Predicting the Neutron and Proton Masses Based on Baryons which are Yang-Mills Magnetic Monopoles and Koide Mass Triplets

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Abstract:

We show how the Koide relationships and associated triplet mass matrices can be generalized to derive the observed sum of the free neutron and proton rest masses in terms of the up and down current quark masses and the Fermi vev to six parts in 10,000, which sum can then be solved for the separate neutron and proton masses using the neutron minus proton mass difference derived by the author in an recent, separate paper. The opposite charges of the up and down quarks are responsible for the appearance of a complex phase exp(iδ) and real rotation angle which leads on an independent basis to mass and mixing matrices similar to that of Cabibbo, Kobayashi and Maskawa (CKM) and which can be used to specify the neutron and proton mass relationships to unlimited accuracy and which are shown within experimental errors to be related to the CKM mixing angles. The Koide generalizations developed here enable these neutron and proton mass relationships to be given a Lagrangian formulation based on neutron and proton field strength tensors that contain vacuum-amplified and current quark wavefunctions and masses. In the course of development, we also uncover new Koide relationships for the neutrinos, the up quarks, and the down quarks.

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1. Introduction

In an earlier paper [1], the author introduced the thesis that baryons are Yang-Mills magnetic monopoles. One of the relationships predicted in this paper, equation [11.22] therein, predicted the electron rest mass as a function of the up and down quark masses, namely:

$$m_e = 3\left(m_d - m_u\right) / \left(2\pi\right)^{\frac{3}{2}}, \tag{1.1}$$

with the factor of $\left(2\pi\right)^{\frac{3}{2}}$ emerging from a three-dimensional Gaussian integration. Based on a "resonant cavity" analysis of the nucleons whereby the energies released or retained during binding are directly dependent upon the masses of the quarks contained within the nucleons, we also predicted that latent, intrinsic binding energies of a neutron and proton, as in [12.12] and [12.13] of [1], are given by:

$$B_P = 2m_u + m_d - \left(m_d + 4\sqrt{m_u m_d} + 4m_u\right) / \left(2\pi\right)^{\frac{3}{2}} = 7.640679 MeV \tag{1.2}$$

$$B_N = 2m_d + m_u - \left(m_u + 4\sqrt{m_u m_d} + 4m_d\right) / \left(2\pi\right)^{\frac{3}{2}} = 9.812358 MeV\ . \tag{1.3}$$

These predict a latent binding energy of 8.7625185 MeV *per nucleon* for a nucleus with an equal number of protons and neutrons, which is remarkably close to what is observed for all but the very lightest nuclides, as well as a total latent binding energy of 493.028394 MeV for ^{56}Fe, in contrast to the empirical binding energy of 492.253892 MeV. This is understood to mean that 99.8429093% of the available binding energy in ^{56}Fe is applied to inter-nucleon binding, with the balance of 0.1570907% retained for the intra-nucleon confinement of quarks. It was also noted that this percentage of energy released for inter-nucleon binding is higher in ^{56}Fe than in any other nuclide, which further explains that although the quarks come closer to de-confinement in ^{56}Fe than in any other nuclide (which also explains the "first EMC effect" [2]), they do always remain confined, as emphasized by the decline in this percentage beyond ^{56}Fe.

In a second paper [3], the author showed how the thesis that baryons are Yang-Mills magnetic monopoles together with the foregoing "resonant cavity" analysis can be used to predict the binding energies of the 1s nuclides, namely ^{2}H, ^{3}H, ^{3}He and ^{4}He, to at least parts per hundred thousand and in most cases parts per million, and also to predict the difference between the neutron and proton masses according to:

$$M_N - M_P = m_u - \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right) / \left(2\pi\right)^{\frac{3}{2}}. \tag{1.4}$$

This relationship, originally predicted in [6.16] of [3] to about seven parts per ten million in AMU, was later taken in [9.1] of [3] to be an *exact* relationship, and all of the other prior mass relationships which had been developed were then nominally adjusted to implement (1.4) as an exact relationship. The review of the solar fusion cycle in section 8 of [3] served to emphasize how effectively this resonant cavity analysis can be used to accurately predict empirical binding energies, and suggested how applying gamma radiation with the right resonant harmonics to a store of hydrogen may well have a catalyzing effect for nuclear fusion. This relationship (1.4) will also play an important role in the development here.

At the heart of these numeric calculations were the two outer products [3.9] and [3.10] in [3] for the neutron and the proton, with components given by [3.11] and related relationships developed throughout section 2 of [3]. In particular, the two matrices which stood at the center of these successful binding calculations were the 3x3 Yang-Mills diagonalized matrices K of mass dimension ½ with components $\mathrm{diag}\left(K_N\right) = \left(\sqrt{m_u}, \sqrt{m_d}, \sqrt{m_d}\right)$ for the neutron and

$\text{diag}(K_P) = (\sqrt{m_d}, \sqrt{m_u}, \sqrt{m_u})$ for the proton, where m_u is the "current" mass of the up quark and m_d is the current mass of the down quark.

What is very intriguing about these *K* matrices (which we designate as such to reference Koide), is that although they originate out of the thesis that baryons are magnetic monopoles, they have a form very similar to matrices which may be used in the so-called Koide mass formula [4] for the charged leptons, namely:

$$R = \frac{\left(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}\right)^2}{m_1 + m_2 + m_3} \cong \frac{3}{2}. \tag{1.5}$$

Above, when we take $m_1 = m_e$, $m_2 = m_\mu$ and $m_3 = m_\tau$ to be the charged lepton masses, the ratio $R \cong 3/2$ gives a very precise relationship among these masses. Indeed, if we use the 2012 PDG data $m_e = 0.510998928 \pm 0.000000011 MeV$, $m_\mu = 105.6583715 \pm 0.0000035 MeV$ and $m_\tau = 1776.82 \pm 0.16 MeV$ [5], we find using the mean experimental data that $R = 1.500022828$ which is very close to 3/2. When we use the extremes of the experimental data ranges, specifically, the largest possible tau mass and the lowest possible mu mass, we obtain *R*=1.5000024968. Although this is an order of magnitude closer to 3/2 than the ratio obtained from the mean data, is still *outside* of experimental errors. This means that while $R \cong 3/2$ is a very close relationship, even accounting for experimental error, it is still approximate. For this to be *within* experimental errors, it would have to be possible to obtain some $R \leq 3/2$ for some combination of masses at the edges of the experimental ranges, and it is not. So in the application of the Koide relationships to various "pole" (low probe energy) mass triplets, the question becomes, not *whether* a triplet has a ratio exactly equal to 3/2, because no triplet does have this exact relationship, but rather, how close to 3/2 any given ratio is, and more importantly, what the meaning is of this ratio and deviations from this ratio.

The similarities between the matrices developed by the author in [1] and [3] and those developed by Koide in [4] are highlighted if we define a Koide matrix *K* generally as:

$$K_{AB} \equiv \begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix}. \tag{1.6}$$

Then, the two latent binding energy relationships (1.2) and (1.3) may be represented as:

$$B_P = K_{AB}K_{BA} - \frac{1}{(2\pi)^{\frac{3}{2}}} K_{AA}K_{BB} = Tr(K^2) - \frac{1}{(2\pi)^{\frac{3}{2}}} Tr(K \otimes K) = 2m_u + m_d - \left(m_d + 4\sqrt{m_u m_d} + 4m_u\right) / (2\pi)^{\frac{3}{2}}$$

$$= Tr\begin{pmatrix} \sqrt{m_d} & 0 & 0 \\ 0 & \sqrt{m_u} & 0 \\ 0 & 0 & \sqrt{m_u} \end{pmatrix}\begin{pmatrix} \sqrt{m_d} & 0 & 0 \\ 0 & \sqrt{m_u} & 0 \\ 0 & 0 & \sqrt{m_u} \end{pmatrix} - \frac{1}{(2\pi)^{\frac{3}{2}}} Tr\begin{pmatrix} \sqrt{m_d} & 0 & 0 \\ 0 & \sqrt{m_u} & 0 \\ 0 & 0 & \sqrt{m_u} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{m_d} & 0 & 0 \\ 0 & \sqrt{m_u} & 0 \\ 0 & 0 & \sqrt{m_u} \end{pmatrix}, \tag{1.7}$$

$$B_N = K_{AB}K_{BA} - \frac{1}{(2\pi)^{\frac{3}{2}}} K_{AA}K_{BB} = Tr(K^2) - \frac{1}{(2\pi)^{\frac{3}{2}}} Tr(K \otimes K) = 2m_d + m_u - \left(m_u + 4\sqrt{m_u m_d} + 4m_d\right) / (2\pi)^{\frac{3}{2}}$$
$$= Tr\begin{pmatrix} \sqrt{m_u} & 0 & 0 \\ 0 & \sqrt{m_d} & 0 \\ 0 & 0 & \sqrt{m_d} \end{pmatrix}\begin{pmatrix} \sqrt{m_u} & 0 & 0 \\ 0 & \sqrt{m_d} & 0 \\ 0 & 0 & \sqrt{m_d} \end{pmatrix} - \frac{1}{(2\pi)^{\frac{3}{2}}} Tr\begin{pmatrix} \sqrt{m_u} & 0 & 0 \\ 0 & \sqrt{m_d} & 0 \\ 0 & 0 & \sqrt{m_d} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{m_u} & 0 & 0 \\ 0 & \sqrt{m_d} & 0 \\ 0 & 0 & \sqrt{m_d} \end{pmatrix}, \quad (1.8)$$

where, starting with (1.6), in (1.7) we have set $m_1 \equiv m_d$ and $m_2 = m_3 \equiv m_u$ and in (1.8) we have set $m_1 \equiv m_u$ and $m_2 = m_3 \equiv m_d$. These originate in the author's thesis in [1] that baryons are Yang-Mills magnetic monopoles. Above, $\otimes$ designates an *outer* matrix product.

On the other hand, setting $m_1 = m_e$, $m_2 = m_\mu$ and $m_3 = m_\tau$ in (1.6), we may write:

$$Tr(K^2) = K_{AB}K_{BA} = m_1 + m_2 + m_3, \quad (1.9)$$

$$Tr(K \otimes K) = K_{AA}K_{BB} = \left(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}\right)^2. \quad (1.10)$$

Then, using (1.9) and (1.10), Koide relationship (1.5) for charged leptons may be written as:

$$R = \frac{\left(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}\right)^2}{m_1 + m_2 + m_3} = \frac{K_{AA}K_{BB}}{K_{AB}K_{BA}} = \frac{Tr(K \otimes K)}{Tr(K^2)} \cong \frac{3}{2}. \quad (1.11)$$

Clearly then, the Koide matrices (1.6) provide a general form for organizing the study of both binding energy and fermion mass relationships which lead to very accurate empirical results. It thus becomes desirable to understand the physical origin of these matrices and tie them to a Lagrangian formulation so that they are no longer just intriguing curiosities that yield tantalizingly-accurate empirical results, but instead can be rooted in fundamental physics principles based on a Lagrangian. And, it is desirable to see if these matrices can be extended in their application to make additional mass predictions and gain a deeper understanding of the particle mass spectrum.

Because the binding energy formulation in (1.7) and (1.8) has its roots in the thesis that baryons are Yang-Mills magnetic monopoles and specifically emerges from the calculation of energies via $E = -\iiint \mathfrak{L} d^3x$, see [11.7] of [1] et. seq., the author's previous findings will provide us with the means to anchor the Koide relationships in a Lagrangian formulation. And, because Koide provides a generalization of the mass matrices derived by the author, these matrices will provide us with the means to derive additional mass relationships as well, in particular, and especially, the neutron and proton rest masses.

Insofar as Koide relations are concerned, in section 2 we shall show how to reformulate these in terms of the statistical variance of the Koide terms across the three generations, which yields some new Koide relationships for the neutrinos, the up quarks, and the down quarks. We shall then show in section 3 how to recast these Koide relationships into a Lagrangian / energy formulation, which addresses the question as to underlying origins of these relationships, so that these relationships are not just curious coincidences, but can rooted in fundamental, physics principles based on a Lagrangian.

Most importantly, in this paper, we shall combine the author's previous work in [1] and [3] as well as [6], using the generalization provided by Koide triplet mass matrices of the form (1.6), to deduce the observed rest masses 938.272046 MeV and 939.565379 MeV of the free neutron and free proton, as a function of the up and down quark masses and electric charges and the Fermi vev. This mass derivation is presented in sections 4 and 5. In section 6 we will examine the "constituent" and "vacuum-amplified" quark masses of the neutron and proton. Finally, in section 7 we develop a Lagrangian formulation for these neutron and proton masses, which underscores that these relationships are not just close numerical coincidences, but originate from fundamental Lagrangian-based physics principles.

