Electromagnetic Mass, Charge and Spin

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ABSTRACT

The electrodynamics is usually considered as a phenomenological theory with respect to the masses and charges of the particles. In this paper we develop theoretical model of electrodynamics that does not contain any phenomenological constants associated with the particles, such as particles' masses and charges. This model can be applied equally to various types of particles, such as photon, charged spin ½ fermions and neutrino, and allows for deriving the values of particles' masses and charges. We avoid using any *ad hoc* particle structures (such as *ad hoc* charge and/or mass distributions) in our model, but only symmetry properties associated with distinctive features of the particles.

CONTENTS

1 Summary

The electrodynamics is usually considered as a phenomenological theory with respect to the masses and charges of the interacting particles.

The quantum electrodynamics (QED) based on Dirac equations can be used to make extremely accurate predictions of quantities like the anomalous magnetic moment of the electron, and the Lamb shift of the energy levels of hydrogen. However, the values of the two phenomenological constants – particle's charge e and it's mass m – cannot be derived from Dirac theory and/or QED.

In this paper we develop theoretical model of electrodynamics that does not contain any phenomenological constants associated with the particles, such as particles' masses and charges. Instead, our model

- allows for deriving the values of particles' masses and charges
- applies equally to various types of particles, such as photon, charged spin ½ fermions and neutrino

At the same time we avoid using any *ad hoc* particle structures (such as *ad hoc* charge and/or mass distributions) in our model, but only symmetry properties associated with distinctive features of the particles.

In particular, we assume that

- *photons* are transverse plane electromagnetic waves
- *charged fermions'* fields are axially symmetric
- *neutrino* field violates parity, and its spinor components satisfy Majorana condition.

Hence, each particle type in our model is associated with some symmetry properties of the field configuration, but basic equations are the same for all types of particles.

In this paper we use the *spinor calculus* developed by B. van der Waerden, G.E. Uhlenbeck and O. Laporte. This is because many spinorial equations are much simpler than the corresponding tensorial equations. This applies equally to Maxwell and Dirac equations.

For instance, the free Dirac equation in spinorial form can be written as

$$\partial^{\mu\dot{\nu}}\eta_{\dot{\nu}} + im\xi^{\mu} = 0, \qquad \partial_{\mu\dot{\nu}}\xi^{\mu} + im\eta_{\dot{\nu}} = 0 \tag{1}$$

All the expressions written in spinorial form are manifestly covariant. In Section 2 we briefly explain all the spinorial notations used in this paper. An excellent introduction to the spinor calculus can also be found in [2-4].

In our approach we consider evolutions of two interacting fields:

- electromagnetic fields $\boldsymbol{E}, \boldsymbol{B}$, and
- spinorial matter fields $\xi, \dot{\eta}$.

The evolution of electromagnetic fields $\boldsymbol{E}, \boldsymbol{B}$ is determined by *Maxwell equations*. In Section 3 we rewrite Maxwell equations, as well as related expressions for the charge densities, stress-energy tensor and Lorentz force density, in spinorial form. In Section 3 we also represent in spinorial form the Maxwell equations for the special cases of transverse plane waves and axially symmetric field configurations.

The evolution of spinorial mater fields $\xi, \dot{\eta}$ is determined by the *matter field equations*. These equations should be consistent with the following requirements:

- they need to be Lorentz invariant
- they should not contain phenomenological constants associated with the type of particle, such as mass and charge
- they should not be based on any *ad hoc* particle structures (such as *ad hoc* charge and/or mass distributions or mass-to-charge density ratios)

In Section 4 we introduce the simplest equations that meet all three requirements:

$$\partial^{\mu\dot{\nu}}\eta_{\dot{\nu}} = +f_{\nu}^{\mu}\xi^{\nu}, \qquad \partial_{\mu\dot{\nu}}\xi^{\mu} = -\dot{f}_{\dot{\nu}}^{\dot{\mu}}\eta_{\dot{\mu}} \tag{2}$$

where f_{ν}^{μ} and $\dot{f}_{\dot{\nu}}^{\dot{\mu}}$ are spinorial forms of electric and magnetic field strengths $\boldsymbol{E}, \boldsymbol{B}$ (see Section 2.5).

In these equations spinorial matter fields ξ^{μ} and $\eta_{\dot{\nu}}$ are coupled *via* electromagnetic field spinors.

Matter field equations (2) replicate the structure of the free Dirac equation (1) if we require that

$$\begin{aligned} f_{\nu}^{\mu}\xi^{\nu} &= \lambda\,\xi^{\mu} \\ \dot{f}_{\dot{\nu}}^{\dot{\mu}}\eta_{\dot{\mu}} &= \bar{\lambda}\,\eta_{\dot{\nu}} \end{aligned} \tag{3}$$

i.e. spinorial fields ξ and $\dot{\eta}$ are *eigenvectors* of the second rank electromagnetic field spinors f_{ν}^{μ} and $\dot{f}_{\dot{\nu}}^{\dot{\mu}}$ correspondingly. Indeed, by applying (3) in (2) we obtain

$$\partial^{\mu\dot{\nu}}\eta_{\dot{\nu}} = +\lambda\,\xi^{\mu}, \qquad \partial_{\mu\dot{\nu}}\xi^{\mu} = -\bar{\lambda}\,\eta_{\dot{\nu}} \tag{4}$$

Eigenvalues λ and $\bar{\lambda}$ in (3) and (4) are shown to be well known invariants of the electromagnetic field (see Section 5):

$$\begin{aligned} \lambda_{\pm} &= \pm\sqrt{E^2 - B^2 - 2i\boldsymbol{EB}} \\ \bar{\lambda}_{\pm} &= \pm\sqrt{E^2 - B^2 + 2i\boldsymbol{EB}} \end{aligned} \tag{5}$$

From the analogy with free Dirac equation (1) we can say that in our model electromagnetic field invariants λ and $\bar{\lambda}$ play the roles of *mass densities*, and the latter have *purely electromagnetic origin*. Unlike *constant* and *real valued* mass terms in Dirac equation, these mass densities are *variable* and *complex valued*.

In Sections 5.1 and 5.2 we demonstrate that, in spite of the complex values of the "mass densities" λ and $\bar{\lambda}$, the *momentum densities* of the matter fields P_{μ} are *real valued* and satisfy the *continuity equation*:

$$\partial_{\mu}P^{\mu} = 0 \tag{6}$$

We also demonstrate that condition (3) is equivalent to the requirement that momentum density of the matter field P_{μ} is an eigenvector of the *stress-energy tensor* of the electromagnetic field.

In Section 5.3 we obtain the following expression for the "mass density square" of the matter field

$$P^{\mu}P_{\mu} = 4|\lambda|^2 \tag{7}$$

This is another manifestation of the *electromagnetic origin* of our "mass terms". In our model the momentum density vector P_μ is always *time-like*, and its time-like component P_0 is always *positive*, hence no solutions with negative energies are allowed.

In Sections 3, 4 and 5 the *Maxwell equations* and the *matter field equations* are considered *separately*. The concept of electromagnetic mass was developed based on matter field equations and property (3).

In the Section 6 we consider both Maxwell and matter field equations *simultaneously* and require that both equations are equivalent, i.e. that matter field equations can be reduced to Maxwell equations, and *vice versa*.

Physically that means that particle's own electromagnetic field evolution is *dynamically balanced* with evolution of its source – particle's spinorial field, so that the total field configuration remains stable in time.

This enables us to develop the concept of *electromagnetic charge*, in a sense that the charge density can also be expressed via electromagnetic field strengths $\boldsymbol{E}, \boldsymbol{B}$.

In the first place (Section 6.1) we consider the case of *transverse plane waves* and show that Maxwell and matter field equations become equivalent if the following relationship between the mass and charge densities is satisfied:

$$J^\mu = \bar{\lambda}\, P^\mu \tag{8}$$

From this we conclude that, in the case of the transverse plane waves, electromagnetic field invariant $\bar{\lambda}$ plays the role of the *charge density* (while λ plays the same role for anti-particles). Generally $\bar{\lambda}$ is complex valued, hence allowing for *both* non-zero electric and magnetic charge densities. It does not, however, contradict the fact that the *total* magnetic charge of all known particle is zero. On the other hand, the non-zero magnetic charge density contributes to the total magnetic moment of the particle.

We also demonstrate that in the case of the transverse plane waves the Lorentz force action on the matter field is zero when $E = B$, i.e. when *real parts* of electromagnetic fields invariants λ and $\bar{\lambda}$ are zero.

Particularly, this applies to electromagnetic waves "in vacuum" ($\boldsymbol{E} \perp \boldsymbol{B}$, $E = B$), where we have $\lambda = \bar{\lambda} = 0$, and matter field equations coincide with Maxwell equations for "source-free" electromagnetic plane waves. In this case the momentum density P^μ of the matter field is non-zero, while the charge density J^μ is zero. In this sense the *photons* are not actually "source-free" electromagnetic waves.

In Section 6.2 we consider *stationary axially symmetric* field configuration corresponding to charged fermions, such as electrons. We demonstrate that in this case consistency of the matter field equations and Maxwell equations require that the "velocity of charge" is not the same as "velocity of mass" anymore. It is shown that this difference results in *centripetal acceleration* of the momentum density P_μ, allowing *rotation* of the particle's field around the symmetry axis. This enables the existence of electromagnetic *spin* of the charged fermions. In Section 6.2 we also derive the equation that enables to establish all possible field configurations of charged fermions, and hence deriving the values of their total masses and charges.

Finally, in Section 7 we consider the field configuration that violates parity. For this reason we associate this field configuration with *neutrino*.

We demonstrate that the spinorial neutrino fields satisfy the *Majorana condition*

$$\begin{aligned} \xi^1 &= -\eta_2 \\ \xi^2 &= +\eta_1 \end{aligned} \tag{9}$$

One of the important consequences of the model is that, in spite of non-zero “mass density terms”, the neutrino field propagates at the *speed of light*.

2 Spinor calculus

2.1 Basic notations

Below we describe the basic notations used in this paper.

The metric of the Minkowski space-time is defined as following:

$$g_{\alpha\beta} = g^{\alpha\beta} = diag\ (+,-,-,-), \qquad \alpha,\beta = 0,1,2,3 \tag{10}$$

We use the following representations for Pauli matrices

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{11}$$

$$\acute{\sigma}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \acute{\sigma}_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \acute{\sigma}_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad \acute{\sigma}_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{12}$$

and Dirac’s gamma matrices:

$$\begin{aligned} \gamma_0 &= \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \quad \gamma_1 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \\ \gamma_2 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \gamma_3 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \end{aligned} \tag{13}$$

2.2 Spinors and co-spinors

In spinor notation the (four-component) wave function of the fermion field ψ is considered as a formal sum of first rank spinor and first rank co-spinor fields:

$$\psi(x) = \{\xi, \dot{\eta}\} = \{\xi, 0\} + \{0, \dot{\eta}\} = \begin{pmatrix} \xi^1(x) \\ \xi^2(x) \\ \eta_1(x) \\ \eta_2(x) \end{pmatrix} \tag{14}$$

where

$$\psi \in \mathbb{C}^2 \oplus \dot{\mathbb{C}}^2;\ \xi \in \mathbb{C}^2;\ \dot{\eta} \in \dot{\mathbb{C}}^2 \tag{15}$$

Under Lorentz transformation first rank spinors and co-spinors are transformed as follows:

$$\begin{gathered} \xi'^{\mu} = u^{\mu}_{\nu}\, \xi^{\nu} \\ \eta'_{\dot{\mu}} = (\overline{u}^{-1})^{\dot{\nu}}_{\dot{\mu}}\, \eta_{\dot{\nu}} \end{gathered} \tag{16}$$

where matrix $u \in SL\,(2,\mathbb{C})$ can be presented in the following form:

$$u = \begin{bmatrix} u^1_1 & u^1_2 \\ u^2_1 & u^2_2 \end{bmatrix} = \exp[\,(-b_k + i\,r_k)\sigma_k\,]\,, \qquad r_k, b_k \in \mathbb{R}, \qquad k = 1,2,3 \tag{17}$$

Any quantities transforming like the products $\xi^{\mu}\xi^{\nu}$, $\eta_{\dot{\mu}}\eta_{\dot{\nu}}$, $\xi^{\mu}\eta_{\dot{\nu}}$ are called second rank spinors and denoted by $a^{\mu\nu}$, $b_{\dot{\mu}\dot{\nu}}$, $c^{\mu}_{\dot{\nu}}$ correspondingly. Analogously one can define the spinors of higher ranks.

