INTEGRAL MEAN ESTIMATES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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ABSTRACT. Let $P(z)$ be a polynomial of degree n having all zeros in $|z| \leq k$ where $k \leq 1$, then it was proved by Dewan *et al* [6] that for every real or complex number α with $|\alpha| \geq k$ and each $r \geq 0$

$$n(|\alpha| - k) \left\{ \int_0^{2\pi} \left| P\left(e^{i\theta}\right) \right|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} \left| 1 + ke^{i\theta} \right|^r d\theta \right\}^{\frac{1}{r}} \underset{|z|=1}{Max} |D_\alpha P(z)|.$$

In this paper, we shall present a refinement and generalization of above result and also extend it to the class of polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, having all its zeros in $|z| \leq k$ where $k \leq 1$ and thereby obtain certain generalizations of above and many other known results.

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree n. It was shown by Turán [12] that if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$n\underset{|z|=1}{Max} |P(z)| \leq 2\underset{|z|=1}{Max} |P'(z)| . \tag{1.1}$$

Inequality (1.1) is best possible with equality holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$. The above inequality (1.1) of Turán [12] was generalized by Malik [10], who proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then

$$\underset{|z|=1}{Max} |P'(z)| \geq \frac{n}{1+k} \underset{|z|=1}{Max} |P(z)| . \tag{1.2}$$

where as for $k \geq 1$, Govil [7] showed that

$$\underset{|z|=1}{Max} |P'(z)| \geq \frac{n}{1+k^n} \underset{|z|=1}{Max} |P(z)| , \tag{1.3}$$

Both the above inequalities (1.2) and (1.3) are best possible, with equality in (1.2) holding for $P(z) = (z + k)^n$, where $k \geq 1$. While in (1.3) the equality holds for the polynomial $P(z) = \alpha z^n + \beta k^n$ where $|\alpha| = |\beta|$.

As a refinement of (1.2), Aziz and Shah [4] proved if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then

$$\underset{|z|=1}{Max} |P'(z)| \geq \frac{n}{1+k} \left\{ \underset{|z|=1}{Max} |P(z)| + \frac{1}{k^{n-1}} \underset{|z|=1}{Min} |P(z)| \right\}. \tag{1.4}$$

Let $D_\alpha P(z)$ denotes the polar derivative of the polynomial $P(z)$ of degree n with respect to the point α, then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is a polynomial of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that

$$\underset{\alpha \to \infty}{Lim} \left[\frac{D_\alpha P(z)}{\alpha} \right] = P'(z).$$

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Aziz and Rather [2] extends (1.2) to polar derivatives of a polynomial and proved that if all the zeros of $P(z)$ lie in $|z| \leq k$ where $k \leq 1$ then for every real or complex number α with $|\alpha| \geq k$,

$$\underset{|z|=1}{Max}\,|D_\alpha P(z)| \geq n\left(\frac{|\alpha|-k}{1+k}\right)\underset{|z|=1}{Max}\,|P(z)|\,. \tag{1.5}$$

For the class of polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, Aziz and Rather [3] proved that if α is real or complex number with $|\alpha| \geq k^\mu$ then

$$\underset{|z|=1}{Max}\,|D_\alpha P(z)| \geq n\left(\frac{|\alpha|-k^\mu}{1+k^\mu}\right)\underset{|z|=1}{Max}\,|P(z)|\,. \tag{1.6}$$

Malik [11] obtained a generalization of (1.1) in the sense that the left-hand side of (1.1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact he proved that if $P(z)$ has all its zeros in $|z| \leq 1$, then for each $q > 0$,

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right\}^{1/q}\underset{|z|=1}{Max}|P'(z)|. \tag{1.7}$$

If we let q tend to infinity in (1.7), we get (1.1).