2. Statistical Reformulation of the Koide Mass Relationship

Let us begin by couching the Koide mass relationship (1.5) for the charged leptons in statistical terms, using $m_1 = m_e$, $m_2 = m_\mu$ and $m_3 = m_\tau$ in (1.6). First, using (1.9), we write the average of the masses $\langle m_i \rangle$ in a Koide mass triplet m_1, m_2, m_3, i.e., the "average of the squares" of the matrix elements in (1.6), as:

$$\left\langle K^2 \right\rangle = Tr(K^2)/3 = K_{AB}K_{BA}/3 = \left(m_1 + m_2 + m_3\right)/3 = \left\langle m_i \right\rangle . \tag{2.1}$$

Next, via (1.10), we write the "square of the average" of these matrix elements as:

$$\left\langle K \right\rangle^2 = \frac{Tr(K \otimes K)}{9} = \frac{K_{AA}K_{BB}}{9} = \left(\frac{\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}}{3} \right)^2 = \frac{\left(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}\right)^2}{9} . \tag{2.2}$$

So, combining (2.1) and (2.2) in the form of (1.5) allows us to write:

$$3\frac{\left\langle K \right\rangle^2}{\left\langle K^2 \right\rangle} = \frac{Tr(K \otimes K)}{Tr(K^2)} = \frac{K_{AA}K_{BB}}{K_{AB}K_{BA}} = \frac{\left(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}\right)^2}{m_1 + m_2 + m_3} = R \cong \frac{3}{2} . \tag{2.3}$$

This allows us to extract the relationship:

$$\left\langle K \right\rangle^2 = \frac{R}{3}\left\langle K^2 \right\rangle \cong \frac{1}{2}\left\langle K^2 \right\rangle , \tag{2.4}$$

which naturally absorbs the 3 from the factor of 3/2.

Now, we simply use (2.4) to form the statistical variance $\sigma(K)$ in the usual way, as:

$$\sigma(K) = \left\langle K^2 \right\rangle - \left\langle K \right\rangle^2 = \left(1 - \frac{R}{3}\right)\left\langle K^2 \right\rangle = \left(\frac{3}{R} - 1\right)\left\langle K \right\rangle^2 = \left(\frac{3}{R} - 1\right)\left\langle m_i \right\rangle \cong \frac{1}{2}\left\langle K^2 \right\rangle = \left\langle K \right\rangle^2 = \left\langle m_i \right\rangle . \tag{2.5}$$

The key relationship here, using the first and last terms, is:

$$\sigma(K) \cong \left\langle m_i \right\rangle . \tag{2.6}$$

So the average $\langle m_i \rangle$ of the charged lepton masses is approximately (and very closely) equal to the statistical variance $\sigma(K)$ of Koide matrix (1.6) for the charged leptons. This is a much simpler and more transparent way to express the Koide mass relationship (1.5), and it completely absorbs the factor of 3/2. The key point: (2.6) is an entirely equivalent, and far more transparent way to restate the Koide mass relationship (1.5).

Of course, as noted after (1.5), this is a very close, but still approximate relationship. The exact relationship, also extracted from (2.5), and using $R = 1.500022828$ based on the mean experimental data, is:

$$\sigma(K) = \left(\frac{3}{R} - 1\right)\left\langle m_i \right\rangle = 0.999969563\left\langle m_i \right\rangle \equiv C\left\langle m_i \right\rangle , \tag{2.7}$$

where we have defined the statistical coefficient *C* and the inverted relationship for *R* as:

$$C \equiv \frac{3}{R} - 1; \quad R \equiv \frac{3}{1+C}. \tag{2.8}$$

Thus, we rewrite the basic Koide relationship (1.5) more generally as:

$$\frac{\left(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}\right)^2}{m_1 + m_2 + m_3} = \frac{3}{1+C} = R. \tag{2.9}$$

In the circumstance where the statistical coefficient C=1, i.e., where the average mass is exactly equal to the statistical variance, we have $R = 3/2$. So the variance of the square roots of the three charged lepton masses is just a tiny touch less ($\times 0.999969563$) than the average of the three masses themselves. But the factor of 3/2, which is somewhat mysterious in (1.5), is now more readily understood when we realize that it corresponds with C=1 in (2.7).

This means that the Koide relationship for *any* given triplet of numbers with mass dimension ½, may be most transparently characterized by the coefficient C. Thus, using (2.7), the coefficient for the charged lepton triplet is (we also include R for comparison):

$$C(e\mu\tau) = 0.999969563 \cong 1; \quad R(e\mu\tau) = 1.500022828 \cong 3/2. \tag{2.10}$$

So what about some other Koide triplets? For the neutrinos, PDG in [7] provides upper limits on the neutrino masses whereby $m_{\nu_e} < 2eV$, $m_{\nu_\mu} < 0.19MeV$ and $m_{\nu_\tau} < 18.2MeV$. If we use these mass limits in a Koide triplet, we find that R=1.202960231. But the significance of this is much more easily seen by using (2.8) to calculate:

$$C(\nu_e \nu_\mu \nu_\tau) = 1.49384803 \cong 3/2; \quad R(\nu_e \nu_\mu \nu_\tau) = 1.202960231 \cong 6/5. \tag{2.11}$$

Here, we have another ratio very close to 3/2, but now it is in the coefficient C rather than the coefficient R. So, for the neutrino mass limits $\sigma(K_\nu) \cong (3/2)\langle m_\nu \rangle$. This in an interesting "coefficient migration" as between the charged and uncharged leptons, wherein for the charged leptons masses $R \cong 3/2$ to parts per 100,000, while for the neutrino lepton upper mass limits, $C \cong 3/2$ within about 0.4%. As we shall see, this is the start of a new Koide pattern.

Turning to quark masses, we use $m_u = 2.223792405MeV$ and $m_d = 4.906470335MeV$ developed in [9.3] and [9.4] of [3] with the conversion 1 u=931.494 061(21) MeV/c^2, as well as $m_c = 1.275 \pm 0.025GeV$, $m_s = 95 \pm 5MeV$, $m_t = 173.5 \pm .6 \pm .8GeV$ and $m_b = 4.18 \pm 0.03GeV$ from PDG's [8]. For Koide triplets of a single electric charge type, we can calculate that:

$$C(uct) = 1.54688 \cong 3/2; \quad R(uct) = 1.177913486 \cong 6/5. \tag{2.12}$$

$$C(dsb) = 1.18741 \cong 6/5; \quad R(dsb) = 1.371483911 \cong 15/11. \tag{2.13}$$

So we now see a distinctive pattern of coefficient migration among (2.10) through (2.13). For the charged leptons in (2.10) which are the lower members of a weak isospin doublet, $R(e\mu\tau) \cong 3/2$. For neutrinos which are the upper members of this doublet, $C(\nu_e \nu_\mu \nu_\tau) \cong 3/2$, which migrates the $3/2$ from the R to the C coefficient. Then, for the up quarks, we find another coefficient migration such that $C(uct) \cong 3/2$, which is same as the C for the neutrinos. Both the up quarks and the neutrinos are the upper members of weak isospin doublets. Finally, we see that the $R(uct) \cong 6/5$ coefficient for the up quarks, now migrates to $C(dsb) \cong 6/5$ for down quarks. So the migration is $R(e\mu\tau) \cong 3/2 \rightarrow C(\nu_e \nu_\mu \nu_\tau) \cong 3/2$ for leptons,

$C(\nu_e\nu_\mu\nu_\tau) \cong 3/2 \to C(uct) \cong 3/2$ providing a "bridge" from "up" leptons to "up" quarks, and then $R(uct) \cong 6/5 \to C(dsb) \cong 6/5$ migrating from the up to the down quarks.

The net upshot of this coefficient migration is that we now have Koide-style close relations for all four sets of fermions (and anti-fermions) of like-electric charge *Q*, namely:

$$R(Q=0) = \frac{\left(\sqrt{m_{\nu(e)}} + \sqrt{m_{\nu(\mu)}} + \sqrt{m_{\nu(\tau)}}\right)^2}{m_{\nu(e)} + m_{\nu(\mu)} + m_{\nu(\tau)}} \cong \frac{6}{5}. \qquad (2.14)$$

$$R(Q=\pm 1) = \frac{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2}{m_e + m_\mu + m_\tau} \cong \frac{3}{2}. \qquad (2.15)$$

$$R(Q=\pm \tfrac{2}{3}) = \frac{\left(\sqrt{m_u} + \sqrt{m_c} + \sqrt{m_t}\right)^2}{m_u + m_c + m_t} \cong \frac{6}{5}. \qquad (2.16)$$

$$R(Q=\pm \tfrac{1}{3}) = \frac{\left(\sqrt{m_d} + \sqrt{m_s} + \sqrt{m_b}\right)^2}{m_d + m_s + m_b} \cong \frac{15}{11}. \qquad (2.17)$$

Each of these relationships takes twelve *a priori* independent fermion masses and reduces by 1, their mutual independence. So with (2.14) through (12.17), to first approximation, we have now eight, rather than twelve independent fermion masses.

For some other commonly-studied Koide triplets we have:

$$C(uds) = 0.69290 \cong 1/\sqrt{2}; \quad R(uds) = 1.772105341 \cong 3\sqrt{2}/\left(1+\sqrt{2}\right). \qquad (2.18)$$

$$C(ctb) = 1.00939 \cong 1; \quad R(ctb) = 1.492994103 \cong 3/2. \qquad (2.19)$$

$$C(usc) = 0.86795; \quad R(usc) = 1.606042302. \qquad (2.20)$$

$$C(csb) = 1.02783 \cong 1; \quad R(csb) = 1.479416975 \cong 3/2 \quad (\text{with } -\sqrt{m_s}). \qquad (2.21)$$

$$C(dcs) = 0.81520; \quad R(dcs) = 1.652718083. \qquad (2.22)$$

We note that the relationship (2.18) for $C(uds) \cong 1/\sqrt{2}$ is accurate to *within experimental errors*. Specifically, given the empirical $m_s = 95 \pm 5 MeV$, (2.18) can be made into an *exact* relationship to ten digits (the accuracy of the up and down masses derived in [3]) if we set $m_s = 98.95303495 MeV$. Of course, even the relationship for the charged leptons is a close but not exact relationship, see the discussion following (1.5), so we ought not expect (2.18) to be exactly $C(uds) = 1/\sqrt{2}$. But, similarly to (1.5), see also (2.10), it may well make sense to regard this as a relationship accurate to the first three or four decimal places, which would improve our knowledge of the strange quark mass by four or five orders of magnitude.

But this main point of the foregoing is not about the specific Koide relationships (though the set of relationships (2.14) through (2.17) are important steps forward in their own right), but about how the ratio parameter *R* which for the charged lepton triplet is $R \cong 3/2$, can be reformulated for *any fermion triplet* into the coefficient *C* in the statistical variance relationship $\sigma(K) = C\langle m_i \rangle$, which, for the charged leptons, is $C \cong 1$. And, as we see in (2.14) through (2.17), this can lead to additional relationships including a cascading migration of coefficients.

Turning back to the neutron and proton triplets $\mathrm{diag}(K_P)=\left(\sqrt{m_d},\sqrt{m_u},\sqrt{m_u}\right)$ and $\mathrm{diag}(K_N)=\left(\sqrt{m_u},\sqrt{m_d},\sqrt{m_d}\right)$ which were so central to obtaining accurate binding energy predictions in [1] and [3], we find using the mass values $m_u=2.223792405MeV$ and $m_d=4.906470335MeV$ obtained in [3] that:

$$C(p=duu)=0.0387876019;\quad R(p=duu)=2.8879821000\,. \tag{2.19}$$

$$C(n=udd)=0.0298844997;\quad R(n=udd)=2.9129480061. \tag{2.20}$$

For these triplets which all have a *small* variance in comparison to the earlier triplets which cross generations, the Koide ratio $R\cong 3$. In the circumstance where the variance is *exactly* zero because all three quarks have the same mass, for example, for the triplets $\Delta^{++}=uuu$ and $\Delta^{-}=ddd$, using the Koide mass relationship for parameterization, we have $C=0$; $R=3$.