Transition from subscript to superscript spinor indices is established by means of Lorentz-invariant spinors $\epsilon_{\mu\nu}$ and $\epsilon^{\dot{\mu}\dot{\nu}}$:

$$\epsilon_{\mu\nu} = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \quad \epsilon^{\dot{\mu}\dot{\nu}} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \tag{18}$$

$$\xi_{\mu} = \epsilon_{\mu\nu}\,\xi^{\nu}, \qquad \eta^{\dot{\mu}} = \epsilon^{\dot{\mu}\dot{\nu}}\eta_{\dot{\nu}} \tag{19}$$

$$\begin{gathered} \xi_1 = \xi^2, \qquad \xi_2 = -\,\xi^1 \\ \eta^{\dot{1}} = -\,\eta_{\dot{2}}, \qquad \eta^{\dot{2}} = \eta_{\dot{1}} \end{gathered} \tag{20}$$

One can show easily that complex conjugates of spinors transform as co-spinors, and *vice versa*, so that we can denote

$$\begin{gathered} \xi_{\dot{\mu}} = \overline{\xi_{\mu}} \\ \eta_{\nu} = \overline{\eta_{\dot{\nu}}} \end{gathered} \tag{21}$$

As in the usual tensor algebra, the only covariant operations are *multiplication* and *contraction*. For instance, from the spinors $a^{\rho}_{\dot{\mu}\dot{\nu}}$ and $b^{\dot{\sigma}}_{\alpha\beta}$ we can form the spinor of the 6th rank

$$c^{\rho\dot{\sigma}}_{\dot{\mu}\dot{\nu}\alpha\beta} = a^{\rho}_{\dot{\mu}\dot{\nu}}\, b^{\dot{\sigma}}_{\alpha\beta} \tag{22}$$

or the spinor of the 4th rank

$$c^{\rho}_{\dot{\mu}\alpha\beta} = a^{\rho}_{\dot{\mu}\dot{\nu}}\, b^{\dot{\nu}}_{\alpha\beta} \tag{23}$$

or the spinor of the 2nd rank

$$c_{\dot{\mu}\beta} = a^{\alpha}_{\dot{\mu}\dot{\nu}}\, b^{\dot{\nu}}_{\alpha\beta} \tag{24}$$

The following two rules are essential for calculations:

$$a_{\mu} b^{\mu} = -a^{\mu} b_{\mu} \tag{25}$$

and

$$a^{\mu} b_{\mu} c_{\nu} + a_{\mu} b_{\nu} c^{\mu} + a_{\nu} b^{\mu} c_{\mu} = 0 \tag{26}$$

An immediate consequence of (16) is that any spinor of odd rank has absolute value zero:

$$a_\mu a^\mu = 0 \quad a_{\lambda\mu\nu} a^{\lambda\mu\nu} = 0 \tag{27}$$

2.3 World vectors and tensors

Any vector and/or tensor of the Minkowski space-time can be expressed in a spinor form:

$$\{x^\alpha\} \to \{S_{\mu\dot{\nu}}\}: (S_{\mu\dot{\nu}}) = x^\alpha \sigma_\alpha^T = x_\alpha \sigma^{\alpha\, T} \tag{28}$$

$$\{x^\alpha\} \to \{S^{\mu\dot{\nu}}\}: (S^{\mu\dot{\nu}}) = x^\alpha \acute{\sigma}_\alpha = x_\alpha \acute{\sigma}^\alpha \tag{29}$$

or, equivalently

$$(S^{\mu\dot{\nu}}) = \begin{pmatrix} S^{1\dot{1}} & S^{1\dot{2}} \\ S^{2\dot{1}} & S^{2\dot{2}} \end{pmatrix} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \tag{30}$$

$$(S_{\mu\dot{\nu}}) = \begin{pmatrix} S_{1\dot{1}} & S_{1\dot{2}} \\ S_{2\dot{1}} & S_{22} \end{pmatrix} = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} = \begin{pmatrix} S^{2\dot{2}} & -S^{2\dot{1}} \\ -S^{1\dot{2}} & S^{1\dot{1}} \end{pmatrix} \tag{31}$$

The determinants of the matrices $S_{\mu\dot{\nu}}$ and $S^{\mu\dot{\nu}}$ are equal to $x^\mu x_\mu$ and remain invariant under $SL(2,C)$ transformations. The following rule is also essential for calculations:

$$S_{\sigma\dot{\nu}} S^{\lambda\dot{\nu}} = \delta_\sigma^\lambda (x^\mu x_\mu) \quad S_{\sigma\dot{\nu}} S^{\sigma\dot{\lambda}} = \delta_{\dot{\nu}}^{\dot{\lambda}} (x^\mu x_\mu) \tag{32}$$

In the spinor notation the gradient co-vector ($\partial_\mu = \frac{\partial}{\partial x^\mu}$) is transformed into the following matrices:

$$(\partial^{\mu\dot{\nu}}) = \begin{bmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{bmatrix} = \partial_0 + \partial_1\sigma_1 + \partial_2\sigma_2 + \partial_3\sigma_3 \tag{33}$$

$$(\partial_{\mu\dot{\nu}}) = \begin{bmatrix} \partial_0 - \partial_3 & -\partial_1 - i\partial_2 \\ -\partial_1 + i\partial_2 & \partial_0 + \partial_3 \end{bmatrix} = \partial_0 - \partial_1\sigma_1^T - \partial_2\sigma_2^T - \partial_3\sigma_3^T \tag{34}$$

From (23) we immediately conclude that

$$\partial_{\mu\dot{\nu}} \partial^{\lambda\dot{\nu}} = \delta_\mu^\lambda (\partial^\mu \partial_\mu) \quad \partial_{\mu\dot{\nu}} \partial^{\mu\dot{\lambda}} = \delta_{\dot{\nu}}^{\dot{\lambda}} (\partial^\mu \partial_\mu) \tag{35}$$

and for any 4-vector V^μ represented (according to (30-31)) by second rank spinor $S^{\mu\dot{\nu}}$ the 4-divergence is written in the following form:

$$\partial_{\mu\dot{\nu}} S^{\mu\dot{\nu}} = \partial_\mu V^\mu \tag{36}$$

2.4 Spinorial currents

Alternatively, any spinor and co-spinor can be used to construct the world vector. We will call such vectors *spinorial currents*.

Consider arbitrary spinor ξ that can be expressed as a matrix with one column and two rows. We denote Hermitian conjugate matrix as ξ^+. Then we can construct the following world vector:

$$p_\mu = \frac{1}{2} (\xi^+ \sigma_\mu \xi) \tag{37}$$

$$
\begin{aligned}
p_0 &= \frac{1}{2}(\xi^+\xi) = \frac{1}{2}\left(\overline{\xi^1}\xi^1 + \overline{\xi^2}\xi^2\right) \quad & p_1 &= \frac{1}{2}(\xi^+\sigma_1\xi) = \frac{1}{2}\left(\overline{\xi^2}\xi^1 + \overline{\xi^1}\xi^2\right) \\
p_2 &= \frac{1}{2}(\xi^+\sigma_2\xi) = \frac{i}{2}\left(\overline{\xi^2}\xi^1 - \overline{\xi^1}\xi^2\right) \quad & p_3 &= \frac{1}{2}(\xi^+\sigma_3\xi) = \frac{1}{2}\left(\overline{\xi^1}\xi^1 - \overline{\xi^2}\xi^2\right)
\end{aligned}
\tag{38}
$$

One can easily check that $p_\mu p^\mu \equiv 0$, and using (16-17) we can see that vector $p_\mu = \frac{1}{2}\left(\xi^+\sigma_\mu\xi\right)$ transforms as *covariant* vector.

Following the general rule (30) the spinor current p_μ can be expressed as

$$
p^{\mu\dot{\nu}} = \begin{bmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{bmatrix} = \begin{bmatrix} \xi^1\overline{\xi^1} & \xi^1\overline{\xi^2} \\ \xi^2\overline{\xi^1} & \xi^2\overline{\xi^2} \end{bmatrix} = \begin{bmatrix} \xi^1\xi^{\dot{1}} & \xi^1\xi^{\dot{2}} \\ \xi^2\xi^{\dot{1}} & \xi^2\xi^{\dot{2}} \end{bmatrix}
\tag{39}
$$

Similarly, we can construct *contravariant* vector from co-spinor $\dot{\eta}$:

$$
\hat{p}^\mu = \frac{1}{2}(\dot{\eta}^+\acute{\sigma}^\mu\dot{\eta})
\tag{40}
$$

$$
\begin{aligned}
\hat{p}^0 &= \frac{1}{2}(\dot{\eta}^+\dot{\eta}) = \frac{1}{2}(\overline{\eta_{\dot{1}}}\eta_{\dot{1}} + \overline{\eta_{\dot{2}}}\eta_{\dot{2}}) \quad & \hat{p}^1 &= \frac{1}{2}(\dot{\eta}^+\acute{\sigma}^1\dot{\eta}) = \frac{1}{2}(\overline{\eta_{\dot{2}}}\eta_{\dot{1}} + \overline{\eta_{\dot{1}}}\eta_{\dot{2}}) \\
\hat{p}^2 &= \frac{1}{2}(\dot{\eta}^+\acute{\sigma}^2\dot{\eta}) = \frac{i}{2}(\overline{\eta_{\dot{2}}}\eta_{\dot{1}} - \overline{\eta_{\dot{1}}}\eta_{\dot{2}}) \quad & \hat{p}^3 &= \frac{1}{2}(\dot{\eta}^+\acute{\sigma}^3\dot{\eta}) = \frac{1}{2}(\overline{\eta_{\dot{1}}}\eta_{\dot{1}} - \overline{\eta_{\dot{2}}}\eta_{\dot{2}})
\end{aligned}
\tag{41}
$$

Vector $\hat{p}^\mu$ is also isotropic: $\hat{p}^\mu\hat{p}_\mu \equiv 0$. Using (30) it can be expressed in spinor form:

$$
\hat{p}^{\mu\dot{\nu}} = \begin{bmatrix} \hat{p}^0 - \hat{p}^3 & -\hat{p}^1 + i\hat{p}^2 \\ -\hat{p}^1 - i\hat{p}^2 & \hat{p}^0 + \hat{p}^3 \end{bmatrix} = \begin{bmatrix} \overline{\eta_{\dot{2}}}\eta_{\dot{2}} & -\overline{\eta_{\dot{2}}}\eta_{\dot{1}} \\ -\overline{\eta_{\dot{1}}}\eta_{\dot{2}} & \overline{\eta_{\dot{1}}}\eta_{\dot{1}} \end{bmatrix} = \begin{bmatrix} \eta^1\eta^{\dot{1}} & \eta^1\eta^{\dot{2}} \\ \eta^2\eta^{\dot{1}} & \eta^2\eta^{\dot{2}} \end{bmatrix}
\tag{42}
$$

Using vectors constructed from spinor and co-spinor, one can form a new vector that will not be isotropic. Such vector is usually defined as a bilinear form of the four-component wave function of the fermion field ψ (*Dirac current*):

$$
P_\mu = -\frac{1}{2}\left(\psi^+\gamma_0\gamma_\mu\psi\right)
\tag{43}
$$

where field ψ^+ is a Hermitian conjugate of ψ:

$$
\psi = \begin{bmatrix} \xi^1 \\ \xi^2 \\ \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{bmatrix}, \qquad \psi^+ = \begin{bmatrix} \overline{\xi}^1 & \overline{\xi}^2 & \overline{\eta}_{\dot{1}} & \overline{\eta}_{\dot{2}} \end{bmatrix}
\tag{44}
$$

