Exact solution of the reduced version of PWE (*paraxial wave equation*) in bipolar coordinate system.

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A new type of exact solution of the reduced 3 dimensional *spatial* PWE (paraxial wave equation) for the case of bipolar coordinates is presented here.
First, we consider a self-similar representation of the solution *in a bipolar coordinate system*, the second we additionally reduce PWE under a proper paraxial assumption.
Analyzing the structure of the final equation, we obtain the simple exact solution which is proved to satisfy to such an equation in *bipolar coordinates*.
Besides, there is a limitation of the components of self-similar solution of a new type.

1. Introduction.

The full 3 dimensional spatial PWE (paraxial wave equation) should be presented in a bipolar coordinate system σ, τ, z as below [1-2]:

$$\Delta_t \psi + 2ik\frac{\partial \psi}{\partial z} = 0, \qquad (1.1)$$

- here ψ - is the slowly varying complex envelope of a paraxial field, satisfying the PWE, $\psi = \psi(\sigma, \tau, z), \sigma \in [0, 2\pi), \tau \in (-\infty, \infty)$; k - is the wave number; $i = \sqrt{-1}$ - is the imaginary unit.

Besides, in bipolar coordinate system [3-4]:

$$\Delta \psi = \frac{(\cosh\tau - \cos\sigma)^2}{a^2}\left(\frac{\partial^2 \psi}{\partial \sigma^2} + \frac{\partial^2 \psi}{\partial \tau^2}\right) + \frac{\partial^2 \psi}{\partial z^2} ,$$

$$\Delta_t \psi = \frac{(\cosh\tau - \cos\sigma)^2}{a^2}\left(\frac{\partial^2 \psi}{\partial \sigma^2} + \frac{\partial^2 \psi}{\partial \tau^2}\right) .$$

In accordance with the assumption of self-similarity, we should find an exact solution of equation (1.1) in a form below:

$$\psi(\sigma, \tau, z) = U(\sigma) \cdot V(\tau) \cdot Z(z), \qquad (1.2)$$

- then having substituted (1.2) into (1.1), we obtain ($\psi \neq 0$):

$$\frac{(\cosh\tau - \cos\sigma)^2}{a^2} \cdot \left(\frac{1}{U(\sigma)}\frac{\partial^2 U(\sigma)}{\partial \sigma^2} + \frac{1}{V(\tau)}\frac{\partial^2 V(\tau)}{\partial \tau^2}\right) + \frac{2ik}{Z(z)} \cdot \frac{\partial Z(z)}{\partial z} = 0, \qquad (1.3)$$

- where for self-similarity we should choose as below (*m* – is the real number):

$$-\frac{2ik\cdot z^2}{Z(z)}\cdot\frac{\partial Z(z)}{\partial z} = m \tag{1.4}$$

Let us express the Cartesian components x, y in bipolar coordinate system [4]:

$$x = a\frac{\sinh\tau}{(\cosh\tau-\cos\sigma)}, \quad y = a\frac{\sin\sigma}{(\cosh\tau-\cos\sigma)},$$

- but *if the paraxial assumption is valid for PWE*, it means

$$\left(\frac{a^2}{z^2}\right)\frac{\sinh^2\tau+\sin^2\sigma}{(\cosh\tau-\cos\sigma)^2} = \frac{x^2+y^2}{z^2} = \varepsilon^2 \to 0, \tag{1.5}$$

Thus, we could present Eq. (1.3) in a form below, taking into account the assumptions (1.4)-(1.5):

$$\begin{aligned}&(\sinh^2\tau+\sin^2\sigma)\cdot\left(\frac{1}{U(\sigma)}\frac{\partial^2 U(\sigma)}{\partial\sigma^2}+\frac{1}{V(\tau)}\frac{\partial^2 V(\tau)}{\partial\tau^2}\right) = \\ &= m\left(\frac{a^2}{z^2}\right)\frac{\sinh^2\tau+\sin^2\sigma}{(\cosh\tau-\cos\sigma)^2} = m\varepsilon^2 \to 0,\end{aligned} \tag{1.6}$$

- besides, we should obtain *the reduced version* of PWE as below:

$$\begin{aligned}&(\sinh^2\tau+\sin^2\sigma)\cdot\left(\frac{1}{U(\sigma)}\frac{\partial^2 U(\sigma)}{\partial\sigma^2}+\frac{1}{V(\tau)}\frac{\partial^2 V(\tau)}{\partial\tau^2}\right) \cong 0 \\ &\Rightarrow \quad \frac{1}{U(\sigma)}\frac{\partial^2 U(\sigma)}{\partial\sigma^2} = c^2 = -\frac{1}{V(\tau)}\frac{\partial^2 V(\tau)}{\partial\tau^2}\end{aligned} \tag{1.7}$$

2. Exact solution of reduced PWE.

Adopting the assumption of self-similarity (1.2) as well as *the paraxial assumption* (1.6), we finally obtain the united system of equations (1.4)-(1.7):

$$\frac{1}{U(\sigma)} \frac{d^2 U(\sigma)}{d\sigma^2} = c^2 ,$$

$$\frac{1}{V(\tau)} \frac{d^2 V(\tau)}{d\tau^2} = -c^2 , \qquad (1.8)$$

$$\frac{2ik \cdot z^2}{Z(z)} \cdot \frac{dZ(z)}{dz} = -m ,$$

- where the 1-st and 2-nd equation of (1.8) has a typical solution as below ($c \neq 0$):

$$U(\sigma) = A_c \cdot ch(\sigma \cdot |c|) + B_c \cdot sh(\sigma \cdot |c|) ,$$

$$V(\tau) = A_{-c} \cdot \cos(\tau \cdot |c|) + B_{-c} \cdot \sin(\tau \cdot |c|) ,$$

- let us remember that $\sigma \in [0, 2\pi)$, $\tau \in (-\infty, \infty)$. It means a limitation of the components of self-similar solution; besides, for the 3-d equation we obtain:

$$Z(z) = Z_0 \cdot \exp\left(-\frac{im}{2kz}\right) .$$

3. Conclusion.

Let us present finally the self-similar solution of the reduced version of PWE (paraxial wave equation, under assumption (1.6)):

$$\psi(\sigma, \tau, z) = U(\sigma) \cdot V(\tau) \cdot Z(z),$$

- where

$$U(\sigma) = A_c \cdot ch(\sigma \cdot |c|) + B_c \cdot sh(\sigma \cdot |c|),$$

$$V(\tau) = A_{-c} \cdot \cos(\tau \cdot |c|) + B_{-c} \cdot \sin(\tau \cdot |c|),$$

$$Z(z) = Z_0 \cdot \exp\left(-\frac{i\,m}{2k\,z}\right),$$

- here $\sigma \in [0, 2\pi)$, $\tau \in (-\infty, \infty)$, $z \in (0, +\infty)$.

References:

1. A.Sommerfeld (1949). *Partial Differential Equations in Physics*. Academic Press, New York.

2. Serway, Moses, and Moyer (2004). *Modern Physics* (*3rd ed.*). Brooks Cole. ISBN 0534493408.

3. E.Kamke (1971). *Hand-book for ordinary differential equations*. Science, Moscow.

4. Svelto, Orazio (2010). *Principles of Lasers* (5th ed.).