3. Lagrangian / Energy Reformulation of the Koide Mass Relationship

The appearance of Koide triplets originating from the thesis that Baryons are Yang-Mills magnetic monopoles can be seen, for example, by considering equation [11.2] of [1] reproduced below, for the field strength tensor of a Yang-Mills magnetic monopole containing a triplet of colored quarks in the zero-perturbation limit:

$$\mathrm{Tr}F^{\mu\nu}=-i\left(\frac{\overline{\psi}_R\left[\gamma^\mu{}_\vee\gamma^\nu\right]\psi_R}{"\not{p}_R-m_R"}+\frac{\overline{\psi}_G\left[\gamma^\mu{}_\vee\gamma^\nu\right]\psi_G}{"\not{p}_G-m_G"}+\frac{\overline{\psi}_B\left[\gamma^\mu{}_\vee\gamma^\nu\right]\psi_B}{"\not{p}_B-m_B"}\right). \tag{3.1}$$

If we generalize this to any three fermion wavefunctions ψ_1,ψ_2,ψ_3 such that (3.1) represents the specific case $\psi_1=\psi_R$, $\psi_2=\psi_G$ and $\psi_3=\psi_B$, and, as we did prior to [11.19] of [1], if we consider the circumstance in which the interactions shown in Figure 1 at the start of section 3 in [1] occur essentially at a point, then $\left[\gamma^\mu{}_\vee\gamma^\nu\right]\to\left[\gamma^\mu,\gamma^\nu\right]$ approaches an ordinary commutator, each of the $\not{p}\to 0$, and the "quoted" denominator becomes an ordinary denominator, see [3.9] through [3.12] of [1] for further background. So also setting $m_1=m_R$, $m_2=m_G$ and $m_3=m_B$, (3.1) generalizes for a point interaction to a Koide-style field strength tensor:

$$\mathrm{Tr}F^{\mu\nu}=-i\left(\frac{\overline{\psi}_1\left[\gamma^\mu,\gamma^\nu\right]\psi_1}{m_1}+\frac{\overline{\psi}_2\left[\gamma^\mu,\gamma^\nu\right]\psi_2}{m_2}+\frac{\overline{\psi}_3\left[\gamma^\mu,\gamma^\nu\right]\psi_3}{m_3}\right). \tag{3.2}$$

Then, we form a pure gauge field Lagrangian $\mathfrak{L}_{\mathrm{gauge}}=-\tfrac{1}{2}Tr\left(F_{\mu\nu}F^{\mu\nu}\right)=-\tfrac{1}{2}Tr\left(F\cdot F\right)$ as in [11.7] of [1]. As discussed in section 2 of [3], we consider both inner and outer products over the Yang-Mills indexes of *F*, i.e., we consider both $TrF^2=Tr\left(F_{AB}\cdot F_{BC}\right)=F_{AB}\cdot F_{BA}$ and $Tr\left(F\otimes F\right)=Tr\left(F_{AB}\cdot F_{CD}\right)=F_{AA}\cdot F_{BB}$. Note carefully the different index structures in $F_{AB}\cdot F_{BA}$ versus $F_{AA}\cdot F_{BB}$, and also contrast this to (1.7) through (1.10) in this paper, which is where we are headed at the moment.

We then use this Lagrangian to calculate energies according to [11.7] of [1], see also [1.1] of [3], which is reproduced below:

$$E=-\iiint\mathfrak{L}_{\mathrm{gauge}}d^3x=\tfrac{1}{2}\mathrm{Tr}\iiint F_{\mu\nu}F^{\mu\nu}d^3x\,. \tag{3.3}$$

In the case where $\psi_1 = \psi_d$, $\psi_2 = \psi_3 = \psi_u$ so that $F^{\mu\nu} = F^{\mu\nu}{}_P$ represents the proton, then depending on whether we contact indexes using $F_{AB} \cdot F_{BA}$ or $F_{AA} \cdot F_{BB}$, we obtain the inner and outer products [2.8] and [2.6] of [3], respectively. When $\psi_1 = \psi_u$, $\psi_2 = \psi_3 = \psi_d$ so $F^{\mu\nu} = F^{\mu\nu}{}_N$ represents the neutron, we obtain the inner and outer products [2.9] and [2.7] of [3], respectively. Using (1.6), the Koide-type generalization of the outer products [2.6] and [2.7] of [3] ($K_{AA}K_{BB}$ index summation) is:

$$E_\otimes = -\iiint \mathfrak{L}_\otimes d^3x = \tfrac{1}{2}\mathrm{Tr}\iiint F_{\mu\nu} \otimes F^{\mu\nu} d^3x = \tfrac{1}{2}\mathrm{Tr}\iiint F_{AB} \cdot F_{CD} d^3x = \tfrac{1}{2}\iiint F_{AA} \cdot F_{BB} d^3x = \tfrac{1}{(2\pi)^{\frac{3}{2}}} K_{AA}K_{BB}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\mathrm{Tr}\left[\begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix}\right] = \frac{1}{(2\pi)^{\frac{3}{2}}}\left(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}\right)^2 \quad ,(3.4)$$

while the Koide generalization of the inner products [2.8] and [2.9] of [3] ($K_{AB}K_{BA}$ index summation) is:

$$E = -\iiint \mathfrak{L} d^3x = \tfrac{1}{2}\mathrm{Tr}\iiint F_{\mu\nu} F^{\mu\nu} d^3x = \tfrac{1}{2}\mathrm{Tr}\iiint F_{AB} \cdot F_{BD} d^3x = \tfrac{1}{2}\iiint F_{AB} \cdot F_{BA} d^3x = \tfrac{1}{(2\pi)^{\frac{3}{2}}} K_{AB}K_{BA}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\mathrm{Tr}\left[\begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix} \begin{pmatrix} \sqrt{m_1} & 0 & 0 \\ 0 & \sqrt{m_2} & 0 \\ 0 & 0 & \sqrt{m_3} \end{pmatrix}\right] = \frac{1}{(2\pi)^{\frac{3}{2}}}\left(m_1 + m_2 + m_3\right) \quad ,(3.5)$$

This means that is now becomes possible to express the Koide relationship (2.9) entirely in terms of energies E derived from the general integral (3.3) of a Lagrangian density $\mathfrak{L} = -\tfrac{1}{2}Tr\left(F \cdot F\right)$ over d^3x. Specifically, combining (2.9) with (3.4) and (3.5) allows us to write:

$$\frac{E_\otimes}{E} = \frac{\iiint \mathfrak{L}_\otimes d^3x}{\iiint \mathfrak{L} d^3x} = \frac{\mathrm{Tr}\iiint F_{\mu\nu} \otimes F^{\mu\nu} d^3x}{\mathrm{Tr}\iiint F_{\mu\nu} F^{\mu\nu} d^3x} = \frac{\mathrm{Tr}\iiint F \otimes F d^3x}{\mathrm{Tr}\iiint F^2 d^3x} = \frac{\iiint F_{AA} \cdot F_{BB} d^3x}{\iiint F_{AB} \cdot F_{BA} d^3x} = \frac{K_{AA}K_{BB}}{K_{AB} \cdot K_{BA}}$$
$$= \frac{\left(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}\right)^2}{m_1 + m_2 + m_3} = \frac{3}{1+C} = R \quad . \quad (3.6)$$

This expresses the Koide mass relationship in multiple forms, in terms of the energy integral of a Lagrangian density of the general form $\mathfrak{L} = -\tfrac{1}{2}Tr\left(F \cdot F\right)$, with the field strength given by (3.2). This means that for *any* Koide triplet of given empirical *R*, there is an energy E_R which vanishes under the condition:

$$E_R = \iiint \left(\mathfrak{L}_\otimes - R\mathfrak{L}\right) d^3x = \mathrm{Tr}\iiint \left(F \otimes F - RF^2\right) d^3x = 0 \,. \quad (3.7)$$

This is the Lagrangian / energy formulation of the Koide relationship (2.9), and although different in appearance, it is entirely equivalent. So, for example, using the symbol $\therefore$ as in figure 1 and Table 3 of [6] to represent the three generations of the fermions for any given charge, the four Koide relationships (2.14) through (2.17) for the "pole" (low-probe energy) masses may be written as in the *entirely equivalent, alternative form*:

$$E_{\nu\therefore} = \iiint \left(\mathfrak{L}_\otimes - \tfrac{6}{5}\mathfrak{L}\right) d^3x = \mathrm{Tr}\iiint \left(F \otimes F - \tfrac{6}{5}F^2\right) d^3x \cong 0 \,. \quad (3.8)$$

$$E_{e\therefore} = \iiint\left(\mathfrak{L}_{\otimes} - \tfrac{3}{2}\mathfrak{L}\right)d^3x = \mathrm{Tr}\iiint\left(F\otimes F - \tfrac{3}{2}F^2\right)d^3x \cong 0\,. \tag{3.9}$$

$$E_{u\therefore} = \iiint\left(\mathfrak{L}_{\otimes} - \tfrac{6}{5}\mathfrak{L}\right)d^3x = \mathrm{Tr}\iiint\left(F\otimes F - \tfrac{6}{5}F^2\right)d^3x \cong 0\,. \tag{3.10}$$

$$E_{d\therefore} = \iiint\left(\mathfrak{L}_{\otimes} - \tfrac{15}{11}\mathfrak{L}\right)d^3x = \mathrm{Tr}\iiint\left(F\otimes F - \tfrac{15}{11}F^2\right)d^3x \cong 0\,. \tag{3.11}$$

Whether these become *exactly* equal to zero for masses at high-probe energies, and whether there is an underlying action principle involved here, are questions beyond the scope of this paper which are worth consideration.

What ties all of this together, is that we *model* the radial behavior of each fermion in the triplet ψ_1, ψ_2, ψ_3 using the Gaussian *ansatz* introduced in [9.9] of [1] which is reproduced below with an added label $i=1,2,3$ for each of the fermions and masses in (3.2):

$$\psi_i(r) = u_i(p)\left(\pi\lambda_i^2\right)^{-\frac{3}{4}}\exp\left(-\frac{1}{2}\frac{\left(r-r_{0i}\right)^2}{\bar{\lambda}_i^2}\right), \tag{3.12}$$

and that we also relate each reduced Compton wavelength $\bar{\lambda}_i$ to its corresponding mass m_i via the DeBroglie relation $\bar{\lambda}_i = \hbar / m_i c$, see [1] following [11.18]. This is what makes it possible to precisely, analytically calculate the energy in integrals of the form (3.3), specifically making use of the basic Gaussian mathematical relation [9.11] of [1]:

$$\iiint\frac{1}{\pi^{\frac{3}{2}}\bar{\lambda}^3}\exp\left(-\frac{\left(r-r_0\right)^2}{\bar{\lambda}^2}\right)d^3x = 1\,, \tag{3.13}$$

and variants thereof. It is (3.12) and (3.13) and $\bar{\lambda}_i = 1/m_i$ (in $\hbar = c = 1$ units) which tie everything together and the "nuts and bolts" mathematical level when (3.2) is employed in (3.3) through (3.7). And this is what leads to the accurate mass relationship (1.1) and binding energy predictions (1.2) and (1.3), as well as the binding energy predictions for ^{2}H, ^{3}H, ^{3}He and ^{4}He and the proton–neutron mass difference (1.4) developed in [3].

The final piece which also ties this together at nuts and bolts level, is the empirical normalization for fermion wavefunctions developed in [11.30] of [1], namely:

$$N^4 = \frac{1}{n_f}\frac{\left(E+m\right)^2}{\left(2m\right)^2} = \frac{1}{24}\frac{\left(E+m\right)^2}{\left(2m\right)^2}\,, \tag{3.14}$$

where $n_f = 24$ is the total number of fermions over three generations including three colors for each quark.

Now, it is important to emphasize that the Gaussian *ansatz* (3.12) is not a *theory*, but rather, it is a *modeling hypothesis* that allows us to perform the necessary integrations and calculate energies that turn out to correlate very well with empirical data. That is, explicitly in [1] and implicitly in [3], we *hypothesized* that the fermion wavefunctions can be modeled as Gaussians with specific Compton wavelengths $\bar{\lambda}_i = 1/m_i$ defined to match the *current* quark masses, we performed the integrations in (3.3), and we found that the energies predicted matched empirical binding data to – in most cases – parts per million. This, in turn, tells us that *for the purpose of predicting binding energies*, it is possible to model the *current* quarks as Gaussians (which means they act as free fermions), with masses and wavelengths based on their undressed, current masses, and to thereby obtain empirically-validated results. But, as also discussed at the end of section 11 in [1], this use of a current quark mass does *not* apply when it comes predicting the short range of the nuclear interaction which we showed at the end of section 10 in [1] is indeed short range with a standard deviation of $\sigma = \frac{1}{\sqrt{2}} \lambdabar$. For, if we use the current quark masses that work so well for binding energies, we find $\lambdabar_u \sim 85.65F$ and $\lambdabar_d \sim 41.04F$, and the predicted short range is still not short enough. If, however, we turn to the *constituent* quark masses which, at the end of section 11, for estimation, we took to be 939 MeV/3=313 MeV, then we have $\lambdabar \sim .63F$ and $\sigma = \frac{1}{\sqrt{2}} \lambdabar \sim .45F$, which tells us that the nuclear interaction virtually ceases at about $4\sigma \approx 3\lambdabar \sim 2F$. This is exactly what *is* observed.

In both cases – for nuclear binding energies and for the nuclear interaction short range – we found that the Gaussian *ansatz* (3.12) does yield empirically-accurate results. But for binding energies, it was the undressed, *current* quark masses which gave us the right results, while for nuclear short range, it was the fully dressed, *constituent* quarks masses that were needed to obtain the correct result. Because we shall momentarily embark on a prediction of the fully dressed rest masses 938.272046 MeV and 939.565379 MeV of the free neutron and free proton, what we learn from this is that while we might also be able to approach the neutron and proton masses using the Gaussian *ansatz* for fermion wavefunctions, we will, however, need to be judicious in the fermions we choose and in the *masses* that we assign to the fermions. That is, the focus of our deliberations will be, not *whether* we can use the Gaussian *ansatz*, but on *how to select the fermions and masses that we <u>do</u> use with the Gaussian ansatz*.

