On The Frequency of Twin Prime Pairs

By Sidharth Ghoshal

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Contents

Abstract:

The goal of the following document is to demonstrate a proof of the Twin Prime Conjecture by determining bounds for the number of twin prime pairs between a number and its square and then proving that the lower bound is always greater than 1 for sufficiently large numbers.

We will use proof by deduction to prove ∃ infinitely many twin primes. Define a twin prime pair H as a pair of integers $(k, k+2)$ s.t. $k, k+2 \in P$ $\{set\ of\ all\ primes\}$, define ∅ as the value $k+4, \forall\ H$, & define $P_{largest} < M$ as the largest prime number S that satisfies $S < M$.

$$\therefore\ k \not\equiv 0\ mod\ 2$$

$$k \not\equiv 0\ mod\ 3$$

$$k \not\equiv 0\ mod\ 5$$

$$\vdots$$

$$k \not\equiv 0\ mod\ P_{largest} < k$$

&

$$\therefore\ k+2 \not\equiv 0\ mod\ 2$$

$$k+2 \not\equiv 0\ mod\ 3$$

$$k+2 \not\equiv 0\ mod\ 5$$

$$\vdots$$

$$k+2 \not\equiv 0\ mod\ P_{largest} < k+2$$

∴ by definition of ∅ we can assert that:

$$\emptyset \not\equiv 2,4\ mod\ 2$$

$$\emptyset \not\equiv 2,4\ mod\ 3$$

$$\emptyset \not\equiv 2,4\ mod\ 5$$

$$\vdots$$

$$\emptyset \not\equiv 2,4\ mod\ P_{largest} < \emptyset - 4 = k$$

Define J' as a on the interval $J = [a, b]$, $P_{m+1} \in P, m > 1$ & consider the interval:

$$Q_1 = [P_{m+1}, \quad {P_{m+1}}^2]$$

It follows from the sieve of Eratosthenes (Sieve of … n.p.) that if $R \in \mathbb{Z}\ s.t.$

$$R \in Q_1\ \&$$

$$R \not\equiv 0\ mod\ 2$$

$$R \not\equiv 0\ mod\ 3$$

$$R \not\equiv 0 \; mod \; 5$$

$$\vdots$$

$$R \not\equiv 0 \; mod \; P_m$$

$\rightarrow R \in P$. Given this definition it follows that $\exists\, H \in Q_1$ iff $\exists\, (k, k+2)$ s.t. k, k+2 $\in P \rightarrow \exists\, H \in Q_1$ iff $\exists\, \emptyset \in Q_1$. $\because$ the largest possible k+2 $\in Q_1$ is ${P_{m+1}}^2 - 2 \rightarrow$ the largest possible $\emptyset \in Q_1$ is ${P_{m+1}}^2$. Note that by definition of $\emptyset$ and primality of $(k, k+2)$ all $\emptyset$ must satisfy:

$$\emptyset \not\equiv 2, 4 \; mod \; 2$$

$$\emptyset \not\equiv 2, 4 \; mod \; 3$$

$$\emptyset \not\equiv 2, 4 \; mod \; 5$$

$$\vdots$$

$$\emptyset \not\equiv 2,4 \; mod \; P_m$$

It is trivial to show that:

$$\exists\, {P_{m+1}}^2 - P_m + 1 \; integers \; \in Q_1 = \left[P_{m+1}, {P_{m+1}}^2\right] \forall\, P_{m+1} > 2 \; (Patrick \; 3)$$

Lemma 1:

$$The \; actual \; number \; of \; \lambda \in Q_1 \; that \; satisfy \; \lambda \not\equiv 2,4 \; mod \; P_r \in P, P_r < P_{m+1}, P_r \neq 2$$
$$\in \left[\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right) - \frac{P_r - 1}{P_r}, \; \left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right) + \frac{P_r - 1}{P_r} \right]$$

$$We \; assume \; no \; knowledge \; of \; the \; primality \; of \; P_{m+1} \; to \; make \; a \; general \; formula:$$