The corresponding generalization of (1.2) which is an extension of (1.7) was obtained by Aziz [1] by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq$ where $k \geq 1$, then for each $q \geq 1$

$$n\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\underset{|z|=1}{Max}|P'(z)|. \tag{1.8}$$

The result is best possible and equality in (1.5) holds for the polynomial $P(z) = \alpha z^n + \beta k^n$ where $|\alpha| = |\beta|$.

As a generalization of inequality 1.5, Dewan *et al* [6] obtained an L^p inequality for the polar derivative of a polynomial and proved the following:

Theorem 1.1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex number α with $|\alpha| \geq k$ and for each $r > 0$,*

$$n(|\alpha|-k)\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi}\left|1+ke^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}}\underset{|z|=1}{Max}|D_\alpha P(z)|. \tag{1.9}$$

In this paper, we consider the class of polynomials $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, having all its zeros in $|z| \leq k$ where $k \leq 1$ and establish some improvements and generalizations of inequalities (1.1),(1.2),(1.5),(1.8) and (1.9).

In this direction, we first present the following interesting results which yields (1.9) as a special case.

Theorem 1.2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex α, β with $|\alpha| \geq k$, $|\beta| \leq 1$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n(|\alpha|-k)\left\{\int_0^{2\pi}\left|P(e^{i\theta})+\beta\frac{m}{k^{n-1}}\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi}|1+ke^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}} \tag{1.10}$$

where $m = Min_{|z|=k}|P(z)|$.

If we take $\beta = 0$, we get the following result.

Corollary 1.3. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex α, with $|\alpha| \geq k$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n(|\alpha| - k)\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1 + ke^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}}. \quad (1.11)$$

Remark 1.4. Theorem 1.1 follows from (1.11) by letting $q \to \infty$ (so that $p \to 1$) in Corollary 1.3. If we divide both sides of inequality (1.11) by $|\alpha|$ and make $\alpha \to \infty$, we get (1.5).

Dividing the two sides of (1.10) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following result.

Corollary 1.5. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex β with $|\beta| \leq 1$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n\left\{\int_0^{2\pi} \left|P(e^{i\theta}) + \beta\frac{m}{k^{n-1}}\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1 + ke^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |P'(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}} \quad (1.12)$$

where $m = Min_{|z|=k}|P(z)|$.

If we let $q \to \infty$ in (1.12), we get the following corollary.

Corollary 1.6. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex β with $|\beta| \leq 1$ and for each $r > 0$, we have*

$$n\left\{\int_0^{2\pi} \left|P(e^{i\theta}) + \beta\frac{m}{k^{n-1}}\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1 + ke^{i\theta}|^{r} d\theta\right\}^{\frac{1}{r}} \underset{|z|=1}{Max}|P'(z)|, \quad (1.13)$$

where $m = Min_{|z|=k}|P(z)|$.

Remark 1.7. If we let $r \to \infty$ in (1.13) and choosing argument of β suitably with $|\beta| = 1$, we obtain (1.4).

Next, we extend (1.9) to the class of polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, having all its zeros in $|z| \leq k$, $k \leq 1$ and thereby obtain the following result.

Theorem 1.8. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for every real or complex α with $|\alpha| \geq k^\mu$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}}. \quad (1.14)$$

Remark 1.9. We let $r \to \infty$ and $p \to \infty$ (so that $q \to 1$) in (1.14), we get inequality (1.6).

If we divide both sides of (1.14) by $|\alpha|$ and make $\alpha \to \infty$, we get the following result.

Corollary 1.10. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n\left\{\int_0^{2\pi} |P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |P'(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}}. \quad (1.15)$$

Letting $q \to \infty$ (so that $p \to 1$) in (1.14), we get the following result:

Corollary 1.11. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, where $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$, where $k \le 1$, then for every real or complex number α with $|\alpha| \ge k^\mu$ and for each $r > 0$,*

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi} \left|1 + k^\mu e^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \underset{|z|=1}{Max}|D_\alpha P(z)|. \tag{1.16}$$

As a generalization of Theorem 1.8, we present the following result:

Theorem 1.12. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$ where $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$ where $k \le 1$, then for every real or complex α with $|\alpha| \ge k^\mu$ and for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi} |P(e^{i\theta}) + \beta m|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}} \tag{1.17}$$

where $m = Min_{|z|=k}|P(z)|$.