Now, based on all of the foregoing development, let us see how to predict the neutron and proton masses.

4. Predicting the Neutron plus Proton Mass Sum to within about 6 Parts in 10,000

Because we can connect any Koide matrix products to a Lagrangian via (3.4) and (3.5), let us work directly with the Koide matrix (1.6) to determine how to assign the masses m_1, m_2, m_3 so as to predict the neutron and proton masses. Then, at the end (in section 7), we can backtrack using the development in section 3 to connect these masses to their associated Lagrangian. In other words, we will first fit the empirical mass data, then we will backtrack to the underlying Lagrangian.

Each of the neutron and proton contains three quarks. The sum of the current quark masses is $2m_d + m_u = 12.0367331 MeV$ for the neutron and $2m_u + m_d = 9.35405514 MeV$ for the proton, using $m_u = 2.223792405 MeV$ and $m_d = 4.906470335 MeV$ earlier introduced after (2.11). For a *free* neutron and proton, none of their rest mass is released as binding energy, and so these quark mass sums are included in $M_P = 938.272046 MeV$ and $M_N = 939.565379 MeV$ respectively, where we use an uppercase *M* to denote these fully-dressed, observed masses. As demonstrated in sections 11 and 12 of [1] and throughout [3], these rest masses are reduced when the neutron and proton fuse with other nucleons. But for *free* protons and neutrons, the entire rest mass is retained and all of the latent binding energy is used to confine quarks. This means the "mass coverings" *m* (using a lowercase *m*) of the neutron and proton may be calculated to be:

$$m_P \equiv M_P - 2m_u - m_d = 928.9179915 MeV, \tag{4.1}$$

$$m_N \equiv M_N - 2m_u - m_d = 927.5286457 MeV. \tag{4.2}$$

That is, these *m* represent observed, fully-dressed neutron and proton masses *M*, less the sum $K_{AB}K_{BA} = m_1 + m_2 + m_3$ of the current quark masses, with $m_1 \equiv m_d$, $m_2 = m_3 \equiv m_u$ for the proton, and $m_1 \equiv m_u$, $m_2 = m_3 \equiv m_d$ for the neutron, see (1.9). One may think of m_P and m_N as the weight of rather heavy "clothing" "covering" bare quarks. The *sum* of these two mass covers is:

$$m_N + m_P = M_N + M_P - 3m_u - 3m_d = 1856.446637 MeV . \tag{4.3}$$

At the end of section 9 of [3], after deriving the neutron minus proton mass difference (1.4), we noted that the individual masses for the neutron and proton could now be obtained by deriving some independent expression related to the *sum* of their masses, and then solving these two simultaneous equations – sum equation and difference equation – for the two target masses – neutron and proton. We shall do exactly that here. In particular, it will be our goal to derive the *sum* $M_N + M_P$ of these two masses, and then use (1.4) as a simultaneous equation to obtain each separate mass. The benefit of this approach using a sum, referring to the so-called mass "toolbox" in [3.11] of [3] and also the discussion of the alpha nuclide following [4.4] of [3], is that in selecting mass terms to consider, we can eliminate any candidates that are not absolutely symmetric under $p \leftrightarrow n$ and $u \leftrightarrow d$ interchange, because the sum $M_N + M_P$ contains three up quarks and three down quarks, as well as one neutron and one proton. Our empirical target, therefore, is $M_N + M_P = 1877.837425 MeV$, or, alternatively, $m_P + m_N = 1856.446637 MeV$ from (4.3) to which we can then readily add $3m_u + 3m_d$. This is what we seek to predict.

Now let us return to the "clues" we laid out in [3.6] through [3.8] of [6]. We start in the simplest way possible by focusing our consideration on [3.8] of [6], reproduced below, but multiplied by a factor of 2 and separated into $\sqrt[4]{v_F m_u}$ and $\sqrt[4]{v_F m_d}$ in the second term:

$$2\sqrt{v_F \cdot \sqrt{m_u m_d}} = 2\sqrt[4]{v_F m_u}\sqrt[4]{v_F m_d} = 2\sqrt[4]{{v_F}^2 m_u d_d} = 1803.670518 \text{ MeV} . \tag{4.4}$$

Here, v_F=246.219651 GeV is the Fermi vev. Because this is about 3% smaller than $m_P + m_N$ in (4.3) and it is closer to $m_P + m_N$ than either [3.6] or [3.7] of [6], and it is symmetric under $u \leftrightarrow d$ interchange, we shall see if (4.4) can be used, by itself, to provide the foundation for reaching the $m_P + m_N = 1856.446637 MeV$ mass target. As we shall, it can be so used!

Now, in (3.11) of [3], we developed a "toolkit" of masses which we used for calculating the binding and fusion release energies of all the 1s nuclides with very close precision. We shall wish to add to this toolkit here, and in particular, will wish to refine our use of the Fermi vev v_F=246.219651 GeV beyond what is shown in (4.4). Specifically, as noted after [3.8] of [6], we need to put (4.4) "and like expressions into the right context and obtain the right coefficients. And where do such coefficients come from? The generators of a GUT!"

Now, we shall use the GUT we developed in [6] to obtain the coefficients needed to bring (4.4) closer to the target mass of $1856.446637 MeV$ in (4.3). Because the vev that seems to bring us into the correct "ballpark" is the Fermi vev, we focus on the electroweak symmetry breaking which occurs at the Fermi vev, and which, in [8.2] of [6], is specified by breaking symmetry using the electric charge generator Q according to:

$$\text{diag}\left(\Phi_F\right) = \text{diag}\left(T^i \varphi_{iF}\right) \equiv v_F\left(0, \tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, -1, -\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) = v_F \text{diag} Q . \tag{4.5}$$

For the proton with a fermion triplet (d,u,u), the corresponding eigenvalue entries in (4.5) above are $\left(-\tfrac{1}{3}v_F, \tfrac{2}{3}v_F, \tfrac{2}{3}v_F\right)$. For the neutron and its (u,d,d) triplet, the entries are $\left(\tfrac{2}{3}v_F, -\tfrac{1}{3}v_F, -\tfrac{1}{3}v_F\right)$. We now wish to use these to establish respective Koide triplet matrices for the neutron and proton which can be used to generate the sum of their masses.

Looking at the vacuum triplets $\left(-\tfrac{1}{3}v_F, \tfrac{2}{3}v_F, \tfrac{2}{3}v_F\right)$ and $\left(\tfrac{2}{3}v_F, -\tfrac{1}{3}v_F, -\tfrac{1}{3}v_F\right)$, we see that to match the mass dimension ½ of the terms with $\sqrt[4]{vm_u}$ and $\sqrt[4]{vm_d}$ in (4.4) and use these as Koide triplets, we will need to take the fourth roots of these vacuum triplets. So we do exactly that, and pair these triplets with the mass triplets (m_d, m_u, m_u) and (m_u, m_d, m_d) for which we also take the fourth root to match (4.4). Thus, use $\left(-\tfrac{1}{3}v_F, \tfrac{2}{3}v_F, \tfrac{2}{3}v_F\right) \to \left(i^{.5}\sqrt[4]{\tfrac{1}{3}v_F m_d}, \sqrt[4]{\tfrac{2}{3}v_F m_u}, \sqrt[4]{\tfrac{2}{3}v_F m_u}\right)$ and $\left(\tfrac{2}{3}v_F, -\tfrac{1}{3}v_F, -\tfrac{1}{3}v_F\right) \to \left(\sqrt[4]{\tfrac{2}{3}v_F m_u}, i^{.5}\sqrt[4]{\tfrac{1}{3}v_F m_d}, i^{.5}\sqrt[4]{\tfrac{1}{3}v_F m_d}\right)$ to define two Koide triplets, one for the neutron and one for the proton, as follows:

$$K_{AB}(N) \equiv \begin{pmatrix} \sqrt[4]{\tfrac{2}{3}v_F m_u} & 0 & 0 \\ 0 & i^{.5}\sqrt[4]{\tfrac{1}{3}v_F m_d} & 0 \\ 0 & 0 & i^{.5}\sqrt[4]{\tfrac{1}{3}v_F m_d} \end{pmatrix}, \tag{4.6}$$

$$K_{AB}(P) \equiv \begin{pmatrix} i^{.5}\sqrt[4]{\tfrac{1}{3}v_F m_d} & 0 & 0 \\ 0 & \sqrt[4]{\tfrac{2}{3}v_F m_u} & 0 \\ 0 & 0 & \sqrt[4]{\tfrac{2}{3}v_F m_u} \end{pmatrix}. \tag{4.7}$$

We see that because of the negative charge of the down quark, each of these triplets contains components with the coefficient $\sqrt[4]{-1} = i^{.5} = \tfrac{1}{\sqrt{2}}(1+i)$, which is a complex number. In recent years, consideration has been given to having *negative* square root terms in Koide mass relations, see for example (2.21) in which one uses $-\sqrt{m_s}$ to derive a close relation for the (csb) triplet. The above, (4.6) and (4.7) take this a step further, because they raise the specter of triplets with *complex* square root coefficients! In the next section we shall explore the implications of these complex components, which arise from the oppositely-signed charges of the up and down quarks. But for the moment, let us ignore $i^{.5}$ in the above so we can look at magnitudes only, and let us form and calculate the following Koide matrix product with $i^{.5}$ excised:

$$K_{AB}(P)K_{BA}(N) = Tr\left[\begin{pmatrix} \sqrt[4]{\tfrac{1}{3}v_F m_d} & 0 & 0 \\ 0 & \sqrt[4]{\tfrac{2}{3}v_F m_u} & 0 \\ 0 & 0 & \sqrt[4]{\tfrac{2}{3}v_F m_u} \end{pmatrix}\begin{pmatrix} \sqrt[4]{\tfrac{2}{3}v_F m_u} & 0 & 0 \\ 0 & \sqrt[4]{\tfrac{1}{3}v_F m_d} & 0 \\ 0 & 0 & \sqrt[4]{\tfrac{1}{3}v_F m_d} \end{pmatrix}\right]. \tag{4.8}$$

$$= 3\cdot\sqrt[4]{\tfrac{2}{9}v_F{}^2 m_u m_d} = \mathbf{1857.570635\ MeV}$$

Comparing to (4.3) which tells us that $m_P + m_N = \mathbf{1856.446637\ MeV}$ we see that we have hit the target to within about 0.06%! That is:

$$\frac{K_{AB}(P)K_{BA}(N)}{(m_N + m_P)_{\text{Observed}}} = \frac{\mathbf{1857.570635\ MeV}}{\mathbf{1856.446637\ MeV}} = \mathbf{1.000605457}\,! \tag{4.9}$$

This is extremely close, and in particular, we now see that to within about 6 parts in 10,000, the sum of the neutron and proton masses may be expressed completely as a function of the up and down quark masses and the Fermi vev! So if we use this close relationship to hypothesize that a meaningful relationship is given by $\left(m_N + m_P\right)_{\text{Predicted}} \cong K_{AB}(P)K_{BA}(N)$, then using the above with (4.3), we now see that to within about 0.06%:

$$M_N + M_P = m_N + m_P + 3m_u + 3m_d \cong 3\cdot\sqrt[4]{\tfrac{2}{9}v_F{}^2 m_u m_d} + 3m_u + 3m_d\ . \tag{4.10}$$

We have now discovered the correct coefficients for the "clue" in (4.4), which yields our neutron plus proton mass sum to 6 parts in 10,000! Further qualifying (4.10) as a proper and not merely coincidental expression for the neutron plus proton mass sum, we see that this is symmetric under $u \leftrightarrow d$ interchange, and that it is formed by taking the inner product $K_{AB}(P)K_{BA}(N)$ of a Koide proton matrix $\mathrm{diag}(K(P)) = \left(\sqrt[4]{\tfrac{1}{3}v_F m_d}, \sqrt[4]{\tfrac{2}{3}v_F m_u}, \sqrt[4]{\tfrac{2}{3}v_F m_u}\right)$ times a Koide neutron matrix $\mathrm{diag}(K(N)) = \left(\sqrt[4]{\tfrac{2}{3}v_F m_u}, \sqrt[4]{\tfrac{1}{3}v_F m_d}, \sqrt[4]{\tfrac{1}{3}v_F m_d}\right)$ and so which product $K_{AB}(P)K_{BA}(N)$ is symmetric under $p \leftrightarrow n$ interchange. Further, both of these fully embed the electric charges and mass magnitudes of the quarks, and. So in sum, (4.10) makes sense on multiple bases: its yields an empirical match to within 6 parts in 10,000, it is the product of a proton matrix with a neutron matrix, the proton matrix contains the masses and charges of two up quarks and one down quark while the neutron matrix contains the same of two down quarks and one up quark, and it is symmetric under both $u \leftrightarrow d$ and $p \leftrightarrow n$ interchange.