Using (25) and (4) one can easily check that

$$
\begin{aligned}
P_0 &= \frac{1}{2}\left(\overline{\xi^1}\xi^1 + \overline{\xi^2}\xi^2\right) + \frac{1}{2}(\overline{\eta_{\dot{1}}}\eta_{\dot{1}} + \overline{\eta_{\dot{2}}}\eta_{\dot{2}}) = p_0 + \hat{p}^0 \\
P_1 &= \frac{1}{2}\left(\overline{\xi^2}\xi^1 + \overline{\xi^1}\xi^2\right) - \frac{1}{2}(\overline{\eta_{\dot{2}}}\eta_{\dot{1}} + \overline{\eta_{\dot{1}}}\eta_{\dot{2}}) = p_1 - \hat{p}^1 \\
P_2 &= \frac{i}{2}\left(\overline{\xi^2}\xi^1 - \overline{\xi^1}\xi^2\right) - \frac{i}{2}(\overline{\eta_{\dot{2}}}\eta_{\dot{1}} - \overline{\eta_{\dot{1}}}\eta_{\dot{2}}) = p_2 - \hat{p}^2 \\
P_3 &= \frac{1}{2}\left(\overline{\xi^1}\xi^1 - \overline{\xi^2}\xi^2\right) - \frac{1}{2}(\overline{\eta_{\dot{1}}}\eta_{\dot{1}} - \overline{\eta_{\dot{2}}}\eta_{\dot{2}}) = p_3 - \hat{p}^3
\end{aligned}
\tag{45}
$$

or

$$P_\mu = p_\mu + g_{\mu\nu}\hat{p}^\nu \tag{46}$$

According to (30), world vector P_μ can be expressed as second rank spinor $P^{\mu\dot{\nu}}$:

$$\{P_\mu\} \to P^{\mu\dot{\nu}} = p^{\mu\dot{\nu}} + \hat{p}^{\mu\dot{\nu}} \tag{47}$$

2.5 Electromagnetic fields

Electromagnetic field strengths are expressed in the form of symmetric second rank spinors that realize irreducible representation of the $SL(2, C)$ group:

$$\left.\begin{aligned} f_{\mu\nu} &= f_{\nu\mu} \\ f_{\dot{\mu}\dot{\nu}} &= f_{\dot{\nu}\dot{\mu}} \end{aligned}\right\} \mathit{symmetry\ condition}$$

$$\left. \dot{f}_{\dot{\nu}}^{\dot{\mu}} = \overline{f_\nu^\mu} \right\} \mathit{neutrality\ condition} \tag{48}$$

In the expression above we used (19) to transform *symmetric* spinors $f_{\mu\nu}$ and $f_{\dot{\mu}\dot{\nu}}$ to *traceless* spinors f_ν^μ and $\dot{f}_{\dot{\nu}}^{\dot{\mu}}$:

$$f_\nu^\mu = \epsilon^{\mu\rho} f_{\rho\nu}, \qquad f_\mu^\mu = 0 \tag{49}$$

Due to symmetry of the spinors the field has only 3 complex components

$$f_{11}, \qquad f_{12} = f_{21}, \qquad f_{22} \tag{50}$$

This property enables us to introduce the structure of 3-dimentional complex space for electromagnetic field spinors

$$f_\nu^\mu = \begin{bmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{bmatrix} = \begin{bmatrix} F^3 & F^1 - iF^2 \\ F^1 + iF^2 & -F^3 \end{bmatrix} = F^k \sigma_k, \qquad k = 1,2,3 \tag{51}$$

where "coordinates" F^k can be decomposed into real and imaginary parts

$$\boldsymbol{F} = \boldsymbol{E} - \boldsymbol{iB} \tag{52}$$

From (51) one can see that matrices f_ν^μ belong to the Lie algebra of the group $SL(2, C)$.

3 Maxwell equations in spinor form

Many spinorial equations are much simpler than the corresponding tensorial equations. This applies equally to Maxwell Dirac equations, as we will demonstrate in the further sections.

Maxwell equations have the following spinor form:

$$\partial^{\nu\dot{\rho}} f_\nu^\mu = S^{\mu\dot{\rho}}, \qquad \partial^{\mu\dot{\rho}} \dot{f}_{\dot{\rho}}^{\dot{\lambda}} = \dot{S}^{\mu\dot{\lambda}} \tag{53}$$

Here we use two spinorial forms of the *electromagnetic current density*: $S^{\mu\dot{\rho}}$ and $\dot{S}^{\mu\dot{\lambda}}$

$$S_{\mu\dot\nu} = \begin{bmatrix} S_{1\dot1} & S_{1\dot2} \\ S_{2\dot1} & S_{2\dot2} \end{bmatrix} = \begin{bmatrix} J^0 + J^3 & J^1 + iJ^2 \\ J^1 - iJ^2 & J^0 - J^3 \end{bmatrix}$$

$$\dot S_{\mu\dot\nu} = \begin{bmatrix} \dot S_{1\dot1} & \dot S_{1\dot2} \\ \dot S_{2\dot1} & \dot S_{2\dot2} \end{bmatrix} = \begin{bmatrix} \dot J^0 + \dot J^3 & \dot J^1 + i\dot J^2 \\ \dot J^1 - i\dot J^2 & \dot J^0 - \dot J^3 \end{bmatrix} \tag{54}$$

These two spinors are Hermitian conjugates of each other

$$\begin{bmatrix} \dot S_{1\dot1} & \dot S_{1\dot2} \\ \dot S_{2\dot1} & \dot S_{2\dot2} \end{bmatrix} = \begin{bmatrix} \overline{S_{1\dot1}} & \overline{S_{2\dot1}} \\ \overline{S_{1\dot2}} & \overline{S_{2\dot2}} \end{bmatrix} \tag{55}$$

Complex vectors J^k and $\dot J^k$ corresponding to spinors $S^{\mu\dot\rho}$ and $\dot S^{\mu\dot\lambda}$ are complex conjugated to each other and can be decomposed into *electric* and *magnetic* current densities

$$J^k = J_e{}^k - iJ_m{}^k \qquad k = 0, 1, 2, 3$$
$$\dot J^k = \dot J_e{}^k - i\dot J_m{}^k = \overline{J^k} \tag{56}$$

To be convinced that any of the complex conjugated spinorial equations (53) explicitly correspond to Maxwell equations, we can rewrite, e.g. the first equation (using (19) and (25)) in the form

$$S_{\mu\dot\nu} = -\partial_{\rho\dot\nu} f^\rho_\mu \tag{57}$$

represent it in matrix form

$$\begin{bmatrix} \partial_{1\dot1} & \partial_{2\dot1} \\ \partial_{1\dot2} & \partial_{2\dot2} \end{bmatrix} \begin{bmatrix} f^1_1 & f^1_2 \\ f^2_1 & f^2_2 \end{bmatrix} = - \begin{bmatrix} S_{1\dot1} & S_{2\dot1} \\ S_{1\dot2} & S_{2\dot2} \end{bmatrix} \tag{58}$$

and then expand this expression using (34), (51), (52), (54) and (56):

$$\begin{aligned}
(\partial_0 - \partial_3)(E^3 - iB^3) + (-\partial_1 + i\partial_2)(E^1 - iB^1 + iE^2 + B^2) &= -\left(J_e{}^0 - iJ_m{}^0 + J_e{}^3 - iJ_m{}^3\right) \\
(\partial_0 - \partial_3)(E^1 - iB^1 - iE^2 - B^2) - (-\partial_1 + i\partial_2)(E^3 - iB^3) &= -\left(J_e{}^1 - iJ_m{}^1 - iJ_e{}^2 - J_m{}^2\right) \\
(-\partial_1 - i\partial_2)(E^3 - iB^3) + (\partial_0 + \partial_3)(E^1 - iB^1 + iE^2 + B^2) &= -\left(J_e{}^1 - iJ_m{}^1 + iJ_e{}^2 + J_m{}^2\right) \\
(-\partial_1 - i\partial_2)(E^1 - iB^1 - iE^2 - B^2) - (\partial_0 + \partial_3)(E^3 - iB^3) &= -\left(J_e{}^0 - iJ_m{}^0 - J_e{}^3 + iJ_m{}^3\right)
\end{aligned} \tag{59}$$

By separating real and imaginary parts of the equations (59), we obtain Maxwell equations in vector form

$$\begin{aligned} div\,\boldsymbol{E} = J_e{}^0 \qquad curl\,\boldsymbol{B} - \dot{\boldsymbol{E}} = \boldsymbol{J_e} \\ div\,\boldsymbol{B} = J_m{}^0 \quad -curl\,\boldsymbol{E} - \dot{\boldsymbol{B}} = \boldsymbol{J_m} \end{aligned} \tag{60}$$

The conservation of charge is a consequence of the Maxwell equations. The continuity equation for the current density reads (see (36) and (35))

$$\partial_k J^k = \partial_{\mu\dot\nu} S^{\mu\dot\nu} = \partial_{\mu\dot\nu} \partial^{\lambda\dot\nu} f^\mu_\lambda = \delta^\lambda_\mu (\partial^\rho \partial_\rho) f^\mu_\lambda = (\partial^\rho \partial_\rho) f^\mu_\mu = 0 \tag{61}$$

since f^μ_ν is a traceless matrix: $f^\mu_\mu = 0$.

3.1 Lorentz force density

Now we can use Maxwell equations (53) to derive the expression for the Lorentz force spinor. We first introduce the *stress-energy density spinor* of the electromagnetic field

$$T^{\delta\dot\rho}_{\mu\dot\nu} = f^\delta_\mu \dot f^{\dot\rho}_{\dot\nu} \tag{62}$$

and then consider the expression

$$\partial_{\delta\dot{\rho}} T^{\delta\dot{\rho}}_{\mu\dot{\nu}} = \partial_{\delta\dot{\rho}} \left[f^{\delta}_{\mu} \dot{f}^{\dot{\rho}}_{\dot{\nu}} \right] = \dot{f}^{\dot{\rho}}_{\dot{\nu}} \left[\partial_{\delta\dot{\rho}} f^{\delta}_{\mu} \right] + f^{\delta}_{\mu} \left[\partial_{\delta\dot{\rho}} \dot{f}^{\dot{\rho}}_{\dot{\nu}} \right] = \Lambda_{\mu\dot{\nu}} \tag{63}$$

where the *force density* spinor

$$\Lambda_{\mu\dot{\nu}} = -\left[\dot{f}^{\dot{\rho}}_{\dot{\nu}} S_{\mu\dot{\rho}} + f^{\delta}_{\mu} \dot{S}_{\delta\dot{\nu}} \right] \tag{64}$$

Of course, the force density spinor $\Lambda_{\mu\dot{\nu}}$ corresponds to the Lorentz force density 4-vector $\mathcal{F}^{\mu}$

$$\Lambda_{\mu\dot{\nu}} = \begin{bmatrix} \Lambda_{1\dot{1}} & \Lambda_{1\dot{2}} \\ \Lambda_{2\dot{1}} & \Lambda_{2\dot{2}} \end{bmatrix} = \begin{bmatrix} \mathcal{F}^0 + \mathcal{F}^3 & \mathcal{F}^1 + i\mathcal{F}^2 \\ \mathcal{F}^1 - i\mathcal{F}^2 & \mathcal{F}^0 - \mathcal{F}^3 \end{bmatrix} \tag{65}$$

3.2 Electromagnetic fields with special symmetries

In further sections we will need the expressions for Maxwell equations for the systems with special symmetries. Particularly we will need such expressions for:

- Transverse plane electromagnetic waves, and
- Stationary fields with axial symmetry

3.2.1 Transverse plane waves

By definition, in transverse plane waves the directions of vectors $\boldsymbol{E}, \boldsymbol{B}$ are orthogonal to the direction of wave propagation. For simplicity we can choose axis $\boldsymbol{e}_3$ parallel to the direction of the wave propagation. In this case we will have:

$$\begin{gathered} F^3 \equiv 0 \\ J^1 = J^2 \equiv 0 \end{gathered} \tag{66}$$

at all times and all points in space.