If we divide both sides by $|\alpha|$ and make $\alpha \to \infty$, we get the following result:

Corollary 1.13. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$ where $k \le 1$, then for for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, we have*

$$n\left\{\int_0^{2\pi} |P(e^{i\theta}) + \beta m|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi} |1 + k^\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |P'(e^{i\theta})|^{qr} d\theta\right\}^{\frac{1}{qr}} \tag{1.18}$$

where $m = Min_{|z|=k}|P(z)|$.

Letting $q \to \infty$ (so that $p \to 1$) in (1.14), we get the following result:

Corollary 1.14. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$ where $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$, where $k \le 1$, then for every real or complex number α with $|\alpha| \ge k^\mu$ and for each $r > 0$,*

$$n(|\alpha| - k^\mu)\left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right) + \beta m\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi} \left|1 + k^\mu e^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \underset{|z|=1}{Max}|D_\alpha P(z)| \tag{1.19}$$

where $m = Min_{|z|=k}|P(z)|$.

2. **Lemmas**

For the proofs of the theorems, we need the following Lemmas:

Lemma 2.1. *If $P(z)$ is a polynomial of degree almost n having all its zeros in in $|z| \le k$ $k \le 1$ then for $|z| = 1$,*

$$|Q'(z)| + \frac{nm}{k^{n-1}} \le k|P'(z)|, \tag{2.1}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$ and $m = Min_{|z|=k}|P(z)|$.

The above Lemma is due to Govil and McTume [8].

Lemma 2.2. *Let $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n, which does not vanish for $|z| < k$, where $k \ge 1$ then for $|z| = 1$,*

$$k^\mu |P'(z)| \le |Q'(z)|, \tag{2.2}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

The above Lemma is due to Chan and Malik [5]. By applying Lemma 2.2 to the polynomial $z^n \overline{P(1/\overline{z})}$, one can easily deduce:

Lemma 2.3. *Let $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, is a polynomial of degree n, having all its zeros in $|z| \le k$, where $k \le 1$ then for $|z| = 1$*

$$k^\mu |P'(z)| \ge |Q'(z)|, \tag{2.3}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

3. Proof of Theorems

Proof of Theorem 1.2. Let $Q(z) = z^n \overline{P(1/\overline{z})}$ then $P(z) = z^n \overline{Q(1/\overline{z})}$ and it can be easily verified that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)| \text{ and } |P'(z)| = |nQ(z) - zQ'(z)|. \tag{3.1}$$

By Lemma (2.1), we have for every β with $|\beta| \le 1$ and $|z| = 1$,

$$\left| Q'(z) + \bar{\beta} \frac{nmz^{n-1}}{k^{n-1}} \right| \le |Q'(z)| + \frac{nm}{k^{n-1}} \le k|P'(z)|. \tag{3.2}$$

Using (3.1) in (3.2), for $|z| = 1$ we have

$$\left| Q'(z) + \bar{\beta} \frac{nmz^{n-1}}{k^{n-1}} \right| \le k|nQ(z) - zQ'(z)|. \tag{3.3}$$

By Lemma 2.3 with $\mu = 1$, for every real or complex number α with $|\alpha| \ge k$ and $|z| = 1$, we have

$$\begin{aligned} |D_\alpha P(z)| &\ge |\alpha||P'(z)| - |Q'(z)| \\ &\ge (|\alpha| - k)|P'(z)|. \end{aligned} \tag{3.4}$$

