x) = x; \quad p_{n+1}(x) = xp_n(x) - b_n p_{n-1}(x); \tag{13}$$

$$b_n = \frac{\langle xp_n, p_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle} = (n\pi/2)^2. \tag{14}$$

The p are special cases of Hahn polynomials of imaginary argument [3, 2]. The vanishing of the odd moments induces

TABLE 1. Reduced coefficients of the orthogonal polynomials. [9, A060338]

n	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	-1	0	1						
3	0	-5	0	1					
4	9	0	-14	0	1				
5	0	89	0	-30	0	1			
6	-225	0	439	0	-55	0	1		
7	0	-3429	0	1519	0	-91	0	1	
8	11025	0	-24940	0	4214	0	-140	0	1
9	0	230481	0	-122156	0	10038	0	-204	0
10	-893025	0	2250621	0	-463490	0	21378	0	-285
11	0	-23941125	0	14466221	0	-1467290	0	41778	0
12	108056025	0	-296266266	0	70548511	0	-4054028	0	76263

- that the general term on the right hand side $x-a$ reduces to x,
- a parity $p_n(x) = (-1)^n p_n(-x)$.

The first of the orthogonal monic polynomials are

$$p_2 = x^2 - \frac{\pi^2}{4}; \tag{15}$$

$$p_3 = x^3 - \frac{5\pi^2}{4}x; \tag{16}$$

$$p_4 = x^4 - \frac{7\pi^2}{2}x^2 + \frac{9\pi^4}{16}; \tag{17}$$

$$p_5 = x^5 - \frac{15\pi^2}{2}x^3 + \frac{89\pi^4}{16}x; \tag{18}$$

$$p_6 = x^6 - \frac{55\pi^2}{4}x^4 + \frac{439\pi^4}{16}x^2 - \frac{225\pi^6}{64}; \tag{19}$$

$$p_7 = x^7 - \frac{91\pi^2}{4}x^5 + \frac{1519\pi^4}{16}x^3 - \frac{3429\pi^6}{64}x; \tag{20}$$

$$p_8 = x^8 - 35\pi^2x^6 + \frac{2107\pi^4}{8}x^4 - \frac{6235\pi^6}{16}x^2 + \frac{11025}{256}\pi^8. \tag{21}$$

One may reduce this information to the integer triangle in Table 1, which displays in row n and column i the value of $[x^i](2/\pi)^{n-i}p_n(x)$, the coefficient in front of x^i of $p_n(x)$ multiplied by $(2/\pi)^{n-i}$. The diagonal of the triangle contains ones, the first sub-diagonal zeros, the second sub-diagonal the negated square pyramidal numbers [9, A000330], and the third sub-diagonal zeros. The first column are alternatingly signed squares of double factorials, [9, A001818].

By simple heuristic one finds

$$\frac{dp_n(x)}{dx} = \sum_{k=1,3,5,\ldots}^{n} (-)^{\lfloor k/2 \rfloor} \binom{n}{k} (k-1)!\, (\pi/2)^{k-1}\, p_{n-k}(x). \tag{22}$$

Remark 2. *If the computation is repeated for integrals of the form $\int_{-\infty}^{\infty} f(x)/\cosh(x\pi/2)$, the modified polynomials satisfy Equation (14) without the factor $(\pi/2)^2$, their coefficients are the integers of Table 1, and the factor $(\pi/2)^{k-1}$ in Equation (22) is absent.*

2.3. **Abscissae and Weights.** The standard further steps are

- normalization of the polynomials such that their norm is unity,

$$p_n^*(x) \equiv \frac{p_n(x)}{\sqrt{\langle p_n, p_n\rangle}}, \tag{23}$$

  where

$$\langle p_n, p_n\rangle = \pi\left[n!(\frac{\pi}{2})^n\right]^2, \tag{24}$$

- computation of all zeros x_i of $p_N(x)$ at some degree N.
- computation of the weights w_i by

$$w_i = -\frac{[x^{N+1}]p_{N+1}^*}{[x^N]p_N^*}\,\frac{1}{p_{N+1}^*(x_i){p_N^*}'(x_i)}, \tag{25}$$

  where $[x^{N+1}]p_{N+1}^*$ and $[x^N]p_N^*$ are the leading coefficients of the two polynomials after normalization, and where the prime at p' denotes the derivative with respect to x.

The final table shows in each line a pair (x_i, w_i) for N between 3 and 128. Because the p_n are even or odd in our case, the x_i emerge in pairs of the same weight; only the non-negative values need to be shown.

```
N=3
0.00000000000000000000000000000e+00 2.513274122871834590770114706624e+00
3.512407365520363196578187321602e+00 3.141592653589793238462643383280e-01

N=4
1.290965896441180073070246826572e+00 1.530492283332155902198177146094e+00
5.733848834599546320086127816633e+00 4.030404346274071703314454554585e-02

N=5
0.00000000000000000000000000000e+00 2.259122807075806373725945803707e+00
2.869916769199472541729524465267e+00 4.370369936830449412430525948935e-01
8.110832925540868955822877150824e+00 4.197929573948491125296194892976e-03

N=6
1.164992698884809070388844640060e+00 1.488812172738848586373384889168e+00
4.712388980384689857693965074919e+00 8.159980918415047372630242553973e-02
1.058977065965418878577677478990e+01 3.843448718975591316343769324640e-04

N=7
0.00000000000000000000000000000e+00 2.110886402412039551302983480629e+00
2.568054183115606545325647437951e+00 5.031080598664201488408040644868e-01
6.724673230156266603527437591123e+00 1.221291450409527079860733266896e-02
1.314219798150161131386382703486e+01 3.215121836142394041855416963962e-05

N=8
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6.586602128556441369494039833563e+00 9.849858301748245794555282828847e-03
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1.773412473502074205768630918435e+02	1.674316729879677075473717777221e-76
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