The Maxwell equations will be reduced to the following expressions:

$$\begin{gathered} (\partial_1 - i\partial_2)(F^1 + iF^2) = J^0 + J^3 \\ (\partial_0 - \partial_3)(F^1 - iF^2) = 0 \\ (\partial_0 + \partial_3)(F^1 + iF^2) = 0 \\ (\partial_1 + i\partial_2)(F^1 - iF^2) = J^0 - J^3 \end{gathered} \tag{67}$$

In absence of charged currents the right hand sides of all the equations vanish, and we obtain

$$\begin{gathered} (\partial_1 - i\partial_2)(F^1 + iF^2) = 0 \\ (\partial_0 - \partial_3)(F^1 - iF^2) = 0 \\ (\partial_0 + \partial_3)(F^1 + iF^2) = 0 \\ (\partial_1 + i\partial_2)(F^1 - iF^2) = 0 \end{gathered} \tag{68}$$

3.2.2 Stationary field configurations with axial symmetry

Now we consider the stationary field configurations with axial symmetry. We introduce the polar cylindrical coordinate system $\{e_1, e_2, e_3\} \to \{e_1, e_\rho, e_\varphi\}$ (see Figure 1):

$$
\begin{aligned}
x^2 &= \rho\cos\varphi \\
x^3 &= \rho\sin\varphi \\
x^1 &= x^1
\end{aligned}
$$

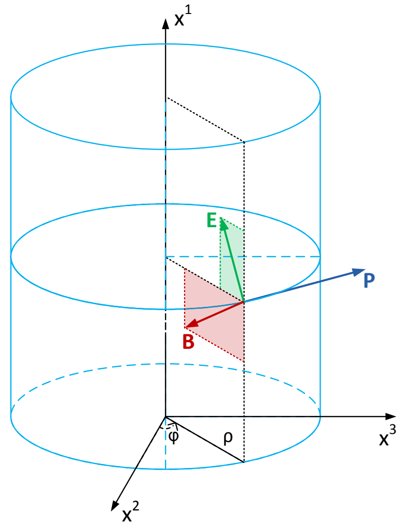

Figure 1. Axially symmetric field configuration

and require that

$$
\begin{aligned}
\partial_0 &\equiv 0 \quad stationarity \\
\partial_\varphi &\equiv 0 \quad axial\ symmetry
\end{aligned}
\tag{69}
$$

This ensures that field configuration is symmetric w.r.t. rotations around axis $\boldsymbol{e}_1$ and is not changing over time.

Due to axial symmetry, electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$ in each point belong to the plane $\varphi = const$ (i.e. $F^\varphi \equiv 0$), and charge density current is parallel to direction of $\boldsymbol{e}_\varphi$ (i.e. $J^\rho = J^1 \equiv 0$).

Then Maxwell equations in cylindrical coordinates can be written as follows:

$$
\begin{aligned}
\frac{1}{\rho}F^\rho + (\partial_1 - i\partial_\rho)(F^1 + iF^\rho) &= J^0 + J^\varphi \\
\left(\partial_0 - \frac{1}{\rho}\partial_\varphi\right)(F^1 - iF^\rho) &= 0 \\
\left(\partial_0 + \frac{1}{\rho}\partial_\varphi\right)(F^1 + iF^\rho) &= 0 \\
\frac{1}{\rho}F^\rho + (\partial_1 + i\partial_\rho)(F^1 - iF^\rho) &= J^0 - J^\varphi
\end{aligned}
\tag{70}
$$

4 Matter field equation

The Dirac equations were first written in spinor form by G.E. Uhlenbeck and O. Laporte in 1931 [2]. According to (14) the four wave functions of Dirac correspond to spinor of the first rank ξ^μ and co-spinor of the first rank $\eta_{\dot\nu}$, and Dirac equations become

$$
(\partial^{\mu\dot\nu} + ie\Phi^{\mu\dot\nu})\eta_{\dot\nu} + im\xi^\mu = 0, \qquad (\partial_{\mu\dot\nu} + ie\Phi_{\mu\dot\nu})\xi^\mu + im\eta_{\dot\nu} = 0, \quad m \in \mathbb{R} \tag{71}
$$

where $\Phi^{\mu\dot\nu}$ is a spinor obtained from electromagnetic potential four-vector A_μ using general rule (30).

The quantum electrodynamics (QED) based on Dirac equations can be used to make extremely accurate predictions of quantities like the anomalous magnetic moment of the electron, and the Lamb shift of the energy levels of hydrogen. However, the values of the two phenomenological constants – particle's charge e and it's mass m – cannot be derived from Dirac theory and/or QED.

If one considers electrodynamics as a phenomenological theory with respect to the mass and charge of the interacting particles, and if one consequently condones the necessity of infinite mass and charge renormalizations, one is tempted to consider quantum electrodynamics as a pretty satisfactory theory (quoted from [1]).

However, the purpose of this paper is to develop the model that allows for deriving the values of particles' masses and charges from the model.

Such model cannot be based on the Dirac equation, because the mass and charge parameters in the Dirac theory need to be identified from experiment for each particle type. Hence we need to assume some other form of matter filed equation. This new equation should be consistent with the following requirements:

- It has to be Lorentz invariant
- It should not contain phenomenological constants associated with the type of particle, such as mass and charge
- It should not be based on any *ad hoc* particle structures (such as *ad hoc* charge and/or mass distributions or mass-to-charge density ratios)

The simplest equation that meets all three requirements is as follows:

$$\partial^{\mu\dot{\nu}}\eta_{\dot{\nu}} = +f^{\mu}_{\nu}\xi^{\nu}, \qquad \partial_{\mu\dot{\nu}}\xi^{\mu} = -\dot{f}^{\dot{\mu}}_{\dot{\nu}}\eta_{\dot{\mu}} \tag{72}$$

In this equation the spinor and co-spinor fields are coupled via electromagnetic field. The conjugate equations can be written as follows:

$$\partial^{\mu\dot{\nu}}\eta_{\mu} = +\dot{f}^{\dot{\nu}}_{\dot{\mu}}\xi^{\dot{\mu}}, \qquad \partial_{\mu\dot{\nu}}\xi^{\dot{\nu}} = -f^{\nu}_{\mu}\eta_{\nu} \tag{73}$$

In this paper we consider equations (72-73) as *fundamental matter field equations* that determine the dynamics of the spinorial matter fields.

In further sections we will demonstrate that, with "supersymmetry" condition (74), these equations can be reduced to Maxwell equations, so that mass and charge densities can be expressed via electromagnetic field strengths. The particle type can then be associated with the type of stable field configuration, and the total masses and charges of the particles will only depend on the corresponding stable configurations of the fields.

5 Electromagnetic mass

In our approach we consider evolutions of two interacting fields:

- electromagnetic fields $\boldsymbol{E}, \boldsymbol{B}$, and
- spinorial fields $\xi, \dot{\eta}$.

The evolution of electromagnetic fields is determined by Maxwell equations (53), while the evolution of spinorial fields is determined by matter field equations (72-73).

We anticipate that there exist *stable configurations* of the fields $\boldsymbol{E}, \boldsymbol{B}$ and $\xi, \dot{\eta}$ that correspond to *elementary particles*, and the *type* of elementary particle depends on the field configuration and its properties.

Physically that means that particle's own electromagnetic field evolution is *dynamically balanced* with evolution of its source – particle's spinorial field, so that the total field configuration remains stable in time.

Mathematically that means that:

- spinorial fields can be expressed via electromagnetic fields, i.e. as functions $\xi = \xi(\boldsymbol{E}, \boldsymbol{B})$ and $\dot{\eta} = \dot{\eta}(\boldsymbol{E}, \boldsymbol{B})$, and as a consequence
- matter field equations become equivalent (or reduced) to Maxwell equations.

We will see that equivalence of Maxwell equations and matter field equations can be achieved if the following Lorentz invariant relationship is satisfied:

$$f_{\nu}^{\mu} \xi^{\nu} = \lambda \, \xi^{\mu}$$
$$\dot{f}_{\dot{\nu}}^{\dot{\mu}} \eta_{\dot{\mu}} = \bar{\lambda} \, \eta_{\dot{\nu}} \tag{74}$$

The meaning of expressions (74) is that spinorial fields ξ and $\dot{\eta}$ are *eigenvectors* of the second rank electromagnetic field spinors f_{ν}^{μ} and $\dot{f}_{\dot{\nu}}^{\dot{\mu}}$ correspondingly. In this section we will also show that direct consequence of the relationship (74) is that the *4-momentum* P_{μ} of the spinorial matter field is an eigenvector of the stress-energy tensor of the electromagnetic field (corresponding to 4-th rank spinor (62)).

With the condition (74) our matter field equations (72) become very simple

$$\partial^{\mu\dot{\nu}} \eta_{\dot{\nu}} = + \lambda \, \xi^{\mu}, \qquad \partial_{\mu\dot{\nu}} \xi^{\mu} = -\bar{\lambda} \, \eta_{\dot{\nu}} \tag{75}$$

and replicate the structure of the *free* Dirac equation (see (1)) where *constant* mass term m is replaced by the *variable* "mass density" terms λ and $\bar{\lambda}$.

Taking account the explicit form of electromagnetic field spinors f_{ν}^{μ} and $\dot{f}_{\dot{\nu}}^{\dot{\mu}}$ (see (51-52)) one can see that eigenvalues of these spinors λ and $\bar{\lambda}$ are well known electromagnetic field invariants:

$$\lambda_{\pm} = \pm\sqrt{(F^1)^2 + (F^2)^2 + (F^3)^2} \qquad {\lambda_{\pm}}^2 = E^2 - B^2 - 2i\boldsymbol{EB}$$
$$\bar{\lambda}_{\pm} = \pm\sqrt{\left(\overline{F^1}\right)^2 + \left(\overline{F^2}\right)^2 + \left(\overline{F^3}\right)^2} \qquad {\bar{\lambda}_{\pm}}^2 = E^2 - B^2 + 2i\boldsymbol{EB} \tag{76}$$

Hence, from the analogy with free Dirac equation we can say that in our model electromagnetic field invariants play the roles of mass densities, and the latter have *purely electromagnetic origin*.

5.1 Momentum density of the matter field

Following the procedure explained in the Section 2.4, the momentum density vector of the spinorial matter field can be defined as a sum of spinorial currents

$$\{P_{\mu} = p_{\mu} + \hat{p}_{\mu}\} \rightarrow P^{\mu\dot{\nu}} = p^{\mu\dot{\nu}} + \hat{p}^{\mu\dot{\nu}} \tag{77}$$

where

$$p^{\mu\dot{\nu}} = \xi^{\mu} \xi^{\dot{\nu}}, \qquad \hat{p}^{\mu\dot{\nu}} = \eta^{\mu} \eta^{\dot{\nu}} \tag{78}$$

When condition (74) is satisfied, the momentum 4-vector P_{μ} becomes an eigenvector of the stress-energy tensor of the electromagnetic field (62):

$$t^{\mu\dot{\rho}}_{\nu\dot{\sigma}}p^{\nu\dot{\sigma}} = \left(\dot{f}^{\dot{\rho}}_{\dot{\sigma}}\xi^{\dot{\sigma}}\right)\left(f^{\mu}_{\nu}\xi^{\nu}\right) = |\lambda|^2 p^{\mu\dot{\rho}}$$
$$t^{\mu\dot{\rho}}_{\nu\dot{\sigma}}\hat{p}^{\nu\dot{\sigma}} = \left(\dot{f}^{\dot{\rho}}_{\dot{\sigma}}\eta^{\dot{\sigma}}\right)\left(f^{\mu}_{\nu}\eta^{\nu}\right) = |\lambda|^2 \hat{p}^{\mu\dot{\rho}} \tag{79}$$

$$t^{\mu\dot{\rho}}_{\nu\dot{\sigma}}P^{\nu\dot{\sigma}} = |\lambda|^2 P^{\mu\dot{\rho}} \tag{80}$$

The eigenvalue of the stress-energy tensor is, of course, expressed via electromagnetic field invariants (76).