Since $P(z)$ has all its zeros in $|z| \le k \le 1$, it follows by Gauss-Lucas Theorem that all the zeros of $P'(z)$ also lie in $|z| \le k \le 1$. This implies that the polynomial

$$z^{n-1} \overline{P'(1/\overline{z})} \equiv nQ(z) - zQ'(z)$$

does not vanish in $|z| < 1$. Therefore, it follows from (3.3) that the function

$$w(z) = \frac{z \left(Q'(z) + \bar{\beta} \dfrac{nmz^{n-1}}{k^{n-1}} \right)}{k\left(nQ(z) - zQ'(z)\right)}$$

is analytic for $|z| \le 1$ and $|w(z)| \le 1$ for $|z| = 1$. Furthermore, $w(0) = 0$. Thus the function $1 + kw(z)$ is subordinate to the function $1 + kz$ for $|z| \le 1$. Hence by a well known property of subordination [9], we have

$$\int_0^{2\pi} \left| 1 + kw(e^{i\theta}) \right|^r d\theta \le \int_0^{2\pi} \left| 1 + ke^{i\theta} \right|^r d\theta, \quad r > 0. \tag{3.5}$$

Now

$$1 + kw(z) = \frac{n \left(Q(z) + \bar{\beta} \dfrac{mz^n}{k^{n-1}} \right)}{nQ(z) - zQ'(z)},$$

and

$$|P'(z)| = |z^{n-1} \overline{P'(1/\overline{z})}| = |nQ(z) - zQ'(z)|, \text{ for } |z| = 1,$$

therefore for $|z|=1$,

$$n\left|Q(z)+\bar{\beta}\frac{mz^n}{k^{n-1}}\right| = |1+kw(z)||nQ(z)-zQ'(z)| = |1+kw(z)||P'(z)|.$$

equivalently,

$$n\left|z^n\overline{P(1/\overline{z})}+\bar{\beta}\frac{mz^n}{k^{n-1}}\right| = |1+kw(z)||P'(z)|.$$

This implies

$$n\left|P(z)+\beta\frac{m}{k^{n-1}}\right| = |1+kw(z)||P'(z)| \quad \text{for} \quad |z|=1. \tag{3.6}$$

From (3.4) and (3.6), we deduce that for $r>0$,

$$n^r(|\alpha|-k)^r\int_0^{2\pi}\left|P(e^{i\theta})+\beta\frac{m}{k^{n-1}}\right|^r d\theta \leq \int_0^{2\pi}|1+kw(e^{i\theta})|^r|D_\alpha P(e^{i\theta})|^r d\theta.$$

This gives with the help of Hölder's inequality and using (3.5), for $p>1$, $q>1$ with $p^{-1}+q^{-1}=1$,

$$n^r(|\alpha|-k)^r\int_0^{2\pi}\left|P(e^{i\theta})+\beta\frac{m}{k^{n-1}}\right|^r d\theta \leq \left(\int_0^{2\pi}|1+ke^{i\theta}|^{pr}d\theta\right)^{1/p}\left(\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr}d\theta\right)^{1/q},$$

equivalently,

$$n(|\alpha|-k^\mu)\left\{\int_0^{2\pi}\left|P(e^{i\theta})+\beta\frac{m}{k^{n-1}}\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi}|1+ke^{i\theta}|^{pr}d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr}d\theta\right\}^{\frac{1}{qr}}$$

which proves the desired result. □

Proof of Theorem 1.8. Since $P(z)$ has all its zeros in $|z|\leq k$, therefore, by using Lemma 2.3 we have for $|z|=1$,

$$|Q'(z)| \leq k^\mu|nQ(z)-zQ'(z)|. \tag{3.7}$$

Now for every real or complex number α with $|\alpha|\geq k^\mu$, we have

$$\begin{aligned}|D_\alpha P(z)| &= |nP(z)+(\alpha-z)P'(z)| \\ &\geq |\alpha||P'(z)|-|nP(z)-zP'(z)|,\end{aligned}$$

by using (3.1) and Lemma 2.3, for $|z|=1$, we get

$$\begin{aligned}|D_\alpha P(z)| &\geq |\alpha||P'(z)|-|Q'(z)| \\ &\geq (|\alpha|-k^\mu)|P'(z)|.\end{aligned} \tag{3.8}$$