Furthermore, if we divide (4.8) by 2, we see that:

$$K_{AB}(P)K_{BA}(N)/2 = \tfrac{3}{2}\sqrt[4]{\tfrac{2}{9}v_F{}^2 m_u m_d} = 928.7853174\,MeV\ . \tag{4.11}$$

This actually falls *between* $m_P = 928.9179915\,MeV$ and $m_N = 927.5286457\,MeV$ from (4.1) and (4.2), and so (4.10) clearly appears to be a correct expression for the leading terms in the neutron and proton masses. Based on this close concurrence and "threading the needle" between the neutron and proton masses with (4.11) and all of the appropriate symmetries noted in the previous paragraph, we now regard (4.10) as a *meaningful* (rather than coincidental) close expression for $M_P + M_N$ to 0.06%.

It will simplify and clarify the calculations from here to define what we shall refer to as "vacuum-amplified" up and down quark masses according to:

$$M_u \equiv \sqrt{\tfrac{2}{3}v_F m_u} = 604.1751345\,MeV\ . \tag{4.12}$$

$$M_d \equiv \sqrt{\tfrac{1}{3}v_F m_d} = 634.5784463\,MeV\ . \tag{4.13}$$

Consequently:

$$\sqrt{M_u M_d} = \sqrt[4]{\tfrac{2}{9}v_F{}^2 m_u m_d} = 619.1902116\,MeV\ . \tag{4.14}$$

This means that the mass sum (4.10) may be rewritten more transparently as:

$$M_N + M_P = m_N + m_P + 3m_u + 3m_d \cong 3\left(\sqrt{M_u M_d} + m_u + m_d\right), \tag{4.15}$$

while the Koide mass matrices (4.6) and (4.7) for the neutron and proton become:

$$K_{AB}(P) = \begin{pmatrix} i^{.5}\sqrt{M_d} & 0 & 0 \\ 0 & \sqrt{M_u} & 0 \\ 0 & 0 & \sqrt{M_u} \end{pmatrix}, \tag{4.16}$$

$$K_{AB}(N)=\begin{pmatrix}\sqrt{M_u} & 0 & 0\\ 0 & i^{.5}\sqrt{M_d} & 0\\ 0 & 0 & i^{.5}\sqrt{M_d}\end{pmatrix}. \tag{4.17}$$

These matrices now restore the $i^{.5}=\frac{1}{\sqrt{2}}(1+i)$ factor that we excised to calculate (4.8). Thus, as in (4.8), but including this complex factor, we now take:

$$\begin{aligned}K_{AB}(P)K_{BA}(N)&=Tr\left[\begin{pmatrix}i^{.5}\sqrt{M_d} & 0 & 0\\ 0 & \sqrt{M_u} & 0\\ 0 & 0 & \sqrt{M_u}\end{pmatrix}\begin{pmatrix}\sqrt{M_u} & 0 & 0\\ 0 & i^{.5}\sqrt{M_d} & 0\\ 0 & 0 & i^{.5}\sqrt{M_d}\end{pmatrix}\right].\\ &=3i^{.5}\sqrt{M_uM_d}=\tfrac{1}{\sqrt{2}}(1+i)\mathbf{1857.570635\ MeV}\end{aligned} \tag{4.18}$$

Having found the right magnitude, we could make use of a $\sqrt{2}$ factor and continue to match the empirical data by writing $\sqrt{2}\,\mathrm{Re}\left(K_{AB}(P)K_{BA}(N)\right)\cong m_P+m_N$. But this just sidesteps understanding the meaning of this complex factor and it does not help us past the 0.06% difference that still remains between the predicted and the empirical data. We need to find a more fundamental way to understand this complex factor. That will be the subject of the discussion in the next section.

5. Exact Characterization of the Neutron and Proton Masses via a Mixing Angle *θ* and Phase Angle *δ*

Let us first represent this factor $i^{.5}=\frac{1}{\sqrt{2}}(1+i)$ in terms of a phase angle δ' such that:

$$i^{.5}=\tfrac{1}{\sqrt{2}}(1+i)=\exp(i\delta')=\cos\delta'+i\sin\delta';\quad \delta'=\pi/4. \tag{5.1}$$

Then, we briefly rename $K\to K'$ and use this phase to rewrite (4.18) as:

$$\begin{aligned}K'_{AB}(P)K'_{BA}(N)&=Tr\left[\begin{pmatrix}e^{i\delta'}\sqrt{M_d} & 0 & 0\\ 0 & \sqrt{M_u} & 0\\ 0 & 0 & \sqrt{M_u}\end{pmatrix}\begin{pmatrix}\sqrt{M_u} & 0 & 0\\ 0 & e^{i\delta'}\sqrt{M_d} & 0\\ 0 & 0 & e^{i\delta'}\sqrt{M_d}\end{pmatrix}\right].\\ &=3\exp(i\delta')\sqrt{M_uM_d}=m'_N+m'_P\end{aligned} \tag{5.2}$$

with similar updates in (4.16). Then, we use this to rewrite the mass sum (4.15) as:

$$M'_N+M'_P=m'_N+m'_P+3m_u+3m_d\cong 3\left(\exp(i\delta')\sqrt{M_uM_d}+m_u+m_d\right), \tag{5.3}$$

where we have also briefly renamed $M\to M'$ and $m_{P,N}\to m'_{P,N}$, all with $\delta'=\pi/4$.

Now, (5.3) gives us the opportunity to define a new Koide matrix E_{AB} which we shall refer to as the “electron generation matrix” E as such:

$$\mathrm{E}_{AB} \equiv \sqrt{3}\begin{pmatrix} \sqrt[4]{M_u M_d} & 0 & 0 \\ 0 & \sqrt{m_u} & 0 \\ 0 & 0 & \sqrt{m_d} \end{pmatrix}. \tag{5.4}$$

Then, making note the phase $\exp(i\delta')$ which multiplies $\sqrt{M_u M_d}$ in (5.3) and keeping in mind how the Kobayashi and Maskawa mixing matrices are formed for three generations, we introduce a new angle θ such that $\theta' = 0$, and form a unitary matrix U_1 including $e^{i\delta'}$, as such:

$$U_{1AB} \equiv \begin{pmatrix} \exp(i\delta') & 0 & 0 \\ 0 & \cos\theta_1' & \sin\theta_1' \\ 0 & -\sin\theta_1' & \cos\theta_1' \end{pmatrix}. \tag{5.5}$$

Of course, with $\delta' = \pi/4$ and $\theta' = 0$, U is diagonal matrix $\mathrm{diag}U = \left(i^{.5},1,1\right)$. So (5.5) multiplied by (5.4) simply generalizes the appearance of the term $i^{.5}\sqrt{M_u M_d}$ in (4.18). But now, let us permit both δ and θ to rotate freely, $\theta' \to \theta$, $\delta' \to \delta$. Then from (5.4) and (5.5), we may form:

$$\begin{aligned} M_N + M_P &\equiv \mathrm{E}_{AB} U_{1BC} \mathrm{E}_{CA} = 3\mathrm{Tr}\begin{pmatrix} \sqrt{M_u M_d}\exp(i\delta) & 0 & 0 \\ 0 & m_u \cos\theta_1 & \sqrt{m_u m_d}\sin\theta_1 \\ 0 & -\sqrt{m_u m_d}\sin\theta_1 & m_d\cos\theta_1 \end{pmatrix} \\ &= 3\mathrm{Tr}\begin{pmatrix} \sqrt[4]{M_u M_d} & 0 & 0 \\ 0 & \sqrt{m_u} & 0 \\ 0 & 0 & \sqrt{m_d} \end{pmatrix}\begin{pmatrix} \exp(i\delta) & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} \sqrt[4]{M_u M_d} & 0 & 0 \\ 0 & \sqrt{m_u} & 0 \\ 0 & 0 & \sqrt{m_d} \end{pmatrix}. \\ &= 3\left(\exp(i\delta)\sqrt{M_u M_d} + m_u\cos\theta_1 + m_d\cos\theta_1\right) \end{aligned} \tag{5.6}$$

For the special case where $\theta \to \theta' = 0$, we precisely reproduce (5.3). But in (5.6) we have removed the approximation sign $\cong$ that was in (5.3), because we are now going to *define* the angles θ, δ so as to *precisely* match up with the *empirical* values of the neutron and proton masses. That is, just as (1.4) is an *exact* formula for the proton–neutron mass difference, we shall now regard (5.6) as an *exact* formula for the neutron plus proton mass sum, with the numerical values of θ, δ *defined by empirical data* so as to make this an exact fit.

Before we proceed, let us recap so we are clear what we have just done: What we have done here is to use the matrix $\mathrm{diag}U = \left(i^{.5},1,1\right)$ implicit in (5.3) as a hint of a matrix $\mathrm{diag}U = \left(\exp(i\delta'),1,1\right)$ with $\delta' = \pi/4$, then use $\mathrm{diag}U = \left(\exp(i\delta'),1,1\right)$ as a further hint of a matrix $\mathrm{diag}U = \left(\exp(i\delta'),\cos\theta',\cos\theta'\right)$ with $\theta' = 0$, then allowed both of these angles to freely rotate yielding (5.5). Then we have used (5.5) to form (5.6) which generalizes (5.3). Now, we will use these angles to permit the otherwise close relationship (5.3) to be fitted exactly by empirically choosing these angles to yield and exact fit.

Before we do this, however, there is a final cascade to this hint, which is to recognize that

(5.5) with the angles free to rotate is one of the three matrices used to define the Cabibbo matrices used for electroweak generation mixing, see [7.11] in [6], and in particular, *is the matrix that is use to introduce the phase angle responsible for CP violation*. We also note that (5.4) is strictly a function of the first (electron generation) quark masses and the Fermi vev which makes the upper left component $\sqrt[4]{M_u M_d}$ containing the "vacuum-enhanced" quark masses substantially larger than the middle and lower right components $\sqrt{m_u}$ and $\sqrt{m_d}$. This is why we named this matrix E_{AB} the electron generation matrix. Because Cabibbo mixing has two more matrices and also mixes two more generations, let us now form two more matrices M and T analogous to (5.4) for the muon and tauon generation of quarks as follows, but following the pattern for mixing in the original parameterization of Kobayashi and Maskawa, we put the large components $\sqrt[4]{M_c M_s}$ and $\sqrt[4]{M_t M_b}$ in the lower right positions. And, as a matter of convention, we keep the up (electric charge = +2/3) series of mass terms in the middle position. Thus we define the muon and tauon generation matrices:

$$\mathrm{M}_{AB} \equiv \sqrt{3}\begin{pmatrix} \sqrt{m_s} & 0 & 0 \\ 0 & \sqrt{m_c} & 0 \\ 0 & 0 & \sqrt[4]{M_c M_s} \end{pmatrix}; \quad \mathrm{T}_{AB} \equiv \sqrt{3}\begin{pmatrix} \sqrt{m_b} & 0 & 0 \\ 0 & \sqrt{m_t} & 0 \\ 0 & 0 & \sqrt[4]{M_t M_b} \end{pmatrix}, \tag{5.7}$$

At the same time, analogously to (4.12) and (4.13), we define:

$$M_c \equiv \sqrt{\tfrac{2}{3} v m_c} = 14,467 MeV, \tag{5.8}$$

$$M_s \equiv \sqrt{\tfrac{1}{3} v m_s} = 2792 MeV, \tag{5.9}$$

$$M_t \equiv \sqrt{\tfrac{2}{3} v m_t} = 168,758 MeV, \tag{5.10}$$

$$M_b \equiv \sqrt{\tfrac{1}{3} v m_b} = 18,522 MeV, \tag{5.11}$$

which yields the higher-generation analogues to (4.14):

$$\sqrt{M_c M_s} = 6356 MeV, \tag{5.12}$$

$$\sqrt{M_t M_b} = 55,908 MeV. \tag{5.13}$$

These values are calculated from the PDG data [8] laid out prior to (2.12). and rounded to the nearest MeV (recognizing substantial experimental uncertainties).