5.2 Conservation of the total momentum

According to (36) the divergence of the momentum density vector P_μ

$$\partial_\mu P^\mu = \partial_{\mu\dot{\nu}}P^{\mu\dot{\nu}} = \partial_{\mu\dot{\nu}}p^{\mu\dot{\nu}} + \partial_{\mu\dot{\nu}}\hat{p}^{\mu\dot{\nu}} \tag{81}$$

Using matter field equations (72-73) we can find that

$$\partial_{\mu\dot{\nu}}p^{\mu\dot{\nu}} = \partial_{\mu\dot{\nu}}\left[\xi^\mu\xi^{\dot{\nu}}\right] = \left(\left[\partial_{\mu\dot{\nu}}\xi^\mu\right]\xi^{\dot{\nu}} + \xi^\mu\left[\partial_{\mu\dot{\nu}}\xi^{\dot{\nu}}\right]\right) = -\left(\dot{f}^{\dot{\mu}}_{\dot{\nu}}\eta_{\dot{\mu}}\xi^{\dot{\nu}} + \xi^\mu f^{\nu}_{\mu}\eta_\nu\right)$$
$$\partial_{\mu\dot{\nu}}\hat{p}^{\mu\dot{\nu}} = \partial^{\mu\dot{\nu}}\left[\eta_\mu\eta_{\dot{\nu}}\right] = \left(\left[\partial^{\mu\dot{\nu}}\eta_\mu\right]\eta_{\dot{\nu}} + \eta_\mu\left[\partial^{\mu\dot{\nu}}\eta_{\dot{\nu}}\right]\right) = +\left(\dot{f}^{\dot{\nu}}_{\dot{\mu}}\xi^{\dot{\mu}}\eta_{\dot{\nu}} + \eta_\mu f^{\mu}_{\nu}\xi^\nu\right) \tag{82}$$

from what we conclude that momentum of the spinorial field is *conserved* due to matter field equations:

$$\partial_\mu P^\mu = 0 \tag{83}$$

5.3 Eigenvectors

Let us now derive the expressions for the eigenvectors of the electromagnetic field spinors f^{μ}_{ν} and $\dot{f}^{\dot{\mu}}_{\dot{\nu}}$.

Consider arbitrary point Q at the space-time. For the sake of convenience we can choose the reference frame (denoted as $M_\perp$) in such a way that the fields $\boldsymbol{E}, \boldsymbol{B}$ at the point Q will be orthogonal to the axis $\boldsymbol{e}_3$. There is, of course, infinite number of such frames, but all the considerations presented in this section are valid for any of these frames.

In the reference frame $M_\perp$ the expression for spinor f^{μ}_{ν} at the point Q will be

$$f^{\mu}_{\nu} = \begin{bmatrix} 0 & F^1 - iF^2 \\ F^1 + iF^2 & 0 \end{bmatrix} \tag{84}$$

because $F^3 = 0$ at the point Q.

One can easily check now that two spinors ξ_+ и ξ_- defined as

$$\xi_\pm = \begin{bmatrix} \xi^1_\pm \\ \xi^2_\pm \end{bmatrix} = \begin{bmatrix} \pm\sqrt{F^1 - iF^2} \\ \sqrt{F^1 + iF^2} \end{bmatrix} \tag{85}$$

will be eigenvectors of the matrix (84) at the point Q

$$f^{\mu}_{\nu}\xi^{\nu}_{\pm} = \lambda_\pm\xi^{\mu}_{\pm} \tag{86}$$

Hence, with the special choice of the reference frame we can write an explicit expression for the components of the spinorial field ξ satisfying condition (74). The expressions for the field components in all other frames can be obtained by the appropriate Lorentz transformations.

Similarly one can show that at the reference frame $M_{\perp}$ two co-spinors $\dot{\eta}_{+}$ and $\dot{\eta}_{-}$ defined as

$$\dot{\eta}_{\pm} = \begin{bmatrix} \eta_{\pm\dot{1}} \\ \eta_{\pm\dot{2}} \end{bmatrix} = \begin{bmatrix} \pm\sqrt{\overline{F^1} - i\overline{F^2}} \\ \sqrt{\overline{F^1} + i\overline{F^2}} \end{bmatrix} \tag{87}$$

will satisfy the condition

$$\dot{f}^{\dot{\mu}}_{\dot{\nu}}\, \eta_{\pm\dot{\mu}} = \bar{\lambda}_{\pm} \eta_{\pm\dot{\nu}} \tag{88}$$

at the point Q.

Using (85) and (87) one can find that in the frame $M_{\perp}$

$$\begin{aligned}
\xi^1_{\pm}\xi^{\dot{1}}_{\pm} &= \sqrt{(F^1 - iF^2)(\overline{F^1} + i\overline{F^2})} = \sqrt{E^2 + B^2 - 2(E^1B^2 - E^2B^1)} = \eta^1_{\pm}\eta^{\dot{1}}_{\pm} \\
\xi^2_{\pm}\xi^{\dot{2}}_{\pm} &= \sqrt{(F^1 + iF^2)(\overline{F^1} - i\overline{F^2})} = \sqrt{E^2 + B^2 + 2(E^1B^2 - E^2B^1)} = \eta^2_{\pm}\eta^{\dot{2}}_{\pm} \\
\xi^1_{\pm}\xi^{\dot{2}}_{\pm} &= \pm\sqrt{(F^1 - iF^2)(\overline{F^1} - i\overline{F^2})} = \pm\sqrt{(E^1)^2 - (E^2)^2 + (B^1)^2 - (B^2)^2 - 2i(E^1E^2 + B^1B^2)} = -\eta^1_{\pm}\eta^{\dot{2}}_{\pm} \\
\xi^2_{\pm}\xi^{\dot{1}}_{\pm} &= \pm\sqrt{(F^1 + iF^2)(\overline{F^1} + i\overline{F^2})} = \pm\sqrt{(E^1)^2 - (E^2)^2 + (B^1)^2 - (B^2)^2 + 2i(E^1E^2 + B^1B^2)} = -\eta^2_{\pm}\eta^{\dot{1}}_{\pm}
\end{aligned} \tag{89}$$

at the point Q. From this we can see that in the frame $M_{\perp}$ the components of the spinorial currents p_{μ} and $\hat{p}^{\mu}$ (see (78)) satisfy the relationships

$$\begin{aligned}
p_{\pm_0} &= \hat{p}_{\pm}{}^{0}, \qquad p_{\pm_1} = \hat{p}_{\pm}{}^{1} \\
p_{\pm_2} &= \hat{p}_{\pm}{}^{2}, \qquad p_{\pm_3} = -\hat{p}_{\pm}{}^{3}
\end{aligned} \tag{90}$$

and the total momentum density P_{μ} has only two non-zero components in the frame $M_{\perp}$

$$\begin{aligned}
P_{\pm_0} &= 2\, p_{\pm_0} = 2\, \hat{p}_{\pm}{}^{0}, \qquad P_{\pm_1} = 0 \\
P_{\pm_2} &= 0, \qquad P_{\pm_3} = 2\, p_{\pm_3} = -2\, \hat{p}_{\pm}{}^{3}
\end{aligned} \tag{91}$$

From (89) we can derive the "mass square" of the momentum density 4-vector P_{μ}, which is invariant under Lorentz transformations and hence has the same value in all reference frames:

$$P^{\mu}P_{\mu} = 4|\lambda|^2 \tag{92}$$

This is another manifestation of the electromagnetic origin of our "mass term". It is worth noting that the momentum density vector P_{μ} is always *time-like*, and its time-like component P_0 is always positive, hence no solutions with negative energies are allowed.

From (89) we can find the values of the following Lorentz invariants:

$$\begin{aligned}
\xi^{\mu}_{\pm}\, \eta_{\pm_{\mu}} &= \pm 2\, \lambda_{\pm} \\
\xi^{\dot{\mu}}_{\pm}\, \eta_{\pm_{\dot{\mu}}} &= \pm 2\, \bar{\lambda}_{\pm}
\end{aligned} \tag{93}$$

In Section 7 we will also use the following identities in connection with *neutrino* model:

$$\xi_{-}^{\mu}\,\eta_{+_{\mu}} = 0$$

$$\xi_{-}^{\dot{\mu}}\,\eta_{+_{\dot{\mu}}} = 0 \qquad (94)$$

5.4 Conservation of spinorial currents

Let us now assume that fields ξ and $\dot{\eta}$ are the eigenvectors of the electromagnetic field spinors f_{ν}^{μ} and $\dot{f}_{\dot{\nu}}^{\dot{\mu}}$, i.e. condition (74) is satisfied. Then using (74) and (93) the expressions (82) can be written as

$$\partial_{\rho} p_{\pm}^{\rho} = -\left(\lambda_{\pm}\xi_{\pm}^{\mu}\eta_{\pm_{\mu}} + \overline{\lambda}_{\pm}\eta_{\dot{\mu}}\xi^{\dot{\mu}}\right) = \mp 2({\lambda_{\pm}}^{2} + {\overline{\lambda}_{\pm}}^{2})$$

$$\partial_{\rho} \hat{p}_{\pm}^{\rho} = +\left(\lambda_{\pm}\xi_{\pm}^{\mu}\eta_{\pm_{\mu}} + \overline{\lambda}_{\pm}\eta_{\dot{\mu}}\xi^{\dot{\mu}}\right) = \pm 2({\lambda_{\pm}}^{2} + {\overline{\lambda}_{\pm}}^{2}) \qquad (95)$$

From (95) we conclude that spinorial currents p^{μ} and $\hat{p}^{\mu}$ are *conserved independently* when $E = B$, i.e. when real parts of the squared electromagnetic field invariants λ^2 and $\overline{\lambda}^2$ are zero. This might be important for considering the case of *neutrino* fields that violate parity (see Section 7).

5.5 Divergence of axial vector current

Axial vector current ${P_A}^{\mu}$ is defined, as usual, as a difference of spinorial currents p^{μ} and $\hat{p}^{\mu}$

$${P_A}^{\mu} = p^{\mu} - \hat{p}^{\mu} \qquad (96)$$

From (95) one can see that the 4-divergence of the axial vector current equals to

$$\partial_{\rho} {P_{A\pm}}^{\rho} = \partial_{\rho} p_{\pm}^{\rho} - \partial_{\rho}\hat{p}_{\pm}^{\rho} = \mp 4\left({\lambda_{\pm}}^{2} + {\overline{\lambda}_{\pm}}^{2}\right) = \mp 8(E^2 - B^2) = \pm 8F^{\mu\nu}F_{\mu\nu} \qquad (97)$$

5.6 Transition to the rest frame

Let us briefly discuss the properties of the momentum densities P^{μ} in the special cases of *orthogonal* ($\boldsymbol{E} \perp \boldsymbol{B}$) electromagnetic fields. The case of *parallel* ($\boldsymbol{E} \parallel \boldsymbol{B}$, $E = B$) electromagnetic fields will be considered in connection with *neutrino* model in Section 7.

Let us first consider the case $\boldsymbol{E} \perp \boldsymbol{B}$, $E > B$.

In this case the squares of invariants of the electromagnetic fields are positive real numbers:

$${\lambda_{\pm}}^{2} = E^2 - B^2 > 0$$

$${\overline{\lambda}_{\pm}}^{2} = E^2 - B^2 > 0 \qquad (98)$$

It is easy to check that in the frame $M_{\perp}$ the non-zero components of the momentum density 4-vector P^{μ} will be

$$P_0 = +4E$$
$$P_3 = -4B \qquad (99)$$

if the pair of vectors $\{\boldsymbol{E}, \boldsymbol{B}\}$ has the same orientation as basis vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, and

$$P_0 = +4E$$
$$P_3 = +4B \qquad (100)$$

for inverse orientation of the pair of vectors $\{\boldsymbol{E}, \boldsymbol{B}\}$.

It is well known that if $\boldsymbol{E} \perp \boldsymbol{B}$, $E > B$, there is a reference frame where magnetic field $\boldsymbol{B}$ vanish [6]. Let's denote this frame as M_E.

From (99-100) we can see that only time-like component P^0 of the momentum density has non-zero value in the frame M_E. In this sense frame M_E might be considered as a "*rest frame*" of the momentum density P^μ.

Similarly we can show that in the case of $\boldsymbol{E} \perp \boldsymbol{B}$, $E < B$ the squares of invariants of the electromagnetic fields are negative real numbers:

$$\lambda_\pm{}^2 = E^2 - B^2 < 0$$
$$\bar{\lambda}_\pm{}^2 = E^2 - B^2 < 0 \tag{101}$$

and in the frame $M_\perp$ the non-zero components of the momentum 4-vector P^μ will be

$$\begin{aligned} P_0 &= +4B \\ P_3 &= -4E \end{aligned} \tag{102}$$

if the pair of vectors $\{\boldsymbol{E}, \boldsymbol{B}\}$ has the same orientation as basis vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, and

$$\begin{aligned} P_0 &= +4B \\ P_3 &= +4E \end{aligned} \tag{103}$$

for inverse orientation of the pair of vectors $\{\boldsymbol{E}, \boldsymbol{B}\}$.