$$Note \; that \; \exists\, {P_{m+1}}^2 - P_m + 1 \; integers \; \in Q_1$$

$$\& \; trivially \; the \; probability \; that \; \lambda \in Q_1, \lambda \not\equiv 2,4 \; mod \; P_r \in P, P_r < P_{m+1}, P_r \neq 2 = \left(\frac{P_r - 2}{P_r}\right)$$

$$\therefore \; \exists \; approximately \left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right) possible \; \lambda$$

$$However \; there \; must \; be \; an \; integral \; number \; of \; \lambda \rightarrow The \; actual \; number \; of \; \lambda$$
$$\in Q_1 \; that \; satisfy \; \lambda \not\equiv 2,4 \; mod \; P_r \in P, P_r < P_{m+1}, P_r \neq 2,$$
$$\in \left[\left\lfloor \left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right) \right\rfloor, \quad \left\lceil \left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right) \right\rceil \right]$$

$$Note\ that\ \left|\left\lfloor\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right)\right\rfloor - \left(\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right)\right)\right| \le \frac{P_r - 1}{P_r}$$

$$\&\ that\left|\left(\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right)\right) - \left\lceil\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right)\right\rceil\right| \le \frac{P_r - 1}{P_r}$$

$$\rightarrow The\ error\ in\ the\ expected\ number\ of\ \lambda \in Q_1\ that\ satisfy\ \lambda \not\equiv 2{,}4\ mod\ P_r \in P, P_r < P_{m+1,}\ P_r \ne 2 \in \left[-\frac{P_r - 1}{P_r}, \frac{P_r - 1}{P_r}\right]\ invariant\ of\ interval\ size$$

$$Therefore: \left[\left\lfloor\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right)\right\rfloor, \quad \left\lceil\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right)\right\rceil\right]$$
$$\in \left[\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right) - \frac{P_r - 1}{P_r},\right.$$
$$\left.\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right) + \frac{P_r - 1}{P_r}\right]$$

$$\because\ The\ total\ number\ of\ \lambda \in Q_1\ that\ satisfy\ \lambda \not\equiv 2{,}4\ mod\ P_r \in P, P_r < P_{m+1,} P_r > 2,$$
$$\in \left[\left\lfloor\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right)\right\rfloor, \quad \left\lceil\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right)\right\rceil\right]$$

$$\&\ \left[\left\lfloor\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right)\right\rfloor, \quad \left\lceil\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right)\right\rceil\right]$$
$$\in \left[\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right) - \frac{P_r - 1}{P_r}, \left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right) + \frac{P_r - 1}{P_r}\right]$$

$$\therefore The\ actual\ of\ \lambda \in Q_1\ that\ satisfy\ \lambda \not\equiv 2{,}4\ mod\ P_r \in P, P_r < P_{m+1,} P_r > 2,$$
$$\in \left[\left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right) - \frac{P_r - 1}{P_r}, \left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{P_r - 2}{P_r}\right) + \frac{P_r - 1}{P_r}\right]$$

$$End\ Lemma:$$

Now reconsider:

$$\emptyset \not\equiv 2, 4\ mod\ 2$$

$$\emptyset \not\equiv 2, 4\ mod\ 3$$

$$\emptyset \not\equiv 2, 4\ mod\ 5$$

$$\vdots$$

$$\emptyset \not\equiv 2{,}4\ mod\ P_m$$

Lemma 2:

$$Actual\ Number\ of\ \emptyset\ \in\ Q_1 \in R_1$$

$$= \left[\left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{2}\right) \prod_{i=2}^{\pi(P_{m+1})-1} \left[\frac{P_i - 2}{P_i}\right] - \pi(P_{m+1}), \right.$$

$$\left. \left({P_{m+1}}^2 - P_{m+1} + 1 \right) \left(\frac{1}{2}\right) \prod_{i=2}^{\pi(P_{m+1})-1} \left[\frac{P_i - 2}{P_i}\right] + \pi(P_{m+1}) \right]$$

$$Note\ from\ Lemma\ 1:$$

$$Probability\ of\ b\ \in\ Q_1, b\ \in\ \mathbb{Z}, b\ \not\equiv 2,4\ \ mod\ P_r \in P, P_r < \ P_{m+1,} P_r > 2, = \left(\frac{P_r - 2}{P_r}\right)$$