We also define two more matrices analogous to (5.5) for the second and third generations in same manner as is used to form the Cabibbo mixing matrices, again see [7.11] in [6]:

$$U_{2AB} \equiv \begin{pmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad U_{3AB} \equiv \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.14}$$

Then, analogously to (5.6), for the second and third generations, respectively, we form:

$$\mathrm{M}_{AB} U_{2BC} \mathrm{M}_{CA} = 3\mathrm{Tr}\begin{pmatrix} m_s\cos\theta_2 & \sqrt{m_s m_c}\sin\theta_2 & 0 \\ -\sqrt{m_s m_c}\sin\theta_2 & m_c\cos\theta_2 & 0 \\ 0 & 0 & \sqrt{M_c M_s} \end{pmatrix} = 3\left(\sqrt{M_c M_s} + \cos\theta_2 m_c + \cos\theta_2 m_s\right), \tag{5.15}$$

$$\mathrm{T}_{AB}U_{3BC}\mathrm{T}_{CA} = 3\mathrm{Tr}\begin{pmatrix} m_b\cos\theta_3 & \sqrt{m_b m_t}\sin\theta_3 & 0 \\ -\sqrt{m_b m_t}\sin\theta_3 & m_t\cos\theta_3 & 0 \\ 0 & 0 & \sqrt{M_t M_b} \end{pmatrix} = 3\left(\sqrt{M_t M_b} + \cos\theta_3 m_t + \cos\theta_3 m_b\right). \quad (5.16)$$

Then, we multiply all three of (5.6), (5.15) and (5.16) together in the same manner that the Cabibbo mixing matrices are formed, again see [7.11] in [6], to obtain a master "mass and mixing matrix" Θ with mass dimension +3, defined as:

$$\Theta \equiv \mathrm{M}\cdot U_2\cdot \mathrm{M}\cdot \mathrm{E}\cdot U_1\cdot \mathrm{E}\cdot \mathrm{T}\cdot U_3\cdot \mathrm{T}$$

$$= 27\begin{pmatrix} \begin{array}{l} -m_u\sqrt{m_s m_c}\sqrt{m_b m_t}\,\mathrm{c}_1\mathrm{s}_2\mathrm{s}_3 \\ +\sqrt{M_u M_d}\,m_s m_b\,\mathrm{c}_2\mathrm{c}_3\mathrm{e}^{i\delta} \end{array} & \begin{array}{l} m_u\sqrt{m_s m_c}\,m_t\,\mathrm{c}_1\mathrm{s}_2\mathrm{c}_3 \\ +\sqrt{M_u M_d}\,m_s\sqrt{m_b m_t}\,\mathrm{c}_2\mathrm{s}_3\mathrm{e}^{i\delta} \end{array} & \sqrt{m_u m_d}\sqrt{m_s m_c}\sqrt{M_t M_b}\,\mathrm{s}_1\mathrm{s}_2 \\ \\ \begin{array}{l} -m_u m_c\sqrt{m_b m_t}\,\mathrm{c}_1\mathrm{c}_2\mathrm{s}_3 \\ -\sqrt{M_u M_d}\sqrt{m_s m_c}\,m_b\,\mathrm{s}_2\mathrm{c}_3\mathrm{e}^{i\delta} \end{array} & \begin{array}{l} m_u m_c m_t\,\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3 \\ -\sqrt{M_u M_d}\sqrt{m_s m_c}\sqrt{m_b m_t}\,\mathrm{s}_2\mathrm{s}_3\mathrm{e}^{i\delta} \end{array} & \sqrt{m_u m_d}\,m_c\sqrt{M_t M_b}\,\mathrm{s}_1\mathrm{c}_2 \\ \\ \sqrt{m_u m_d}\sqrt{M_c M_s}\sqrt{m_b m_t}\,\mathrm{s}_1\mathrm{s}_3 & -\sqrt{m_u m_d}\sqrt{M_c M_s}\,m_t\,\mathrm{s}_1\mathrm{c}_3 & m_d\sqrt{M_c M_s}\sqrt{M_t M_b}\,\mathrm{c}_1 \end{pmatrix}. \quad (5.17)$$

This matrix contains all six of the quark masses in all three generations, all three of the real mixing angles and the one CP violating phase angle that appears when the three generations are mixed, and implied in the vacuum-enhanced mass terms, the Fermi vev and the electric charges of all of these quarks. If all of the masses are set to equal 1, this reduces to the usual generational mixing matrix in the original parameterization of Kobayashi and Maskawa, seen in, e.g., see [7.11] in [6]. In the circumstance where $s_2 = 0$, $s_3 = 0$, this reduces to:

$$\Theta = 27\begin{pmatrix} \sqrt{M_u M_d}\,m_s m_b\,\mathrm{e}^{i\delta} & 0 & 0 \\ 0 & m_u m_c m_t\cos\theta_1 & \sqrt{m_u m_d}\,m_c\sqrt{M_t M_b}\sin\theta_1 \\ 0 & -\sqrt{m_u m_d}\sqrt{M_c M_s}\,m_t\sin\theta_1 & m_d\sqrt{M_c M_s}\sqrt{M_t M_b}\cos\theta_1 \end{pmatrix}. \quad (5.18)$$

and in the further circumstance where all of the second and third generation masses are set to 1, this further reduces to 9 times the matrix shown in (5.6):

$$\Theta = 27\begin{pmatrix} \sqrt{M_u M_d}\,\mathrm{e}^{i\delta} & 0 & 0 \\ 0 & m_u\cos\theta_1 & \sqrt{m_u m_d}\sin\theta_1 \\ 0 & -\sqrt{m_u m_d}\sin\theta_1 & m_d\cos\theta_1 \end{pmatrix}. \quad (5.19)$$

So that comparing with (5.6), in this particular special case, (5.17) even contains the neutron plus proton mass sum:

$$\tfrac{1}{9}\mathrm{Tr}\Theta = 3\left(\sqrt{M_u M_d}\exp(i\delta) + m_u\cos\theta_1 + m_d\cos\theta_1\right) = M_N + M_P\,! \quad (5.20)$$

So this puts this nucleon (baryon) mass sum in a broader context that includes all of the generational mixing angles and all of the quark masses and their electric charges and the Fermi vev. Certainly, (5.17) can and will be used therefore to gain substantial new insights into fermion and baryon masses. And all of this emerges in cascade fashion from the simple hint of a matrix with $\mathrm{diag}U = \left(i^{.5},1,1\right)$ in the neutron plus proton mass formula (5.3), with the $i^{.5}$ itself having emerged from the simple fact that up and down quarks have opposite charges which led to terms containing $\sqrt[4]{-1}$ when we formed Koide matrices to represent masses.

With this important contextual digression, we now solve (1.4) and (5.6) as *simultaneous equations*, that is, we solve the equation set:

$$\begin{cases} M_P + M_N = 3\left(\sqrt{M_u M_d}\exp(i\delta) + m_u\cos\theta + m_d\cos\theta\right) \\ M_N - M_P = m_u - \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}} \end{cases}. \tag{5.21}$$

It now requires no more than elementary algebra to determine that the neutron and proton masses, separately, are each given by:

$$\begin{aligned} M_N &= \tfrac{1}{2}\left(3\left(\sqrt{M_u M_d}\exp(i\delta) + \cos\theta_1(m_u + m_d)\right) + m_u - \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}}\right) \\ M_P &= \tfrac{1}{2}\left(3\left(\sqrt{M_u M_d}\exp(i\delta) + \cos\theta_1(m_u + m_d)\right) - m_u + \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}}\right) \end{aligned}. \tag{5.22}$$

These can be made into *exact* theoretical expressions for the neutron and proton mass by solving for θ, δ, to find their *empirical* values based on the neutron and proton masses. Let's do so.

Because each of (5.22) contains a complex phase, we will need to form the square modulus magnitude $|M|^2 = M^* M$ of these masses. So first we deduce:

$$\begin{aligned} 4|M_N|^2 &= 9M_u M_d + 6\cos\delta\sqrt{M_u M_d}\left(3\cos\theta_1(m_u + m_d) + m_u - \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}}\right) \\ &\quad + \left(3\cos\theta_1(m_u + m_d) + m_u - \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}}\right)^2 \\ 4|M_P|^2 &= 9M_u M_d + 6\cos\delta\sqrt{M_u M_d}\left(3\cos\theta_1(m_u + m_d) - m_u + \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}}\right) \\ &\quad + \left(3\cos\theta_1(m_u + m_d) - m_u + \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}}\right)^2 \end{aligned}. \tag{5.23}$$

Now we solve these as simultaneous equations for θ_1 and δ. First we restructure in terms of δ:

$$\begin{aligned} \cos\delta &= \frac{4|M_N|^2 - 9M_u M_d - \left(3\cos\theta_1(m_u + m_d) + m_u - \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}}\right)^2}{6\sqrt{M_u M_d}\left(3\cos\theta_1(m_u + m_d) + m_u - \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}}\right)} \\ \cos\delta &= \frac{4|M_P|^2 - 9M_u M_d - \left(3\cos\theta_1(m_u + m_d) - m_u + \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}}\right)^2}{6\sqrt{M_u M_d}\left(3\cos\theta_1(m_u + m_d) - m_u + \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}}\right)} \end{aligned}. \tag{5.24}$$

We now set these equal to one another to eliminate δ and solve for θ. It will be easier to see the underlying structure of these equations as well as solve them if we write the above as:

$$C\cos\delta = \frac{N - (B\cos\theta_1 + A)^2}{(B\cos\theta_1 + A)} = \frac{P - (B\cos\theta_1 - A)^2}{(B\cos\theta_1 - A)} \tag{5.25}$$

using the following substitution of variables:

$$\begin{aligned} &N \equiv 4|M_N|^2 - 9M_u M_d; \quad P \equiv 4|M_P|^2 - 9M_u M_d \\ &A \equiv m_u - \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right)/(2\pi)^{\frac{3}{2}}; \quad B \equiv 3(m_u + m_d); \quad C \equiv 6\sqrt{M_u M_d} \end{aligned}. \tag{5.26}$$

Next, we reduce (5.25) successively in five steps as follows:

$$1: \quad \left(N-\left(B\cos\theta_1+A\right)^2\right)\left(B\cos\theta_1-A\right)=\left(P-\left(B\cos\theta_1-A\right)^2\right)\left(B\cos\theta_1+A\right)$$

$$2: \quad N\left(B\cos\theta_1-A\right)-\left(B^2\cos^2\theta_1-A^2\right)\left(B\cos\theta_1+A\right)=P\left(B\cos\theta_1+A\right)-\left(B^2\cos^2\theta_1-A^2\right)\left(B\cos\theta_1-A\right)$$

$$3: \quad \frac{N\left(B\cos\theta_1-A\right)}{\left(B^2\cos^2\theta_1-A^2\right)}-A=\frac{P\left(B\cos\theta_1+A\right)}{\left(B^2\cos^2\theta_1-A^2\right)}+A \tag{5.27}$$

$$4: \quad N\left(B\cos\theta_1-A\right)-A\left(B^2\cos^2\theta_1-A^2\right)=P\left(B\cos\theta_1+A\right)+A\left(B^2\cos^2\theta_1-A^2\right)$$

$$5: \quad 0=2AB^2\cos^2\theta_1-\left(N-P\right)B\cos\theta_1+\left(N+P\right)A-2A^3$$

In the final line, we arrive at a quadratic. We obtain the solution via the quadratic equation. Then, we use the variables (5.26) including the empirical masses of the neutron and proton, to calculate that:

$$\cos\theta_1=\frac{N-P-\sqrt{\left(N-P\right)^2-8\left(A^2\left(N+P\right)-2A^4\right)}}{4AB}=0.9474541242; \quad \sin\theta_1=0.31989167 \tag{5.28}$$

In the above, we use the negative root, because this yields a $-1\le\cos\theta_1\le1$. This means that the *empirically-determined* value of θ_1 is:

$$\theta=0.32561515\text{ rad}=18.65637386°=\pi/9.64817715. \tag{5.29}$$

Now, we use (5.28) in (5.25) to solve for δ, and calculate to find that:

$$\begin{aligned}\cos\delta&=\frac{N-\left(\left(N-P-\sqrt{\left(N-P\right)^2-8\left(A^2\left(N+P\right)-2A^4\right)}\right)\Big/4A+A\right)^2}{C\left(\left(N-P-\sqrt{\left(N-P\right)^2-8\left(A^2\left(N+P\right)-2A^4\right)}\right)\Big/4A+A\right)}\\&=\frac{P-\left(\left(N-P-\sqrt{\left(N-P\right)^2-8\left(A^2\left(N+P\right)-2A^4\right)}\right)\Big/4A-A\right)^2}{C\left(\left(N-P-\sqrt{\left(N-P\right)^2-8\left(A^2\left(N+P\right)-2A^4\right)}\right)\Big/4A-A\right)}=1\end{aligned} \tag{5.30}$$

The numerical calculation reveals that $\cos\delta=1$, *exactly*, so the phase factor $\delta=0$. This means that when the variables in (5.26) are substituted into (5.30), the extremely unwieldy-looking resulting expression will reduce to 1 identically! So to the extent that δ is a CP-violating phase, and given that $\delta=0$ is a deduced result for the neutron and proton masses (5.22), this tells us that there is are no CP-violating effects associated with neutron and proton. This is validated by the empirical data which shows that the mass of the antiproton is equal to that of the proton, and the mass of the antineutron is equal to that of the neutron, see, e.g., [9], [10]. So, we take (5.22) to now be *exact* formulations of the neutron and proton masses, in the circumstance where the empirically-determined angle $\theta=0.32561515$ and the CP-violating phase $\delta=0$.