Similarly, in the reference frame M_B where electric field $\boldsymbol{E}$ vanish, only time-like component P^0 of the momentum density has non-zero value.

In the case of $\boldsymbol{E} \perp \boldsymbol{B}$, $E = B$ we have $\lambda = \bar{\lambda} = 0$, and the momentum density is *isotropic* in all reference frames: $P^\mu P_\mu = 4|\lambda|^2 = 0$.

6 Electromagnetic charge

In this section we will see how matter field equations can be reduced to Maxwell equations. For simplicity we will consider two systems with special symmetries:

- Transverse plane waves corresponding to *photons*,
- Stationary fields with axial symmetry, that can be associated with *charged massive fermions* at their rest frames.

Due to chosen symmetries of the field configurations, at every point of the space-time the electric and magnetic field vectors $\boldsymbol{E}, \boldsymbol{B}$ are orthogonal to one of the basis vectors.

In particular, for the transverse plane waves we direct axis $\boldsymbol{e}_3$ parallel to the direction of wave propagation, so that electric and magnetic fields are orthogonal to $\boldsymbol{e}_3$.

For axially symmetric configuration we introduce cylindrical polar coordinates (see Figure 1, Section 3.2.2) in such a way that fields $\boldsymbol{E}, \boldsymbol{B}$ are orthogonal to the basis vector $\boldsymbol{e}_\varphi$ in each point.

This enables us to use in our calculations the explicit expressions for spinorial filed components (85) and (87).

6.1 Transverse plane waves

As mentioned above, we choose the coordinates system in such a way that in each point

$$\boldsymbol{E},\boldsymbol{B} \perp \boldsymbol{e}_3, \quad F^3 = \bar{F}^3 \equiv 0 \tag{104}$$

Let us now consider the matter wave equations (75) written in the form

$$\begin{gathered}\partial^{\mu\dot{\nu}}\eta_{\dot{\nu}} = +\lambda\xi^{\mu} \\ \partial_{\mu\dot{\nu}}\xi^{\dot{\nu}} = -\lambda\,\eta_{\mu}\end{gathered} \tag{105}$$

We can expand these equations using (33), (34), (85) and (87)

$$\begin{array}{lr}(\partial_0+\partial_3)\sqrt{\bar{F}^1-i\bar{F}^2}+(\partial_1-i\partial_2)\sqrt{\bar{F}^1+i\bar{F}^2} = +\lambda\sqrt{F^1-iF^2} & \text{(i)} \\ (\partial_1+i\partial_2)\sqrt{\bar{F}^1-i\bar{F}^2}+(\partial_0-\partial_3)\sqrt{\bar{F}^1+i\bar{F}^2} = +\lambda\sqrt{F^1+iF^2} & \text{(ii)} \\ (\partial_0-\partial_3)\sqrt{\bar{F}^1+i\bar{F}^2}+(-\partial_1-i\partial_2)\sqrt{\bar{F}^1-i\bar{F}^2} = -\lambda\sqrt{F^1+iF^2} & \text{(iii)} \\ (-\partial_1+i\partial_2)\sqrt{\bar{F}^1+i\bar{F}^2}+(\partial_0+\partial_3)\sqrt{\bar{F}^1-i\bar{F}^2} = -\lambda\sqrt{F^1-iF^2} & \text{(iv)}\end{array} \tag{106}$$

By adding (i) and (iv) we obtain

$$(\partial_0+\partial_3)(\bar{F}^1-i\bar{F}^2) = 0 \tag{107}$$

Similarly, by adding (ii) and (iii) we obtain

$$(\partial_0-\partial_3)(\bar{F}^1+i\bar{F}^2) = 0 \tag{108}$$

With the two remaining equations the whole system can be written as

$$\begin{gathered}(\partial_1+i\partial_2)(\bar{F}^1-i\bar{F}^2) = +2\lambda\sqrt{F^1+iF^2}\sqrt{\bar{F}^1-i\bar{F}^2} \\ (\partial_0-\partial_3)(\bar{F}^1+i\bar{F}^2) = 0 \\ (\partial_0+\partial_3)(\bar{F}^1-i\bar{F}^2) = 0 \\ (\partial_1-i\partial_2)(\bar{F}^1+i\bar{F}^2) = +2\lambda\sqrt{F^1-iF^2}\sqrt{\bar{F}^1+i\bar{F}^2}\end{gathered} \tag{109}$$

and complex conjugated equations will have the form

$$\begin{gathered}(\partial_1-i\partial_2)(F^1+iF^2) = +2\bar{\lambda}\sqrt{F^1+iF^2}\sqrt{\bar{F}^1-i\bar{F}^2} \\ (\partial_0-\partial_3)(F^1-iF^2) = 0 \\ (\partial_0+\partial_3)(F^1+iF^2) = 0 \\ (\partial_1+i\partial_2)(F^1-iF^2) = +2\bar{\lambda}\sqrt{F^1-iF^2}\sqrt{\bar{F}^1+i\bar{F}^2}\end{gathered} \tag{110}$$

With expressions (89-91) for the momentum density of the matter field P_{μ} in the frame $M_{\perp}$, we can rewrite (110) in the following form:

$$\begin{gathered}(\partial_1-i\partial_2)(F^1+iF^2) = \bar{\lambda}(P^0+P^3) \\ (\partial_0-\partial_3)(F^1-iF^2) = 0 \\ (\partial_0+\partial_3)(F^1+iF^2) = 0 \\ (\partial_1+i\partial_2)(F^1-iF^2) = \bar{\lambda}(P^0-P^3)\end{gathered} \tag{111}$$

From comparison of (110) and (67) we conclude that in the case of the transverse plane waves:

- Matter field equations are reduced to the Maxwell equations, and

- The charge density current J^{μ} is expressed via electromagnetic field invariant $\bar{\lambda}$ and momentum density P^{μ} in the following way:

$$J^{\mu} = \bar{\lambda}\, P^{\mu} \tag{112}$$

Hence we conclude that, in the case of the transverse plane waves, electromagnetic field invariant $\bar{\lambda}$ plays the role of the electromagnetic *charge density* (it is clear that λ plays the same role for anti-particles). Generally $\bar{\lambda}$ is complex valued, hence allowing for *both* non-zero electric and magnetic charge densities.

Now we can find the expression for the Lorentz force density (64) acting on matter fields.

(112) can be written as

$$\begin{aligned} S_{\mu\dot{\nu}} &= \bar{\lambda} P_{\mu\dot{\nu}} = \bar{\lambda}\left(p_{\mu\dot{\nu}} + \hat{p}_{\mu\dot{\nu}}\right) = \bar{\lambda}\left(\xi_{\mu}\xi_{\dot{\nu}} + \eta_{\mu}\eta_{\dot{\nu}}\right) \\ \dot{S}_{\mu\dot{\nu}} &= \lambda P_{\mu\dot{\nu}} = \lambda\left(p_{\mu\dot{\nu}} + \hat{p}_{\mu\dot{\nu}}\right) = \lambda\left(\xi_{\mu}\xi_{\dot{\nu}} + \eta_{\mu}\eta_{\dot{\nu}}\right) \end{aligned} \tag{113}$$

Using (74) we find that

$$\begin{aligned} f_{\mu}^{\delta}\dot{S}_{\delta\dot{\nu}} &= \lambda\left(f_{\mu}^{\delta}\xi_{\delta}\xi_{\dot{\nu}} + f_{\mu}^{\delta}\eta_{\delta}\eta_{\dot{\nu}}\right) = \lambda^{2}\left(-\xi_{\mu}\xi_{\dot{\nu}} + \eta_{\mu}\eta_{\dot{\nu}}\right) \\ \dot{f}_{\dot{\nu}}^{\dot{\rho}} S_{\mu\dot{\rho}} &= \bar{\lambda}\left(\xi_{\mu}\dot{f}_{\dot{\nu}}^{\dot{\rho}}\xi_{\dot{\rho}} + \eta_{\mu}\dot{f}_{\dot{\nu}}^{\dot{\rho}}\eta_{\dot{\rho}}\right) = \bar{\lambda}^{2}\left(-\xi_{\mu}\xi_{\dot{\nu}} + \eta_{\mu}\eta_{\dot{\nu}}\right) \end{aligned} \tag{114}$$

and the Lorentz force density becomes

$$\Lambda_{\mu\dot{\nu}} = -\left[\dot{f}_{\dot{\nu}}^{\dot{\rho}} S_{\mu\dot{\rho}} + f_{\mu}^{\delta}\dot{S}_{\delta\dot{\nu}}\right] = \left(\lambda^{2} + \bar{\lambda}^{2}\right)\left(\xi_{\mu}\xi_{\dot{\nu}} - \eta_{\mu}\eta_{\dot{\nu}}\right) = \left(\lambda^{2} + \bar{\lambda}^{2}\right)P_{A_{\mu\dot{\nu}}} \tag{115}$$

It is interesting that the Lorentz force $\mathcal{F}^{\mu}$ is proportional to the axial vector current ${P_A}^{\mu}$ (see Section 5.5). From (115) we can see that Lorentz force vanishes if $E = B$, i.e. when real parts of electromagnetic fields invariants λ and $\bar{\lambda}$ are zero. Particularly, this is the case of electromagnetic waves "in vacuum" (i.e. *photons*, see below) and *neutrino* (see Section 7). When Lorentz force is zero, the momentum density of the matter field remains constant in the course of particle's motion, hence allowing for *uniform* motion of the particle.

In the case of plane electromagnetic waves "in vacuum" ($\boldsymbol{E} \perp \boldsymbol{B}$, $E = B$) we have $\lambda = \bar{\lambda} = 0$, and matter field equations (111) coincide with the "source-free" Maxwell equations (68). In this case the momentum density P^{μ} of the matter field is non-zero, while the charge density J^{μ} is zero. In this sense our system is not actually "source-free".

Thus we have demonstrated that our model can be used for description of bosonic particles such as *photons*. In further sections we will apply our model to the field configurations corresponding to *charged fermions* (Section 6.2) and *neutrino* (Section 7).

6.2 Stationary field configurations with axial symmetry

In Cartesian basis matter field equations

$$\begin{aligned} \partial^{\mu\dot{\nu}}\eta_{\dot{\nu}} &= +\lambda\xi^{\mu} \\ \partial_{\mu\dot{\nu}}\xi^{\mu} &= -\bar{\lambda}\eta_{\dot{\nu}} \end{aligned} \tag{116}$$

can be written in the following matrix form

$$(\partial_0 + \partial_1\sigma_1 + \partial_2\sigma_2 + \partial_3\sigma_3)\dot{\eta} = +\lambda\xi$$
$$(\partial_0 - \partial_1\sigma_1 - \partial_2\sigma_2 - \partial_3\sigma_3)\xi = -\bar{\lambda}\dot{\eta} \tag{117}$$

where

$$\xi = \begin{bmatrix}\xi^1\\ \xi^2\end{bmatrix}, \qquad \dot{\eta} = \begin{bmatrix}\eta_{\dot{1}}\\ \eta_{\dot{2}}\end{bmatrix} \tag{118}$$

By introducing polar cylindrical coordinates (see Figure 1)

$$\begin{gathered} x^2 = \rho\cos\varphi \\ x^3 = \rho\sin\varphi \\ x^1 = x^1 \end{gathered} \tag{119}$$

we will have the following expressions for partial derivatives

$$\begin{gathered} \partial_0 = \partial_0 \\ \partial_2 = \cos\varphi\,\partial_\rho - \frac{\sin\varphi}{\rho}\partial_\varphi \\ \partial_3 = \sin\varphi\,\partial_\rho + \frac{\cos\varphi}{\rho}\partial_\varphi \\ \partial_1 = \partial_1 \end{gathered} \tag{120}$$

By using (120) we obtain

$$\partial_2\sigma_2 + \partial_3\sigma_3 = (\sigma_2\cos\varphi + \sigma_3\sin\varphi)\partial_\rho + \frac{1}{\rho}(-\sigma_2\sin\varphi + \sigma_3\cos\varphi)\partial_\varphi \tag{121}$$

with consequent expressions for matter field equations (117) in the new coordinate system.