Since $P(z)$ has all its zeros in $|z|\leq k\leq 1$, it follows by Gauss-Lucas Theorem that all the zeros of $P'(z)$ also lie in $|z|\leq k\leq 1$. This implies that the polynomial

$$z^{n-1}\overline{P'(1/\overline{z})} \equiv nQ(z)-zQ'(z)$$

does not vanish in $|z|<1$. Therefore, it follows from (3.7) that the function

$$w(z)=\frac{zQ'(z)}{k^\mu\left(nQ(z)-zQ'(z)\right)}$$

is analytic for $|z|\leq 1$ and $|w(z)|\leq 1$ for $|z|=1$. Furthermore, $w(0)=0$. Thus the function $1+k^\mu w(z)$ is subordinate to the function $1+k^\mu z$ for $|z|\leq 1$. Hence by a well known property of subordination [9], we have

$$\int_0^{2\pi}\left|1+k^\mu w(e^{i\theta})\right|^r d\theta \leq \int_0^{2\pi}\left|1+k^\mu e^{i\theta}\right|^r d\theta, \quad r>0. \tag{3.9}$$

Now

$$1+k^{\mu}w(z)=\frac{nQ(z)}{nQ(z)-zQ'(z)},$$

and

$$|P'(z)|=|z^{n-1}\overline{P'(1/\overline{z})}|=|nQ(z)-zQ'(z)|,\ \text{ for } |z|=1,$$

therefore, for $|z|=1$,

$$n|Q(z)|=|1+k^{\mu}w(z)||nQ(z)-zQ'(z)|=|1+k^{\mu}w(z)||P'(z)|. \tag{3.10}$$

From (3.8) and (3.10), we deduce that for $r>0$,

$$n^r(|\alpha|-k^{\mu})^r\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\le\int_0^{2\pi}|1+k^{\mu}w(e^{i\theta})|^r|D_\alpha P(e^{i\theta})|^r d\theta.$$

This gives with the help of Hölder's inequality and (3.9), for $p>1$, $q>1$ with $p^{-1}+q^{-1}=1$,

$$n^r(|\alpha|-k^{\mu})^r\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\le\left(\int_0^{2\pi}|1+k^{\mu}e^{i\theta}|^{pr}d\theta\right)^{1/p}\left(\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr}d\theta\right)^{1/q},$$

equivalently,

$$n(|\alpha|-k^{\mu})\left\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}\le\left\{\int_0^{2\pi}|1+k^{\mu}e^{i\theta}|^{pr}d\theta\right\}^{\frac{1}{pr}}\left\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{qr}d\theta\right\}^{\frac{1}{qr}}$$

which proves the desired result.

□

Proof of Theorem 1.12. Let $m=Min_{|z|=k}|P(z)|$, so that $m\le|P(z)|$ for $|z|=k$. If $P(z)$ has a zero on $|z|=k$ then $m=0$ and result follows from Theorem 1.8. Henceforth we suppose that all the zeros of $P(z)$ lie in $|z|<k$. Therefore for every β with $|\beta|<1$, we have $|m\beta|<|P(z)|$ for $|z|=k$. Since $P(z)$ has all its zeros in $|z|<k\le1$, it follows by Rouche's theorem that all the zeros of $F(z)=P(z)+\beta m$ lie in $|z|<k\le1$. If $G(z)=z^n\overline{F(1/\overline{z})}=Q(z)+\bar{\beta}mz^n$, then by applying Lemma 2.3 to polynomial $F(z)=P(z)+\beta m$, we have for $|z|=1$,

