Prespacetime Journal| July 2013 | Volume 4| Issue 6 | pp. 604-608

Fractional Lagrangian and Hamiltonian Formulations in Field Theory

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Abstract

The fractional variational principle represents an important part of fractional calculus and has found many applications. There are several versions of fractional variational principles and so different kinds of fractional Euler-Lagrange equations. In this paper, we propose the fractional sine-Gordon Lagrangian density. Then using the fractional Euler-Lagrange equations we obtain fractional sine-Gordon equation.

Keywords: Fractional Variational Calculus; Fractional Euler-Lagrange Equation

PACS: 11.10.Ef

1. Introduction

Fractional calculus generalizes the classical calculus [1] and has many applications in various fields of physics, from classical and quantum mechanics and electrodynamics to field theory and cosmology [2-14]**.**The fractional variational principle represents an important part of fractional calculus. This subject was initiated by Riewe [15, 16]. Riewe developed fractional Lagrangian, fractional Hamiltonian, and fractional mechanics. He has shown that Lagrangian with fractional derivative lead directly to equations of motion with non-conservative classical forces such as friction. Klimek [17, 18] and Agrawal [19] brought this subject to the main stream and initiated the field of fractional variational calculus. Recently Agrawal has written a comprehensive reviewing paper on this subject that can be found in [20] and discussed about various features of fractional variational calculus. Applications of fractional variational calculus in the field of physics have gained considerable popularity and many important results were obtained during the last years [21-27]. As a new application, in this paper we propose the fractional sine-Gordon Lagrangian density. Then using the fractional Euler-Lagrange equations, we obtain fractional sine-Gordon equation. It is well known that sine-Gordon equation is one of the basic equations of modern nonlinear wave and field theory and it arises in a large number of areas of theoretical and applied physics [28-36]. For example, it includes: Josephson junction theory (propagation of fluxons in Josephson junctions between two superconductors), dislocations in solid state physics, motion of Bloch magnetic walls in magnetic crystals, stability of fluid motions, nonlinear optics etc. Fractional dynamics is related to the study of applications of fractional calculus to describe systems with long-term memory, non-local and fractal properties and it is interesting to study nonlinear behavior of phenomena in such systems. For this purpose, in recent years researchers of the field of fractional dynamics have proposed some fractional generalizations of sine-Gordon equation for modeling above mentioned phenomena in complex media [37-40].
In the following, mathematical tools are briefly reviewed. Then in Sec. 3 we present a new fractional sine-Gordon Lagrangian density. Then using the fractional Euler-Lagrange equations

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we obtain fractional sine-Gordon equation that has the same form as the fractional sine-Gordon equation obtained in [39]. At last, in Sec. 4, we will present some conclusions.

2. Mathematical Tools

The fractional derivative has different definitions such as: Grünwald–Letnikov, Riemann-Liouville, Weyl, Riesz, Hadamard and Caputo fractional derivative [1], however in the papers cited above, the problems have been formulated mostly in terms of two types of fractional derivatives, namely Riemann-Liouville (RL) and Caputo. Among mathematicians, RL fractional derivatives have been popular largely because they are amenable to many mathematical manipulations. However, the RL derivative of a constant is not zero, and in many applications it requires fractional initial conditions which are generally not specified. Many believe that fractional initial conditions are not physical. In contrast, Caputo derivative of a constant is zero, and a fractional differential equation defined in terms of Caputo derivatives require standard boundary conditions. For these reasons, Caputo fractional derivatives have been popular among engineers and scientists. In this section we briefly present some fundamental definitions. The left and the right partial Riemann–Liouville and Caputo fractional derivatives of order α_k , $0<\alpha_k<1$ of a function f depending on n variables, $x_1,...,x_n$ defined over the domain $\Omega=\prod_{i=1}^{n}[a_i,b_i]$ with respect to x_k are as follow [25]:

The Left (Forward) RL fractional derivative

$$\left({}_{+}\partial_k^{\alpha}f\right)(x)=\frac{1}{\Gamma(1-\alpha_k)}\partial x_k\int_{a_k}^{x_k}\frac{f(x_1,...,x_{k-1},u,x_{k+1},...,x_n)}{(x_k-u)^{\alpha_k}}du \tag{1}$$

The Right (Backward) RL fractional derivative

$$\left({}_{-}\partial_k^{\alpha}f\right)(x)=\frac{-1}{\Gamma(1-\alpha_k)}\partial x_k\int_{x_k}^{b_k}\frac{f(x_1,...,x_{k-1},u,x_{k+1},...,x_n)}{(u-x_k)^{\alpha_k}}du \tag{2}$$