To complete the transition to the polar coordinate system, we need to account for *change of spinor components* due to change of basis vectors in each point: $\{\boldsymbol{e}_2, \boldsymbol{e}_3\} \to \{\boldsymbol{e}_\rho, \boldsymbol{e}_\varphi\}$.

The pair of vectors $\{\boldsymbol{e}_\rho, \boldsymbol{e}_\varphi\}$ can be obtained by rotating the pair of Cartesian basis vectors $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$ by the angle φ around the axis $\boldsymbol{e}_1$ at every point with coordinates (ρ, φ, x^1). This rotation results in the following transformation of the spinor components:

$$\begin{gathered} \xi' = \exp\left[i\frac{\varphi}{2}\sigma_1\right]\xi = \left[\cos\frac{\varphi}{2} + i\sigma_1\sin\frac{\varphi}{2}\right]\xi = S\xi \\ \dot{\eta}' = \exp\left[i\frac{\varphi}{2}\sigma_1\right]\dot{\eta} = \left[\cos\frac{\varphi}{2} + i\sigma_1\sin\frac{\varphi}{2}\right]\dot{\eta} = S\dot{\eta} \end{gathered} \tag{122}$$

with the following transition operators

$$S = \exp\left[i\frac{\varphi}{2}\sigma_1\right] \quad S^{-1} = \exp\left[-i\frac{\varphi}{2}\sigma_1\right] \tag{123}$$

By applying operators S and S^{-1} to the equations (117) we obtain

$$\begin{gathered} S\,(\partial_0 + \partial_1\sigma_1 + \partial_2\sigma_2 + \partial_3\sigma_3)S^{-1}S\dot{\eta} = +\lambda S\xi \\ S(\partial_0 - \partial_1\sigma_1 - \partial_2\sigma_2 - \partial_3\sigma_3)S^{-1}S\xi = -\bar{\lambda}S\dot{\eta} \end{gathered} \tag{124}$$

or

$$\begin{gathered} S\,(\partial_0 + \partial_1\sigma_1 + \partial_2\sigma_2 + \partial_3\sigma_3)S^{-1}\dot{\eta}' = +\lambda\xi' \\ S(\partial_0 - \partial_1\sigma_1 - \partial_2\sigma_2 - \partial_3\sigma_3)S^{-1}\xi' = -\bar{\lambda}\dot{\eta}' \end{gathered} \tag{125}$$

Since operator S commutes with σ_0 and σ_1, we will have

$$\begin{aligned} S\,(\partial_0+\partial_1\sigma_1+\partial_2\sigma_2+\partial_3\sigma_3)S^{-1} &= \partial_0+\partial_1\sigma_1+S(\partial_2\sigma_2+\partial_3\sigma_3)S^{-1} \\ S(\partial_0-\partial_1\sigma_1-\partial_2\sigma_2-\partial_3\sigma_3)S^{-1} &= \partial_0-\partial_1\sigma_1-S(\partial_2\sigma_2+\partial_3\sigma_3)S^{-1} \end{aligned} \tag{126}$$

It is now easy to check that

$$S(\partial_2\sigma_2+\partial_3\sigma_3)S^{-1}=\sigma_2\partial_\rho+\frac{1}{\rho}\sigma_3\partial_\varphi+\frac{1}{2\rho}\sigma_2 \tag{127}$$

and we complete transition of the matter field equations to polar cylindrical coordinates:

$$\begin{aligned} \left[\partial_0+\sigma_1\partial_1+\sigma_2\partial_\rho+\frac{1}{\rho}\sigma_3\partial_\varphi+\frac{1}{2\rho}\sigma_2\right]\dot{\eta}' &= +\lambda\xi' \\ \left[\partial_0-\sigma_1\partial_1-\sigma_2\partial_\rho-\frac{1}{\rho}\sigma_3\partial_\varphi-\frac{1}{2\rho}\sigma_2\right]\xi' &= -\bar{\lambda}\dot{\eta}' \end{aligned} \tag{128}$$

Now we can use (85) and (87) to express components of the spinors ξ' and $\dot{\eta}'$ via components of the fields $\boldsymbol{E},\boldsymbol{B}$ in *cylindrical coordinates* (i.e. in the basis $\{\boldsymbol{e}_\rho,\boldsymbol{e}_\varphi,\boldsymbol{e}_1\}$):

$$\begin{aligned} \xi' &= \begin{bmatrix}\xi'^1\\ \xi'^2\end{bmatrix}=\begin{bmatrix}\sqrt{F^1-iF^\rho}\\ \sqrt{F^1+iF^\rho}\end{bmatrix} \\ \dot{\eta}' &= \begin{bmatrix}\eta'_{\dot{1}}\\ \eta'_{\dot{2}}\end{bmatrix}=\begin{bmatrix}\sqrt{\bar{F}^1-i\bar{F}^\rho}\\ \sqrt{\bar{F}^1+i\bar{F}^\rho}\end{bmatrix} \end{aligned} \tag{129}$$

and write matter field equations as follows:

$$\begin{aligned} \begin{bmatrix}\partial_0+\frac{1}{\rho}\partial_\varphi & \partial_1-i\partial_\rho-i\frac{1}{2\rho}\\ \partial_1+i\partial_\rho+i\frac{1}{2\rho} & \partial_0-\frac{1}{\rho}\partial_\varphi\end{bmatrix}\begin{bmatrix}\sqrt{\bar{F}^1-i\bar{F}^\rho}\\ \sqrt{\bar{F}^1+i\bar{F}^\rho}\end{bmatrix} &= +\lambda\begin{bmatrix}\sqrt{F^1-iF^\rho}\\ \sqrt{F^1+iF^\rho}\end{bmatrix} \\ \begin{bmatrix}\partial_0-\frac{1}{\rho}\partial_\varphi & -\partial_1+i\partial_\rho+i\frac{1}{2\rho}\\ -\partial_1-i\partial_\rho-i\frac{1}{2\rho} & \partial_0+\frac{1}{\rho}\partial_\varphi\end{bmatrix}\begin{bmatrix}\sqrt{F^1-iF^\rho}\\ \sqrt{F^1+iF^\rho}\end{bmatrix} &= -\bar{\lambda}\begin{bmatrix}\sqrt{\bar{F}^1-i\bar{F}^\rho}\\ \sqrt{\bar{F}^1+i\bar{F}^\rho}\end{bmatrix} \end{aligned} \tag{130}$$

After expanding expressions (130) and applying complex conjugation to the first two equations, we obtain

$$\begin{aligned} &\left(\partial_0+\frac{1}{\rho}\partial_\varphi\right)\sqrt{F^1+iF^\rho}+\left(\partial_1+i\partial_\rho+i\frac{1}{2\rho}\right)\sqrt{F^1-iF^\rho}=+\bar{\lambda}\sqrt{\bar{F}^1+i\bar{F}^\rho} && \text{(i)} \\ &\left(\partial_1-i\partial_\rho-i\frac{1}{2\rho}\right)\sqrt{F^1+iF^\rho}+\left(\partial_0-\frac{1}{\rho}\partial_\varphi\right)\sqrt{F^1-iF^\rho}=+\bar{\lambda}\sqrt{\bar{F}^1-i\bar{F}^\rho} && \text{(ii)} \\ &\left(\partial_0-\frac{1}{\rho}\partial_\varphi\right)\sqrt{F^1-iF^\rho}+\left(-\partial_1+i\partial_\rho+i\frac{1}{2\rho}\right)\sqrt{F^1+iF^\rho}=-\bar{\lambda}\sqrt{\bar{F}^1-i\bar{F}^\rho} && \text{(iii)} \\ &\left(-\partial_1-i\partial_\rho-i\frac{1}{2\rho}\right)\sqrt{F^1-iF^\rho}+\left(\partial_0+\frac{1}{\rho}\partial_\varphi\right)\sqrt{F^1+iF^\rho}=-\bar{\lambda}\sqrt{\bar{F}^1+i\bar{F}^\rho} && \text{(iv)} \end{aligned} \tag{131}$$

By adding (i) and (iv) we obtain

$$\left(\partial_0 + \frac{1}{\rho}\partial_\varphi\right)(F^1 + iF^\rho) = 0 \tag{132}$$

Similarly, by adding (ii) and (iii) we obtain

$$\left(\partial_0 - \frac{1}{\rho}\partial_\varphi\right)(F^1 - iF^\rho) = 0 \tag{133}$$

Naturally, (132) and (133) are consistent with assumed stationarity and axial symmetry of the field configuration:

$$\begin{aligned} \partial_0 &\equiv 0 \quad \textit{stationarity} \\ \partial_\varphi &\equiv 0 \quad \textit{axial symmetry} \end{aligned} \tag{134}$$

With the two remaining equations the whole system can be written as

$$\begin{gathered} (\partial_1 - i\partial_\rho)(F^1 + iF^\rho) - i\frac{1}{\rho}(F^1 + iF^\rho) = +2\bar{\lambda}\sqrt{\bar{F}^1 - i\bar{F}^\rho}\sqrt{F^1 + iF^\rho} \\ \left(\partial_0 - \frac{1}{\rho}\partial_\varphi\right)(F^1 - iF^\rho) = 0 \\ \left(\partial_0 + \frac{1}{\rho}\partial_\varphi\right)(F^1 + iF^\rho) = 0 \\ (\partial_1 + i\partial_\rho)(F^1 - iF^\rho) + i\frac{1}{\rho}(F^1 - iF^\rho) = +2\bar{\lambda}\sqrt{\bar{F}^1 + i\bar{F}^\rho}\sqrt{F^1 - iF^\rho} \end{gathered} \tag{135}$$

From comparison with (70) we conclude that (135) will coincide with Maxwell equations if we define:

$$\begin{aligned} J^0 - J^\varphi &= \bar{\lambda}(P^0 - P^\varphi) - i\frac{1}{\rho}F^1 \\ J^0 + J^\varphi &= \bar{\lambda}(P^0 + P^\varphi) + i\frac{1}{\rho}F^1 \end{aligned} \tag{136}$$

where J^0, J^φ and P^0, P^φ are non-zero components of charge density and momentum density correspondingly.

It is interesting that in axially symmetric case

$$\begin{aligned} J^0 &= \bar{\lambda}P^0 \\ J^\varphi &= \bar{\lambda}P^\varphi + 2i\frac{F^1}{\rho} \end{aligned} \tag{137}$$

and hence the ratio $\frac{J^\varphi}{J^0}$ is not equal to the ratio $\frac{P^\varphi}{P^0}$. Consequently, the "velocity of charge" is not the same as "velocity of mass" anymore.

To understand the meaning of the new term in (137) added to J^φ let us consider the Lorentz force acting on matter fields.