So we now return to (5.22), set $\delta=0$, and so obtain our final expressions for the neutron and proton masses:

$$\begin{aligned}M_N&=\tfrac{1}{2}\left(3\left(\sqrt{M_uM_d}+\cos\theta_1\left(m_u+m_d\right)\right)+m_u-\left(3m_d+2\sqrt{m_\mu m_d}-3m_u\right)/\left(2\pi\right)^{\frac{3}{2}}\right)\\M_P&=\tfrac{1}{2}\left(3\left(\sqrt{M_uM_d}+\cos\theta_1\left(m_u+m_d\right)\right)-m_u+\left(3m_d+2\sqrt{m_\mu m_d}-3m_u\right)/\left(2\pi\right)^{\frac{3}{2}}\right)\end{aligned}, \tag{5.31}$$

which are *exact* relations with the empirical substitution $\theta=0.32561515=\pi/9.64817715$.

These relationships, in turn, now enable us to go back to the masses for the 1s nuclides predicted to high accuracy and rewrite [7.6], [7.1], [7.3] and [7.5] of [3], respectively, as:

$$ {}^{2}_{1}M = M_P + M_N - m_u = 3\left(\sqrt{M_u M_d} + \cos\theta_1 \left(m_u + m_d\right)\right) - m_u, \tag{5.32}$$

$$\begin{aligned} {}^{3}_{1}M &= M_P + 2M_N - 4m_u + 2\sqrt{m_\mu m_d} \;/\; (2\pi)^{\frac{3}{2}} \\ &= \tfrac{1}{2}\left(9\left(\sqrt{M_u M_d} + \cos\theta_1\left(m_u + m_d\right)\right) - 7m_u - \left(3m_d - 2\sqrt{m_\mu m_d} - 3m_u\right)\left(2\pi\right)^{\frac{3}{2}}\right) \end{aligned} . \tag{5.33}$$

$$\begin{aligned} {}^{3}_{2}M &= 2M_P + M_N - 2m_u - \sqrt{m_u m_d} \\ &= \tfrac{1}{2}\left(9\left(\sqrt{M_u M_d} + \cos\theta_1\left(m_u + m_d\right)\right) - 5m_u - 2\sqrt{m_u m_d} + \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right) / \left(2\pi\right)^{\frac{3}{2}}\right) \end{aligned} . \tag{5.34}$$

$$\begin{aligned} {}^{4}_{2}M &= 2M_P + 2M_N - 6m_u - 6m_d + \left(10m_d + 10m_u + 16\sqrt{m_u m_d}\right) / \left(2\pi\right)^{\frac{3}{2}} + 2\sqrt{m_u m_d} \\ &= 6\left(\sqrt{M_u M_d} + \cos\theta_1\left(m_u + m_d\right) - m_u - m_d\right) + 2\sqrt{m_u m_d} + \left(10m_d + 10m_u + 16\sqrt{m_u m_d}\right) / \left(2\pi\right)^{\frac{3}{2}} \end{aligned} \tag{5.35}$$

The binding energies ${}^{A}_{Z}B_0 = Z \cdot {}^{1}_{1}M + N \cdot {}^{1}_{0}M - {}^{A}_{Z}M = ZM_P + NM_N - {}^{A}_{Z}M$ for any given nuclide with Z protons and N neutrons and A=Z+N nucleons thus $N - Z = A - 2Z$ may also be rewritten generally in relation to their nuclear weights using (5.31), in the form:

$$ {}^{A}_{Z}B_0 + {}^{A}_{Z}M = \frac{1}{2}\left(3A\left(\sqrt{M_u M_d} + \cos\theta_1\left(m_u + m_d\right)\right) + \left(A - 2Z\right)\left(m_u - \left(\frac{3m_d + 2\sqrt{m_\mu m_d} - 3m_u}{\left(2\pi\right)^{\frac{3}{2}}}\right)\right)\right) \tag{5.36}$$

One final exercise of interest is to return to the mass and mixing matrix Θ in (5.17) and set $\theta_2 = \theta_3 = \delta = 0$ while using $\cos\theta_1 = 0.9474541242$ found in (5.28). In this circumstance, (5.17) reduces to:

$$\Theta = 27\begin{pmatrix} \sqrt{M_u M_d}\, m_s m_b & 0 & 0 \\ 0 & m_u m_c m_t \cos\theta_1 & 0 \\ 0 & 0 & m_d \sqrt{M_c M_s}\sqrt{M_t M_b}\cos\theta_1 \end{pmatrix}. \tag{5.37}$$

This is in dimensions of mass^3. If we take the cubed root, and divide by 2 (because we know that this originated with the neutron plus proton mass *sum*) to get mass numbers that should be related to individual baryons, we find $\frac{1}{2}\mathrm{diag}\Theta = \left(939.72MeV, 1163MeV, 1773MeV\right)$. This first entry is very close to the neutron mass which would not be expected *a priori*, but this is because $\sqrt{m_s m_b} = 630MeV$ which is not too far from $\sqrt{M_u M_d} = 619MeV$. Perhaps this is yet another close relationship among fermion masses. The second entry at 1163 MeV, which would only become smaller when $\theta_2 \neq 0$, $\theta_3 \neq 0$, is only about 4% larger than the mass of the $\Lambda_0(uds) = 1115.683MeV$ baryon, which could readily be compensated by non-zero θ_2, θ_3 angles as well as measurement errors in the charm and top quark masses. The final entry at 1773 MeV, is perhaps suggestive of the $\Omega\_(sss) = 1672.45MeV$ baryon mass, however, there are no omitted angles and somewhere we should expect to come across a baryon with a third generation quark. These relationships just noted are simply pointed out in an exploratory spirit, and it is to be noted that Θ in (5.17) is just one representation of a mass / mixing matrix and that one can also vary the way in which one sets up the Koide triplets (5.4) and (5.7), so as to be able to obtain this Θ matrix in several different forms. Whatever the correct fits may turn out to be with various higher-generation baryons, it should be clear that the matrix (5.17) and like matrices that can be similarly constructed are an exceedingly useful tool for trying to develop and fit mutual relationships among mixing angles, CP violating phases, and quark and baryon masses.

It is also interesting to see if the empirical $\cos\theta_1 = 0.9474541242$ found in (5.28) turns out to be related to the empirically-known Cabibbo, Kobayashi and Maskawa (CKM) mixing angles in some representation, which could then relate neutron and proton masses to the CKM angles, which is preferable to $\cos\theta_1$ being a new, separate parameter. Toward this end, we first write the CKM matrix with the "standard choice" of angles and its *empirical* values from [11]:

$$\begin{aligned} V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \\ &= \begin{pmatrix} 0.97427 \pm 0.00015 & 0.22534 \pm 0.00065 & 0.00351^{+0.00015}_{-0.00014} \\ -0.22520 \pm 0.00065 & 0.97344 \pm 0.00016 & 0.0412^{+0.0011}_{-0.0005} \\ -0.00867^{+0.00029}_{-0.00031} & -0.0404^{+0.0011}_{-0.0005} & 0.999146^{+0.000021}_{-0.000046} \end{pmatrix} \end{aligned} . \tag{5.38}$$

(Note a negative sign for the three lower-left matrix entries.) Now, while $\cos\theta_1 = 0.9474541242$ does not fit any particular one of these elements, what is of interest is the determinant which is:

$$|V| = V_{ud}V_{cs}V_{tb} + V_{us}V_{cb}V_{td} + V_{ub}V_{cd}V_{ts} - V_{ub}V_{cs}V_{td} - V_{us}V_{cd}V_{tb} - V_{ud}V_{cb}V_{ts} = 1 , \tag{5.39}$$

and which contains invariant expressions of interest. (See also [12] which cleverly connects this determinant, when real as in the standard angle choice (5.38), to the Jarlskog determinant.) Specifically, if we employ the mean experimental values in (5.38), we find that sum of the three positively-signed (+) terms in the determinant, $|V|_+$, *which is an invariant containing all nine matrix elements*, is given by:

$$|V|_+ = V_{ud}V_{cs}V_{tb} + V_{us}V_{cb}V_{td} + V_{ub}V_{cd}V_{ts} = 0.947535 . \tag{5.40}$$

This is *very close* to $\cos\theta_1 = 0.947454$ determined from the proton and neutron masses, truncated to the known precision of $|V|_+$. In fact we find $|V|_+ = 0.947192 = \cos\theta_1 - 0.000262$ if we use the lower bounds of all the experimental error ranges in (5.38), and $|V|_+ = 0.947854 = \cos\theta_1 + 0.000400$ if we use upper bounds. Therefore, using $\cos\theta_1 = 0.947454$ as the baseline against which to compare $|V|_+$, we find that:

$$|V|_+ = \cos\theta_{1\,-0.000262}^{\;+0.000400} = 0.947454^{+0.000400}_{-0.000262} . \tag{5.41}$$

This means that in related to the invariant scalar $|V|_+$:

$$\cos\theta_1 = |V|_+ = V_{ud}V_{cs}V_{tb} + V_{us}V_{cb}V_{td} + V_{ub}V_{cd}V_{ts} \tag{5.42}$$

well within experimental errors! If we now take this to be a meaningful relationship given that it falls well within experimental errors, this means that we can go back to (5.31) and use (5.42) to rewrite the neutron and proton masses completely in terms of the CKM matrix elements, as:

$$\begin{aligned} M_N &= \tfrac{1}{2}\left(3\left(\sqrt{M_u M_d} + |V|_+ \cdot (m_u + m_d)\right) + m_u - \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right) / (2\pi)^{\frac{3}{2}}\right) \\ M_P &= \tfrac{1}{2}\left(3\left(\sqrt{M_u M_d} + |V|_+ \cdot (m_u + m_d)\right) - m_u + \left(3m_d + 2\sqrt{m_\mu m_d} - 3m_u\right) / (2\pi)^{\frac{3}{2}}\right) \end{aligned} . \tag{5.43}$$

This now connects the proton and neutron masses to an invariant of the CKM matrix *V*.

Further, because $\left|V\right|_+$ injects into the proton and neutron masses an imaginary term with a Jarlskog determinant $J = c_{13}{}^2 c_{12} c_{23} s_{12} s_{13} s_{23} \sin\delta_{CKM}$ (which may be calculated using the angles in (5.38) with $\delta \to \delta_{CKM}$), and if we wish to maintain the proton and neutron masses to be entirely real based on $\cos\delta = 1$ (the "nucleon phase angle" $\delta \neq \delta_{CKM}$) deduced in (5.30), then we can achieve this by restoring the phase to the vacuum-enhanced mass term as in (5.21), i.e., by restoring $\sqrt{M_u M_d} \to \sqrt{M_u M_d}\exp(i\delta)$, and then choosing δ in $i\sqrt{M_u M_d}\sin\delta$ to absorb the terms with the Jarlskog determinant, again see [12] which shows how the Jarlskog determinant is "the imaginary part of any one element among the six components of determinant of *V* . . . when the whole determinant is made real" as it is in (5.39). Specifically, referring to (5.43), this means that one would set $i\sin\delta \cdot \sqrt{M_u M_d} + \mathrm{Im}\left|V\right|_+ \cdot (m_u + m_d) = 0$ to maintain CP symmetry for the neutron and proton, and given that $\mathrm{Im}\left|V\right|_+ = -3J$, this means that:

$$\sin\delta \equiv 3J\frac{m_u + m_d}{\sqrt{M_u M_d}} = 3c_{13}{}^2 c_{12} c_{23} s_{12} s_{13} s_{23} \sin\delta_{CKM}\frac{m_u + m_d}{\sqrt{M_u M_d}}. \qquad (5.44)$$

will define a very tiny phase in the term $\sqrt{M_u M_d}\exp(i\delta)$ in the proton and neutron masses such that these masses remain real and thus obey CP symmetry. This could provide additional insight into the so-called "strong CP problem."

Finally, as regards fermion masses, if we write each elementary fermion mass m_f in terms of the Fermi vev using a dimensionless coupling G_f as $\sqrt{2}m_f \equiv G_f v_F$, see, e.g., [15.32] of [13], then use these relationships in (5.17) for Θ or a similarly-formed matrix in a CKM representation (such as (5.38)), we find that the matrix entries will contain terms of the form $G_f{}^3 v_F{}^3$, $G_f{}^3 v_F{}^4$ and depending on representation, $G_f{}^3 v_F{}^5$. This may assist us to gain further insight into fermion masses as well as high-order vacuum terms ϕ^3, ϕ^4, ϕ^5 in the Lagrangian.

6. Vacuum-Amplified and Constituent Quark Masses

In (4.12) through (4.14) we defined three very helpful mass values all between 604 MeV and 635 MeV. It is natural therefore to inquire whether these "vacuum-amplified" masses might be related to the so-called "constituent" quark masses which specify how much mass each quark contributes to total mass of a nucleon or baryon, as opposed to the bare "current" quark masses. Specifically, recalling that these were the ingredients in the neutron plus proton mass *sum*, we note that $M_u / 2 = 302.0875673 MeV$ and $M_d / 2 = 317.2892232 MeV$ in (4.12) and (4.13), which is about 1/3 of the neutron and proton masses. This suggests that (4.12) to (4.14) may be related to the *constituent* masses of the up and down quarks which specify how much of the neutron and proton masses arise from each of the quarks and their interactions with the vacuum. The question we now ask, referring to the neutron and proton mass formulas (5.31), is how much does each up quark contribute, and how much does each down quark contribute, to these total masses? In other words, what are the "constituent" masses of the up quarks and down quarks in each of the neutron and proton, as opposed to their bare "current" masses?