The Left (Forward) Caputo fractional derivative

$$\left({}_{+}^{C}\partial_k^{\alpha}f\right)(x)=\frac{1}{\Gamma(1-\alpha_k)}\int_{a_k}^{x_k}\frac{\partial_u f(x_1,...,x_{k-1},u,x_{k+1},...,x_n)}{(x_k-u)^{\alpha_k}}du \tag{3}$$

The Right (Backward) Caputo fractional derivative

$$\left({}_{-}^{C}\partial_k^{\alpha}f\right)(x)=\frac{-1}{\Gamma(1-\alpha_k)}\int_{x_k}^{b_k}\frac{\partial_u f(x_1,...,x_{k-1},u,x_{k+1},...,x_n)}{(u-x_k)^{\alpha_k}}du \tag{4}$$

The fractional variational principle represents an important part of fractional calculus and has found many applications in physics. As it is mentioned in [20] there are several versions of fractional variational principles and fractional Euler-Lagrange equations due to the fact that we have several definitions for the fractional derivatives. In this work we use new approach presented in [25] where authors developed the action principle for field systems described in terms of fractional derivatives, by use of a functional $S(\phi)$ as:

$$S(\phi)=\int L\left(\phi(x_k),\left({}_{+}^{C}\partial_k^{\alpha}\right)\phi(x_k),\left({}_{-}^{C}\partial_k^{\alpha}\right)\phi(x_k),x_k\right)(dx_k) \tag{5}$$

where $L\left(\phi(x_k),\left({}_{+}^{C}\partial_k^{\alpha}\right)\phi(x_k),\left({}_{-}^{C}\partial_k^{\alpha}\right)\phi(x_k),x_k\right)$ is a Lagrangian density function. Accordingly, x_k represents n variables x_1 to x_n, $\phi(x_k)\equiv\phi(x_1,...,x_n)$, $L\left(*,{}_{+}^{C}\partial_k^{\alpha},*,*\right)\equiv L\left(*,{}_{+}^{C}\partial_1^{\alpha},...,{}_{+}^{C}\partial_n^{\alpha},*,*\right)$, $(dx_k)\equiv dx_1...dx_n$ and the integration is taken over the entire domain Ω. From these definitions, we can obtain the fractional Euler-Lagrange equation as:

$$\frac{\partial L}{\partial\phi}+\sum_{k=1}^{n}{}_{-}\partial_k^{\alpha}\frac{\partial L}{\partial\left({}_{+}^{C}\partial_k^{\alpha}\phi\right)}+\sum_{k=1}^{n}{}_{+}\partial_k^{\beta}\frac{\partial L}{\partial\left({}_{-}^{C}\partial_k^{\beta}\phi\right)}=0 \tag{6}$$

Above equation is the Euler–Lagrange equation for the fractional field system and for $\alpha,\beta\to 1$, gives the usual Euler–Lagrange equations for classical fields. Also we can study the Hamiltonian formulation of the field systems [25]. For this, consider the fractional Lagrangian given in equation (5). Then the fractional canonical momentum densities π_{α_k} and π_{β_k} are:

$$\pi_{\alpha_k}=\frac{\partial L}{\partial\left({}_{+}^{C}\partial_k^{\alpha}\phi\right)},\quad \pi_{\beta_k}=\frac{\partial L}{\partial\left({}_{-}^{C}\partial_k^{\beta}\phi\right)} \tag{7}$$

So the fractional canonical Hamiltonian density is:

$$H=\sum_{k=1}^{n}\left(\pi_{\alpha_k}\left({}_{+}^{C}\partial_k^{\alpha}\phi\right)+\pi_{\beta_k}\left({}_{-}^{C}\partial_k^{\beta}\phi\right)\right)-L \tag{8}$$

With this fractional canonical Hamiltonian density one can obtain the fractional Hamilton equation of motion as:

$$\frac{\partial H}{\partial\phi}=\sum_{k=1}^{n}\left({}_{-}\partial_k^{\alpha}\pi_{\alpha_k}+{}_{+}\partial_k^{\beta}\pi_{\beta_k}\right) \tag{9}$$

One can prove that above equations can lead to the correct Euler–Lagrange equation of motion.