From (136) we can see that the charge density spinor $S_{\mu\dot{\nu}}$ can be split into two components

$$S_{\mu\dot{\nu}} = \bar{\lambda}P_{\mu\dot{\nu}} + M_{\mu\dot{\nu}} \tag{138}$$

where $M_{\mu\dot{\nu}}$ is as follows:

$$\left(M_{\mu\dot{\nu}}\right) = \begin{bmatrix} M_{1\dot{1}} & M_{1\dot{2}} \\ M_{2\dot{1}} & M_{2\dot{2}} \end{bmatrix} = \begin{bmatrix} +i\dfrac{1}{\rho}F^1 & 0 \\ 0 & -i\dfrac{1}{\rho}F^1 \end{bmatrix} \tag{139}$$

Similarly, Hermitian conjugate charge density spinor

$$\dot{S}_{\mu\dot{\nu}} = \lambda P_{\mu\dot{\nu}} + \dot{M}_{\mu\dot{\nu}} \tag{140}$$

where

$$\left(\dot{M}_{\mu\dot{\nu}}\right) = \begin{bmatrix} \dot{M}_{1\dot{1}} & \dot{M}_{1\dot{2}} \\ \dot{M}_{2\dot{1}} & \dot{M}_{2\dot{2}} \end{bmatrix} = \begin{bmatrix} +i\dfrac{1}{\rho}F^1 & 0 \\ 0 & -i\dfrac{1}{\rho}F^1 \end{bmatrix} \tag{141}$$

Using (64) we find that the additional term makes the following contribution to the Lorentz force density spinor

$$\Lambda_{\mathrm{M}\,\mu\dot{\nu}} = -\left[\dot{f}_{\dot{\nu}}^{\dot{\rho}} M_{\mu\dot{\rho}} + f_{\mu}^{\delta} \dot{M}_{\delta\dot{\nu}}\right] = \begin{bmatrix} 0 & -2\dfrac{i}{\rho}F^1(\bar{F}^1 + i\bar{F}^\rho) \\ +2\dfrac{i}{\rho}F^1(\bar{F}^1 - i\bar{F}^\rho) & 0 \end{bmatrix} \tag{142}$$

or, in terms of the world 4-vectors,

$$\left(\mathcal{F}_{M\mu}\right) = \begin{bmatrix} \mathcal{F}_{M0} \\ \mathcal{F}_{M1} \\ \mathcal{F}_{M\rho} \\ \mathcal{F}_{M\varphi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{2}{\rho}F^1\bar{F}^1 \\ 0 \end{bmatrix} \tag{143}$$

That means that additional term contributes to *centripetal acceleration* of the momentum density P_μ.

In principle, equations (135) can be resolved with respect to F^1 and F^ρ, therefore allowing for calculation of the particle's total mass M and charge Q:

$$M = \int_V P^0 dV, \qquad Q = \int_V J^0 dV \tag{144}$$

7 Neutrino model

7.1 General considerations

So far we have always been considering the matter field equations in the following forms:

$$\partial^{\mu\dot{\nu}}\eta_{\pm\dot{\nu}} = +\lambda_{\pm}\,\xi_{\pm}{}^{\mu}, \qquad \partial_{\mu\dot{\nu}}\xi_{\pm}{}^{\mu} = -\bar{\lambda}_{\pm}\,\eta_{\pm\dot{\nu}} \tag{145}$$

In (145) spinor ξ_+ is coupled with co-spinor $\dot{\eta}_+$, and spinor ξ_- is coupled with co-spinor $\dot{\eta}_-$.

Let us now consider a field configuration where spinor ξ_- is coupled with co-spinor $\dot{\eta}_+$. The matter field equations will be written as

$$\partial^{\mu\dot{\nu}}\eta_{+\dot{\nu}} = +\lambda_{-}\,\xi_{-}{}^{\mu}, \qquad \partial_{\mu\dot{\nu}}\xi_{-}{}^{\mu} = -\bar{\lambda}_{+}\,\eta_{+\dot{\nu}} \tag{146}$$

or, taking account that $\lambda_{\pm} = -\lambda_{\mp}$ (see (76))

$$\partial^{\mu\dot{\nu}} \eta_{+\dot{\nu}} = +\lambda_{-}\, \xi_{-}{}^{\mu}, \qquad \partial_{\mu\dot{\nu}} \xi_{-}{}^{\mu} = +\bar{\lambda}_{-}\, \eta_{+\dot{\nu}} \tag{147}$$

According to (85) and (87) the components of the spinor and co-spinor fields will be

$$\begin{aligned} \xi_{-}^{1} &= -\sqrt{F^1 - iF^2} \quad & \eta_{+1} &= +\sqrt{\overline{F^1} - i\overline{F^2}} \\ \xi_{-}^{2} &= +\sqrt{F^1 + iF^2} \quad & \eta_{+2} &= +\sqrt{\overline{F^1} + i\overline{F^2}} \end{aligned} \tag{148}$$

and hence the Lorentz invariant *Majorana condition* will be satisfied:

$$\begin{aligned} \xi^1 &= -\eta_2 \\ \xi^2 &= +\eta_1 \end{aligned} \tag{149}$$

We put (149) in the matter field equations (147)

$$\begin{bmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{bmatrix} \begin{bmatrix} \overline{\xi^2} \\ -\overline{\xi^1} \end{bmatrix} = +\lambda_{-} \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix}$$
$$\begin{bmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} = +\bar{\lambda}_{-} \begin{bmatrix} \overline{\xi^2} \\ -\overline{\xi^1} \end{bmatrix} \tag{150}$$

and after expansion of the formulas and complex conjugation of the first pair of equations we obtain

$$\left.\begin{aligned} (\partial_0 + \partial_3)\xi^2 - (\partial_1 + i\partial_2)\xi^1 &= +\bar{\lambda}_{-}\, \overline{\xi^1} \\ (\partial_1 - i\partial_2)\xi^2 - (\partial_0 - \partial_3)\xi^1 &= +\bar{\lambda}_{-}\, \overline{\xi^2} \end{aligned}\right\}$$
$$\left.\begin{aligned} (\partial_0 - \partial_3)\xi^1 - (\partial_1 - i\partial_2)\xi^2 &= +\bar{\lambda}_{-}\, \overline{\xi^2} \\ -(\partial_1 + i\partial_2)\xi^1 + (\partial_0 + \partial_3)\xi^2 &= +\bar{\lambda}_{-}\, \overline{\xi^1} \end{aligned}\right\} \tag{151}$$

From (151) it is clear that, due to Majorana condition, the two matter field equations (146) become equivalent to each other, hence only one of these equations is *independent*.

Let us now find the expressions for divergences of spinorial currents and momentum density.

With matter field equations (147) one can easily find that

$$\partial_{\mu\dot{\nu}} p^{\mu\dot{\nu}} = \partial_{\mu\dot{\nu}} \left[\xi^{\mu} \xi^{\dot{\nu}}\right] = \left(\left[\partial_{\mu\dot{\nu}} \xi^{\mu}\right]\xi^{\dot{\nu}} + \xi^{\mu}\left[\partial_{\mu\dot{\nu}} \xi^{\dot{\nu}}\right]\right) = \left(-\bar{\lambda}_{+}\, \eta_{+\dot{\nu}} \xi_{-}{}^{\dot{\nu}} - \lambda_{+} \eta_{+\mu} \xi_{-}{}^{\mu}\right) = 0$$
$$\partial_{\mu\dot{\nu}} \hat{p}^{\mu\dot{\nu}} = \partial^{\mu\dot{\nu}} \left[\eta_{\mu} \eta_{\dot{\nu}}\right] = \left(\left[\partial^{\mu\dot{\nu}} \eta_{\mu}\right]\eta_{\dot{\nu}} + \eta_{\mu}\left[\partial^{\mu\dot{\nu}} \eta_{\dot{\nu}}\right]\right) = \left(\bar{\lambda}_{-}\, \eta_{+\dot{\nu}} \xi_{-}{}^{\dot{\nu}} + \lambda_{-}\, \xi_{-}{}^{\mu} \eta_{+\mu}\right) = 0 \tag{152}$$

In (152) we used the invariant properties (94).

Consequently we conclude that both spinorial currents p_{μ} and $\hat{p}_{\mu}$, as well as momentum density current P_{μ} are conserved.

7.2 longitudinal plane waves

In Section 5.4 we have demonstrated that spinorial currents p^{μ} and $\hat{p}^{\mu}$ are conserved independently when $E = B$, i.e. when real parts of the squared electromagnetic fields invariants λ^2 and $\bar{\lambda}^2$ are zero.

It is known (see [6]) that if $E = B$ and electric and magnetic fields vectors $\{\boldsymbol{E}, \boldsymbol{B}\}$ are *not* orthogonal to each other, there exist a reference frame where these vectors are parallel to each other: $\boldsymbol{E} \parallel \boldsymbol{B},\ E = B$.

Let's denote this frame as $M_{\parallel}$ and assume for simplicity that both $\boldsymbol{E}$ and $\boldsymbol{B}$ are directed along the axis $\boldsymbol{e}_1$:

$$F^1 \neq 0 \quad F^2 = F^3 = 0 \tag{153}$$

In the frame $M_{\parallel}$ the components of the spinors (148) will have the form:

$$\begin{aligned} \xi_-^1 &= -\sqrt{F^1} \quad \eta_{+\dot{1}} = +\sqrt{F^1} \\ \xi_-^2 &= +\sqrt{F^1} \quad \eta_{+\dot{2}} = +\sqrt{F^1} \end{aligned} \tag{154}$$

If we denote

$$\zeta = \sqrt{F^1} \tag{155}$$

then spinorial currents (78) can be written as

$$\begin{matrix} p_0 = +\zeta\bar{\zeta} & p_1 = -\zeta\bar{\zeta} \\ p_2 = 0 & p_3 = 0 \end{matrix} \tag{156}$$

and

$$\begin{matrix} \hat{p}^0 = +\zeta\bar{\zeta} & \hat{p}^1 = +\zeta\bar{\zeta} \\ \hat{p}^2 = 0 & \hat{p}^3 = 0 \end{matrix} \tag{157}$$

Consequently, spatial parts of both spinorial currents p_μ and $\hat{p}_\mu$, as well as momentum density vector P_μ, are *opposite in direction to the axis* $\boldsymbol{e}_1$, while the momentum density 4-vector is isotropic: $P^\mu P_\mu = 0$. This is the first indication that, *in spite of* non-zero "mass term" λ in the matter field equations, the neutrino field is "moving" at the speed of light.

Let us now rewrite the matter field equations (151) in the frame $M_{\parallel}$.

$$\begin{aligned} (\partial_0 + \partial_3)\zeta + (\partial_1 + i\partial_2)\zeta &= -\bar{\lambda}_-\,\bar{\zeta} \\ (\partial_1 - i\partial_2)\zeta + (\partial_0 - \partial_3)\zeta &= +\bar{\lambda}_-\,\bar{\zeta} \end{aligned} \tag{158}$$

By adding and subtracting these equations we obtain:

$$\begin{aligned} (\partial_0 + \partial_1)\zeta &= 0 \\ (\partial_3 + i\partial_2)\zeta &= -2\,\bar{\lambda}_-\,\bar{\zeta} \end{aligned} \tag{159}$$

The first equation means that the field ζ is "moving" at the speed of light in the direction opposite to the axis $\boldsymbol{e}_1$. That means that in the frame $M_{\parallel}$ the neutrino field is a *longitudinal* wave, i.e. the wave propagating parallel to the direction of the fields $\boldsymbol{E}, \boldsymbol{B}$.

The second equation in (159) can be further expressed in terms of the field ζ taking account that $\bar{\lambda}_- = -(\bar{\zeta})^2$:

$$(\partial_3 + i\partial_2)\zeta = +2\,(\bar{\zeta})^3 \tag{160}$$

The Maxwell equations in the chosen frame $M_{\parallel}$ will have the form

$$
\begin{aligned}
(\partial_1 + i\partial_2)F^1 &= J^0 - J^3 && \text{(i)} \\
(\partial_0 - \partial_3)F^1 &= -J^1 + iJ^2 && \text{(ii)} \\
(\partial_0 + \partial_3)F^1 &= -J^1 - iJ^2 && \text{(iii)} \\
(\partial_1 - i\partial_2)F^1 &= J^0 + J^3 && \text{(iv)}
\end{aligned} \tag{161}
$$

By adding all equations we will obtain:

$$(\partial_0 + \partial_1)F^1 = J^0 - J^1 \tag{162}$$

By adding and subtracting equations $[(\mathrm{i}) - (\mathrm{ii}) + (\mathrm{iii}) - (\mathrm{iv})]$ we obtain

$$(\partial_3 + i\partial_2)F^1 = -J^3 - iJ^2 \tag{163}$$

Hence the Maxwell equations will be consistent with matter field equations if the following relationships are satisfied:

$$
\begin{aligned}
J^0 - J^1 &= 0 \\
-J^3 - iJ^2 &= 4\zeta(\bar{\zeta})^3
\end{aligned} \tag{164}
$$

In Section 6.1 we have demonstrated that in the case of the transverse plane electromagnetic waves “in vacuum” the charge density was zero while the momentum density of the matter field was non-zero.

From (164) we conclude that in our model of neutrino the components of the charge density J^2 and J^3might be non-zero while the components of the momentum density P^2 and P^3 are both zero.

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