$$|G'(z)|\le k^{\mu}|F'(z)|.$$

This gives

$$|Q'(z)+nm\bar{\beta}z^{n-1}|\le k^{\mu}|P'(z)|. \tag{3.11}$$

Using (3.1) in (3.11), for $|z|=1$ we have

$$|Q'(z)+nm\bar{\beta}z^{n-1}|\le k^{\mu}|nQ(z)-zQ'(z)| \tag{3.12}$$

Since $P(z)$ has all its zeros in $|z|<k\le1$, it follows by Gauss-Lucas Theorem that all the zeros of $P'(z)$ also lie in $|z|<k\le1$. This implies that the polynomial

$$z^{n-1}\overline{P'(1/\overline{z})}\equiv nQ(z)-zQ'(z)$$

does not vanish in $|z|<1$. Therefore, it follows from (3.12) that the function

$$w(z)=\frac{z(Q'(z)+nm\bar{\beta}z^{n-1})}{k^{\mu}\left(nQ(z)-zQ'(z)\right)}$$

is analytic for $|z|\le1$ and $|w(z)|\le1$ for $|z|=1$. Furthermore, $w(0)=0$. Thus the function $1+k^{\mu}w(z)$ is subordinate to the function $1+k^{\mu}z$ for $|z|\le1$. Hence by a well known property of subordination [9], we have

$$\int_0^{2\pi}\left|1+k^{\mu}w(e^{i\theta})\right|^r d\theta\le\int_0^{2\pi}\left|1+k^{\mu}e^{i\theta}\right|^r d\theta,\ \ r>0. \tag{3.13}$$

Now

$$1+k^{\mu}w(z)=\frac{n(Q(z)+m\bar{\beta}z^{n})}{nQ(z)-zQ'(z)},$$

and

$$|P'(z)|=|z^{n-1}\overline{P'(1/\overline{z})}|=|nQ(z)-zQ'(z)|,\ \text{ for }\ |z|=1,$$

therefore, for $|z|=1$,

$$n|Q(z)+m\bar{\beta}z^{n}|=|1+k^{\mu}w(z)||nQ(z)-zQ'(z)|=|1+k^{\mu}w(z)||P'(z)|.$$

This implies

$$n|G(z)|=|1+k^{\mu}w(z)||nQ(z)-zQ'(z)|=|1+k^{\mu}w(z)||P'(z)|. \tag{3.14}$$

Since $|F(z)|=|G(z)|$ for $|z|=1$, therefore, from (3.14) we get

$$n|P(z)+\beta m|=|1+k^{\mu}w(z)||P'(z)|\ \text{ for }\ |z|=1. \tag{3.15}$$

From (3.8) and (3.15), we deduce that for $r>0$,

$$n^{r}(|\alpha|-k^{\mu})^{r}\int_{0}^{2\pi}|P(e^{i\theta})+\beta m|^{r}d\theta\leq\int_{0}^{2\pi}|1+k^{\mu}w(e^{i\theta})|^{r}|D_{\alpha}P(e^{i\theta})|^{r}d\theta.$$

This gives with the help of Hölder's inequality in conjunction with (3.13) for $p>1$, $q>1$ with $p^{-1}+q^{-1}=1$,

$$n^{r}(|\alpha|-k^{\mu})^{r}\int_{0}^{2\pi}|P(e^{i\theta})+\beta m|^{r}d\theta\leq\left(\int_{0}^{2\pi}|1+k^{\mu}e^{i\theta}|^{pr}d\theta\right)^{1/p}\left(\int_{0}^{2\pi}|D_{\alpha}P(e^{i\theta})|^{qr}d\theta\right)^{1/q},$$

equivalently,

$$n(|\alpha|-k^{\mu})\left\{\int_{0}^{2\pi}|P(e^{i\theta})+\beta m|^{r}d\theta\right\}^{\frac{1}{r}}\leq\left\{\int_{0}^{2\pi}|1+k^{\mu}e^{i\theta}|^{pr}d\theta\right\}^{\frac{1}{pr}}\left\{\int_{0}^{2\pi}|D_{\alpha}P(e^{i\theta})|^{qr}d\theta\right\}^{\frac{1}{qr}}$$

which proves the desired result. □

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