3. Fractional sine-Gordon equation

In this section we apply fractional Euler-Lagrange equation to obtain fractional sine-Gordon equation. One of the classical Lagrangians where one does this, is

$$L=\partial_\mu\phi\partial^\mu\phi-U(\phi) \tag{10}$$

Using the standard Euler-Lagrange equation:

$$\frac{\partial L}{\partial\phi}-\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\right)=0 \tag{11}$$

It is easy to verify that this Lagrangian density's corresponding equations of motion are:

$$\partial_\mu\phi\partial^\mu\phi+\left(\frac{\partial U}{\partial\phi}\right)=0 \tag{12}$$

For the sine-Gordon equation, we now just take $U=\cos(\phi)$ or $U=1-\cos(\phi)$, which yield identical equations of motion, namely:

$$\partial_\mu\phi\partial^\mu\phi+\sin(\phi)=0 \tag{13}$$

If we restrict ourselves to one space and one time dimension, it is:

$$\frac{\partial^2}{\partial t^2}\phi-\frac{\partial^2}{\partial x^2}\phi+\sin(\phi)=0 \tag{14}$$

This equation is one of the basic equations of modern nonlinear wave and field theory and it arises in a large number of areas of physics [28-36].Now let us consider following Lagrangian which contains fractional Caputo derivative:

$$L_f = \frac{1}{2}\left\{\eta^{2(\alpha-1)}\left({}_{+}^{C}\partial_t^{\alpha}\phi\right)^2 - \eta'^{2(\alpha-1)}\left({}_{+}^{C}\partial_x^{\alpha}\phi\right)^2\right\} + \cos(\phi) \tag{15}$$

Also, note that we have introduced arbitrary quantities η with dimension of [second] and η' with dimension of [meter] to ensure that all quantities have correct dimensions. Using equations (6), the Euler–Lagrange equations for variable ϕ is given as:

$$\sin(\phi) + \left(\eta^{2(\alpha-1)}\ {}_{-}\partial_t^{\alpha}\left({}_{+}^{C}\partial_t^{\alpha}\phi\right) - \eta'^{2(\alpha-1)}\ {}_{-}\partial_x^{\alpha}\left({}_{+}^{C}\partial_x^{\alpha}\phi\right)\right) = 0 \tag{16}$$

The above equation has the same form as the fractional sine-Gordon equation obtained in [39].Now we want to construct the Hamiltonian formulation for this problem. By using (7) the fractional canonical momenta are:

$$\pi_\alpha = \frac{1}{\eta^{\alpha-1}}\frac{\partial L}{\partial\left({}_{+}^{C}\partial_t^{\alpha}\phi\right)} = \eta^{(\alpha-1)}\left({}_{+}^{C}\partial_t^{\alpha}\phi\right), \quad \pi'_\alpha = \frac{1}{\eta'^{\alpha-1}}\frac{\partial L}{\partial\left({}_{+}^{C}\partial_x^{\beta}\phi\right)} = -\eta'^{(\alpha-1)}\left({}_{+}^{C}\partial_x^{\alpha}\phi\right) \tag{17}$$

Making use of (8, 15, 17) we can obtain the fractional canonical Hamiltonian as:

$$H_f = \eta^{(\alpha-1)}\pi_\alpha\left({}_{+}^{C}\partial_t^{\alpha}\phi\right) + \eta'^{(\alpha-1)}\pi'_\alpha\left({}_{+}^{C}\partial_x^{\alpha}\phi\right) - L_f = \frac{\pi_\alpha}{2} - \frac{\pi'_\alpha}{2} + \cos(\phi) \tag{18}$$

Substituting (18) in (9), we can easily obtain the corresponding equation of motion that is the same as the equation we obtained using fractional Euler-Lagrange equation.

4. Conclusion

The fractional variational principle represents an important part of fractional calculus and has found many applications in physics. As it is mentioned in [20] there are different kinds of fractional variational calculus and fractional Euler-Lagrange equations due to the fact that we have several definitions for the fractional derivatives**.** As an example, in this paper, we have proposed the fractional sine-Gordon Lagrangian density. Then using the fractional Euler-Lagrange equations, we have obtained fractional sine-Gordon equation. We can see that this result is the same as the sine-Gordon equation obtained in [39]. Also we showed that equation of motion obtained by using fractional Hamiltonian formulation has the same form as the equation we obtained using fractional Euler-Lagrange equation.

As we know, sine-Gordon equation is one of the basic equations of modern nonlinear wave and field theory and it arises in a large number of areas of theoretical and applied physics such as: Josephson junction theory (propagation of fluxons in Josephson junctions between two superconductors), dislocations in solid state physics, motion of Bloch magnetic walls in magnetic crystals, stability of fluid motions, nonlinear optics etc. In recent years some fractional generalizations of sine-Gordon equation have been introduced for modeling the above mentioned phenomena in complex media. In this paper we did not study the application of our new model for these systems. We hope to report on these subjects in the future.

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