We now consider the case of 2:

$$(b\ \ \not\equiv 2\ mod\ 2) \equiv (b\ \ \not\equiv 4\ mod\ 2)\ \because 2 \equiv 4\ mod\ 2 \rightarrow$$

$$Probability\ of\ b\ \not\equiv 2,4\ mod\ 2 = \left(\frac{1}{2}\right)$$

$$Note\ that\ for\ each\ incongruence\ we\ can\ write\ a\ probability\ of\ satisfaction$$

$$P(b \not\equiv 2,4\ mod\ 2) = \left(\frac{1}{2}\right)$$

$$P(b \not\equiv 2,4\ mod\ 3) = \left(\frac{1}{3}\right)$$

$$P(b \not\equiv 2,4\ mod\ 5) = \left(\frac{3}{5}\right)$$

$$\vdots$$

$$P(b \not\equiv 2,4\ mod\ P_m) = \left(\frac{P_m - 2}{P_m}\right)$$

$$\rightarrow P\big(b \in \{\emptyset\}(NOT\ THE\ EMPTY\ SET)\big) = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{3}{5}\right) \ldots \left(\frac{P_m - 2}{P_m}\right) = \left(\frac{1}{2}\right) \prod_{i=2}^{m} \left[\frac{P_m - 2}{P_m}\right]$$

$$\therefore Expected\ number\ of\ \emptyset\ \in\ Q_1 = (P_{m+1}^2 - \ P_{m+1} + \ 1) \left(\frac{1}{2}\right) \prod_{i=2}^{m} \left[\frac{P_m - 2}{P_m}\right]$$

$$Note\ that\ each\ term\ introduces\ an\ \ error\ independent\ of\ interval\ size:$$

$$Error\ Expected\ number(b \not\equiv 2,4\ mod\ P_m) \in \left[-\frac{P_m - 1}{P_m}, \frac{P_m - 1}{P_m}\right]$$

$$\therefore Error\ of\ Expected\ number\ of\ \emptyset\ \in\ Q_1$$

$$\in \left[-\left(\frac{1}{2}\right)-\left(\frac{2}{3}\right)-\left(\frac{4}{5}\right)\ldots-\left(\frac{P_m-1}{P_m}\right),\qquad \left(\frac{1}{2}\right)+\left(\frac{2}{3}\right)+\left(\frac{4}{5}\right)\ldots+\left(\frac{P_m-1}{P_m}\right)\right]$$

$$=\left[-\sum_{i=1}^{m}\left[\frac{P_i-1}{P_i}\right],\sum_{i=1}^{m}\left[\frac{P_i-1}{P_i}\right]\right]$$

$$Note: \left|\frac{P_i-1}{P_i}\right| < 1\ \forall\ P_i\ \in P$$

$$\therefore\ \left[-\sum_{i=1}^{m}\left[\frac{P_i-1}{P_i}\right],\sum_{i=1}^{m}\left[\frac{P_i-1}{P_i}\right]\right]\ \in \left[-\sum_{i=1}^{m}[1],\sum_{i=1}^{m}[1]\right]=[-m,m]=[-\pi(P_m),\pi(P_m)]$$

$$Note\ that\ \pi(P_m)=\pi(P_{m+1})-1 \rightarrow\ \ \pi(P_m)<\ \pi(P_{m+1})$$

$$\rightarrow [-\pi(P_m),\pi(P_m)] \in\ [-\pi(P_{m+1}),\pi(P_{m+1})]$$

$$Combining\ the\ expression\ of\ expected\ value\ with\ error:$$

$$Actual\ number\ of\ \emptyset\ \in\ Q_1\ \in\ R_1$$

$$=\left[(P_{m+1}^2-\ P_{m+1}+\ 1)\left(\frac{1}{2}\right)\prod_{i=2}^{\pi(P_{m+1})-1}\left[\frac{P_i-2}{P_i}\right]-\ \pi(P_{m+1})\ ,\right.$$
$$\left.(P_{m+1}^2-\ P_{m+1}+\ 1)\left(\frac{1}{2}\right)\prod_{i=2}^{\pi(P_{m+1})-1}\left[\frac{P_i-2}{P_i}\right]+\pi(P_{m+1})\right]$$

$$End\ Lemma:$$

Lemma 3:

$$\left(\frac{1}{3}\right)\left(\frac{3}{5}\right)\left(\frac{5}{7}\right)\ldots\left(\frac{P_n-2}{P_n}\right)=\prod_{i=2}^{n}\left[\frac{P_i-2}{P_i}\right]\geq\left(\frac{1}{3}\right)\prod_{i=3}^{n}\left[\frac{P_{i-1}}{P_i}\right]=\left(\frac{1}{3}\right)\left(\frac{3}{5}\right)\left(\frac{5}{7}\right)\ldots\left(\frac{P_{n-1}}{P_n}\right)=\left(\frac{1}{P_n}\right)$$

$$Note\ it\ is\ trival\ to\ show: \left(\frac{1}{3}\right)\prod_{i=3}^{n}\left[\frac{P_{i-1}}{P_i}\right]=\left(\frac{1}{3}\right)\left(\frac{3}{5}\right)\left(\frac{5}{7}\right)\ldots\left(\frac{P_{n-1}}{P_n}\right)=\left(\frac{1}{P_n}\right)$$

$$\because \left(\frac{1}{3}\right)\left(\frac{3}{5}\right)=\left(\frac{1}{5}\right)$$

$$\left(\frac{1}{3}\right)\left(\frac{3}{5}\right)\left(\frac{5}{7}\right)=\left(\frac{1}{7}\right)$$

$$and\ by\ induction: \left(\frac{1}{3}\right)\left(\frac{3}{5}\right)\left(\frac{5}{7}\right)\ldots\left(\frac{P_{n-1}}{P_n}\right)=\left(\frac{1}{P_n}\right)$$

$$Note\ P_{n-1} > 2\ \in \{odd\ numbers\} \rightarrow\ P_n -\ P_{n-1} \geq 2\ \&\ P_n -\ P_{n-1} \in \{even\ numbers\}\ \forall\ n > 1$$

$$if\ P_n = 11 \rightarrow P_{n-1} = 7 \rightarrow \frac{P_{n-1}}{P_n} = \left(\frac{7}{11}\right) \leq \frac{9}{11} = \frac{P_n - 2}{P_n} \rightarrow \frac{P_{n-1}}{P_n} \leq\ \frac{P_n - 2}{P_n}\ if\ P_n = 11.$$

$$\therefore if\ P_n -\ P_{n-1} = 2\ \forall\ P_n\ \neq 11, \prod_{i=2}^{n}\left[\frac{P_i - 2}{P_i}\right] \geq \prod_{i=3}^{n}\left[\frac{P_{i-1}}{P_i}\right] =\ \left(\frac{1}{P_n}\right)$$

$$if\ P_n -\ P_{n-1} \neq 2\ \forall\ P_n, \prod_{i=2}^{n}\left[\frac{P_i - 2}{P_i}\right] \geq \prod_{i=3}^{n}\left[\frac{P_{i-1}}{P_i}\right] =\ \left(\frac{1}{P_n}\right)$$

$$\therefore \prod_{i=2}^{n}\left[\frac{P_i - 2}{P_i}\right] \geq \left(\frac{1}{3}\right)\prod_{i=3}^{n}\left[\frac{P_{i-1}}{P_i}\right] =\ \left(\frac{1}{P_n}\right)$$

$$End\ Lemma:$$

$$\therefore\ R_1' = \left({P_{m+1}}^2 -\ P_{m+1} + 1\right)\left(\frac{1}{2}\right)\prod_{i=2}^{\pi(P_{m+1})-1}\left[\frac{P_i - 2}{P_i}\right] - \pi(P_{m+1}) \geq\ R_2$$

$$= \left({P_{m+1}}^2 -\ P_{m+1} + 1\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\prod_{i=3}^{\pi(P_{m+1})-1}\left[\frac{P_{i-1}}{P_i}\right] -\ \pi(P_{m+1})$$

$$= \left({P_{m+1}}^2 -\ P_{m+1} + 1\right)\left(\frac{1}{2}\right)\left(\frac{1}{P_m}\right) - \pi(P_{m+1})$$

Now we make an additional note that $P_{m+1} > \ P_m \because P$ is an ordered infinitely large set due to the work of Euclid[1].

$$\therefore R_2 =\ \left({P_{m+1}}^2 -\ P_{m+1} + 1\right)\left(\frac{1}{2}\right)\left(\frac{1}{P_m}\right) - \pi(P_{m+1}) \geq\ R_3$$

$$= \left({P_{m+1}}^2 -\ P_{m+1} + 1\right)\left(\frac{1}{2}\right)\left(\frac{1}{P_{m+1}}\right) - \pi(P_{m+1})$$

Note that:

$$\pi(x) <\ 1.25506\frac{x}{\log{(x)}}\ \forall\ x\ \in\ \mathbb{R}, x \geq 17\ (Rosser, Schoenfield\ 2)$$

$$\therefore\ R_3 =\ \left({P_{m+1}}^2 -\ P_{m+1} + 1\right)\left(\frac{1}{2}\right)\left(\frac{1}{P_{m+1}}\right) - \pi(P_{m+1}) \geq R_4$$

$$= \left({P_{m+1}}^2 -\ P_{m+1} + 1\right)\left(\frac{1}{2}\right)\left(\frac{1}{P_{m+1}}\right) -\ 1.25506\frac{P_{m+1}}{\log{(P_{m+1})}}\ \ \forall\ P_{m+1} \geq 17$$

Lemma 4:

$$\forall\ P_{m+1} \geq 17\ R_4 = \left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{1}{2}\right)\left(\frac{1}{P_{m+1}}\right) - 1.25506\frac{P_{m+1}}{\log\left(P_{m+1}\right)} \geq 1, \forall\, P_{m+1} \geq 341.$$

$$Let\ f \in \mathbb{R}$$

$$(f^2 - f + 1)\left(\frac{1}{2}\right)\left(\frac{1}{f}\right) - 1.25506\frac{f}{\log(f)} \geq 1\ \ for\ f = 341$$

$$\frac{d}{dP_{m+1}}\left[(f^2 - f + 1)\left(\frac{1}{2}\right)\left(\frac{1}{f}\right) - 1.25506\frac{f}{\log(f)}\right] \geq 0\ \forall\, f \geq 341$$

$$\rightarrow\ (f^2 - f + 1)\left(\frac{1}{2}\right)\left(\frac{1}{f}\right) - 1.25506\frac{f}{\log(f)} \geq 1\ \ \forall\, f \geq 341$$

$$\rightarrow\ \left({P_{m+1}}^2 - P_{m+1} + 1\right)\left(\frac{1}{2}\right)\left(\frac{1}{P_{m+1}}\right) - 1.25506\frac{P_{m+1}}{\log\left(P_{m+1}\right)} \geq 1, \forall\, P_{m+1} \geq f \geq 341$$

$$End\ Lemma$$

Now note:

$$R_4 \geq 1\ \forall\, P_{m+1} \geq 341$$

$$R_3 \geq\ R_4 \geq 1\ \forall\, P_{m+1} \geq 341$$

$$R_2 \geq\ R_3 \geq\ R_4 \geq 1\ \forall\, P_{m+1} \geq 341$$

$$R_1' \geq\ R_2 \geq\ R_3 \geq\ R_4 \geq 1\ \ \forall\, P_{m+1} \geq 341\ \rightarrow\ R_1' \geq 1\ \forall\, P_{m+1} \geq 341$$

$$\rightarrow\ The\ actual\ number\ of\ \emptyset\ \in\ Q_1\ \geq 1\ \forall\ \ P_{m+1} \geq 341$$

Conclusion:

$$\because \exists\ infinite\ Q\ \in P\ of\ arbitrarily\ large\ size \rightarrow \exists\ infinite\ \ \ P_{m+1} \geq 341$$

$$\therefore\ \exists\ infinitely\ many\ \emptyset$$

$$\therefore\ \exists\ infinitely\ many\ H$$

$$\therefore\ \exists\ infinitely\ many\ twin\ prime\ pairs$$

$$Q.E.D.$$

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