Referring to the neutron and proton masses (5.31), for the square root terms $\sqrt{M_u M_d}$ and $\sqrt{m_\mu m_d}$, we cannot directly *segregate* the up quark mass contribution from that of the down quark. In these square root terms, the up and down are coequal mass contributors. So we shall *allocate* instead. For the term $3\cdot\sqrt{M_u M_d}$ in the neutron we shall allocate a $1\cdot\sqrt{M_u M_d}$ contribution to the one up quark and a total $2\cdot\sqrt{M_u M_d}$ contribution to the two down quarks. For the proton, we allocate $1\cdot\sqrt{M_u M_d}$ to the one down quark and $2\cdot\sqrt{M_u M_d}$ to the two up quarks. We similarly allocate the $\sqrt{m_\mu m_d}$ terms. But as to the terms which contain m_u alone, or m_d alone, we segregate these and apply them directly to the up and down quarks respectively. Thus, we identically rewrite each of (5.31) as follows, while defining the respective constituent quark mass sums $U_N + 2D_N$ and $2U_P + D_P$:

$$M_N = \frac{1}{2}\begin{pmatrix} \sqrt{M_u M_d} + 3m_u\cos\theta_1 + m_u - \dfrac{2\sqrt{m_\mu m_d}}{3(2\pi)^{\frac{3}{2}}} + \dfrac{3m_u}{(2\pi)^{\frac{3}{2}}} \\ 2\sqrt{M_u M_d} + 3m_d\cos\theta_1 - \dfrac{4\sqrt{m_\mu m_d}}{3(2\pi)^{\frac{3}{2}}} - \dfrac{3m_d}{(2\pi)^{\frac{3}{2}}} \end{pmatrix} \equiv U_N + 2D_N, \tag{6.1}$$

$$M_P = \frac{1}{2}\begin{pmatrix} 2\sqrt{M_u M_d} + 3m_u\cos\theta_1 - m_u + \dfrac{4\sqrt{m_\mu m_d}}{3(2\pi)^{\frac{3}{2}}} - \dfrac{3m_u}{(2\pi)^{\frac{3}{2}}} \\ \sqrt{M_u M_d} + 3m_d\cos\theta_1 + \dfrac{2\sqrt{m_\mu m_d}}{3(2\pi)^{\frac{3}{2}}} + \dfrac{3m_d}{(2\pi)^{\frac{3}{2}}} \end{pmatrix} \equiv 2U_P + D_P, \tag{6.2}$$

with the up and down quark contributions respectively specified in the upper and lower lines of each of (6.1) and (6.2). That is, the above represent a deconstruction of the neutron and proton masses into the separate contributions emanating from up and down quarks. We then separate out the constituent quark masses and calculate them using $\cos\theta_1 = 0.9474541242$, as follows:

$$U_N = \frac{1}{2}\left(\sqrt{M_u M_d} + 3m_u\cos\theta_1 + m_u - \frac{2\sqrt{m_\mu m_d}}{3(2\pi)^{\frac{3}{2}}} + \frac{3m_u}{(2\pi)^{\frac{3}{2}}}\right) = 314.0092987\,MeV, \tag{6.3}$$

$$D_N = \frac{1}{2}\left(\sqrt{M_u M_d} + \frac{3}{2}m_d\cos\theta_1 - \frac{2\sqrt{m_\mu m_d}}{3(2\pi)^{\frac{3}{2}}} - \frac{3m_d}{2(2\pi)^{\frac{3}{2}}}\right) = 312.7780400\,MeV, \tag{6.4}$$

$$U_P \equiv \frac{1}{2}\left(\sqrt{M_u M_d} + \frac{3}{2}m_u\cos\theta_1 - m_u + \frac{2\sqrt{m_\mu m_d}}{3(2\pi)^{\frac{3}{2}}} - \frac{3m_u}{2(2\pi)^{\frac{3}{2}}}\right) = 310.0274283\,MeV. \tag{6.5}$$

$$D_P = \frac{1}{2}\left(\sqrt{M_u M_d} + 3m_d\cos\theta_1 + \frac{2\sqrt{m_\mu m_d}}{3(2\pi)^{\frac{3}{2}}} + \frac{3m_d}{(2\pi)^{\frac{3}{2}}}\right) = 318.2171900\,MeV. \tag{6.6}$$

The first expression (6.3) for U_N is the constituent contribution of the up quark *to the mass of the neutron*. The second expression (6.4) for D_N is the constituent contribution of *each* of the two down quarks to the mass of the *neutron*. U_P in (6.5) is the constituent contribution of *each* of the two up quarks *to the mass of the proton*. Finally, D_P in (6.6) is the constituent contribution of the down quark to the mass of the *proton*. One can verify that $M_N = U_N + 2D_N$ and $M_P = 2U_P + D_P$, numerically and analytically. It is important to observe that $U_N \neq U_P$ and $D_N \neq D_P$, which is to say that the constituent contribution of each quark to the mass of a nucleon is *not* the same for different nucleons, but rather is dependent upon the particular nucleon in question, in this case, a proton or a neutron. So the lone up quark in neutron makes a slightly greater contribution to the overall neutron mass than each of the two down quarks, and the lone down quark in the proton makes a slightly greater contribution to the proton mass than each of the two up quarks.

This sort of context-dependent variable behavior depending upon nuclide is to be expected based not only on what we uncovered throughout [3], but more generally based on the fact that when nucleons bind together, they release binding energy, so that different nuclides have different weights per nucleon, and indeed, different nucleons within a given nuclide should be expected to have different weights *from one another* based on their shell characterization. Constituent mass equations (6.3) through (6.6) tell us that along these same lines, the *constituent* mass contributions from each quark will differ depending upon the particular nuclide in question, and indeed, upon the particular nucleon with which a quark is associated within that nuclide. The above, (6.3) through (6.6), make the point that this type of variable mass behavior already starts to appear of individual quarks *even as between the free neutron and proton*.

We also see that the “vacuum-amplified” quark masses (4.12) through (4.14), although related thereto, are *not* synonymous with constituent quark masses. These vacuum-amplified masses are ingredients which are used as part of the calculation of the constituent quark masses. While the constituent quark masses vary from one nucleon and nuclide and nucleon within a nuclide to the next, the vacuum-amplified quark masses do not vary. They are mass constants (to the same degree that current quark masses are constants, recognizing mass screening) which do *not* change from one nucleon or nuclide to the next, and which are used as ingredients for calculating the varying constituent quark masses, as we see in (6.3) through (6.6), as well as for calculating neutron and proton masses (5.31) and nuclear weights (5.32) through (5.36).

7. The Lagrangian Formulation of the Neutron plus Proton Mass Sum

Now we revert to the start of section 4, where we noted that we can connect *any* Koide matrix products to a Lagrangian via (3.4) and (3.5). Now that we have obtained a theoretical expression for the neutron and proton masses, it is time to backtrack using the development in section 3 to connect these masses to their associated Lagrangian expression, simply to put all of the foregoing into a more formal physics context so that it is understood as going beyond simply playing with mass numbers to make them numerically fit an equation with opaque origins. We shall develop such a Lagrangian formulation for the *neutron plus proton mass sum* (5.6), recognizing that a Lagrangian connection for the separate masses of the neutron and proton can then be developed using Yang-Mills matrix expressions such as [4.3], [4.4], [5.3] and [6.20] of [3] to also develop a Lagrangian formulation of the neutron minus proton mass difference (1.4).

Using the Pauli spin matrix T_2, a unitary rotation matrix may of course be written:

$$\exp\left(iT_2\theta\right)=1+iT_2\theta+\tfrac{1}{2!}\left(iT_2\theta\right)^2+\tfrac{1}{3!}\left(iT_2\theta\right)^3+\tfrac{1}{4!}\left(iT_2\theta\right)^4+\dots$$
$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&\theta\\-\theta&0\end{pmatrix}-\tfrac{1}{2!}\begin{pmatrix}\theta^2&0\\0&\theta^2\end{pmatrix}+\tfrac{1}{3!}\begin{pmatrix}0&-\theta^3\\\theta^3&0\end{pmatrix}+\tfrac{1}{4!}\begin{pmatrix}\theta^4&0\\0&\theta^4\end{pmatrix}+\dots\ . \tag{7.1}$$
$$=\begin{pmatrix}1-\tfrac{1}{2!}\theta^2+\tfrac{1}{4!}\theta^4+\dots & \theta-\tfrac{1}{3!}\theta^3+\dots\\ -\left(\theta-\tfrac{1}{3!}\theta^3\right)+\dots & 1-\tfrac{1}{2!}\theta^2+\tfrac{1}{4!}\theta^4+\dots\end{pmatrix}=\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}$$

Consequently, the square root of this rotation matrix is:

$$\sqrt{\exp\left(iT_2\theta\right)}=\exp\left(\tfrac{1}{2}iT_2\theta\right)=\begin{pmatrix}\cos\tfrac{1}{2}\theta&\sin\tfrac{1}{2}\theta\\-\sin\tfrac{1}{2}\theta&\cos\tfrac{1}{2}\theta\end{pmatrix}. \tag{7.2}$$

With this in mind we start with the expression (5.6) including the phase $\exp\left(i\delta\right)$ which we later found in (5.30) is $\exp\left(i\delta\right)=1$, and write the neutron plus proton mass sum using a square root rotation matrix as:

$$M_N+M_P=\mathrm{E}_{AB}U_{1BC}\mathrm{E}_{CA}=\mathrm{E}_{AB}\sqrt{U_1}_{BC}\sqrt{U_1}_{CD}\mathrm{E}_{DA}\equiv\overline{\mathrm{E}}'_{AB}\mathrm{E}'_{BA}$$
$$=3\mathrm{Tr}\begin{pmatrix}\sqrt[4]{M_uM_d}\exp\left(\tfrac{1}{2}i\delta\right)&0&0\\0&\sqrt{m_u}\cos\tfrac{1}{2}\theta_1&\sqrt{m_u}\sin\tfrac{1}{2}\theta_1\\0&-\sqrt{m_d}\sin\tfrac{1}{2}\theta_1&\sqrt{m_d}\cos\tfrac{1}{2}\theta\end{pmatrix}\begin{pmatrix}\sqrt[4]{M_uM_d}\exp\left(\tfrac{1}{2}i\delta\right)&0&0\\0&\sqrt{m_u}\cos\tfrac{1}{2}\theta_1&\sqrt{m_d}\sin\tfrac{1}{2}\theta_1\\0&-\sqrt{m_u}\sin\tfrac{1}{2}\theta_1&\sqrt{m_d}\cos\tfrac{1}{2}\theta_1\end{pmatrix}, \tag{7.3}$$
$$=3\left(\exp\left(i\delta\right)\sqrt{M_uM_d}+m_u\cos\theta_1+m_d\cos\theta_1\right)$$

in combination with a rotated "electron generation matrix" E' defined via left multiplication with $\sqrt{U_1}$ as:

$$\mathrm{E}'_{AB}=\sqrt{3}\begin{pmatrix}\sqrt[4]{M_uM_d}\exp\left(\tfrac{1}{2}i\delta\right)&0&0\\0&\sqrt{m_u}\cos\tfrac{1}{2}\theta_1&\sqrt{m_d}\sin\tfrac{1}{2}\theta_1\\0&-\sqrt{m_u}\sin\tfrac{1}{2}\theta_1&\sqrt{m_d}\cos\tfrac{1}{2}\theta_1\end{pmatrix}, \tag{7.4}$$
$$=\sqrt{U_1}_{AC}\mathrm{E}_{CB}=\sqrt{3}\begin{pmatrix}\exp\left(\tfrac{1}{2}i\delta\right)&0&0\\0&\cos\tfrac{1}{2}\theta_1&\sin\tfrac{1}{2}\theta_1\\0&-\sin\tfrac{1}{2}\theta_1&\cos\tfrac{1}{2}\theta_1\end{pmatrix}\begin{pmatrix}\sqrt[4]{M_uM_d}&0&0\\0&\sqrt{m_u}&0\\0&0&\sqrt{m_d}\end{pmatrix}$$

and an adjoint matrix defined via right-multiplication with $\sqrt{U_1}$ as:

$$\overline{\mathrm{E}}'_{AB}\equiv\sqrt{3}\begin{pmatrix}\sqrt[4]{M_uM_d}\exp\left(\tfrac{1}{2}i\delta\right)&0&0\\0&\sqrt{m_u}\cos\tfrac{1}{2}\theta_1&\sqrt{m_u}\sin\tfrac{1}{2}\theta_1\\0&-\sqrt{m_d}\sin\tfrac{1}{2}\theta_1&\sqrt{m_d}\cos\tfrac{1}{2}\theta\end{pmatrix}. \tag{7.5}$$
$$=\mathrm{E}_{AC}\sqrt{U_1}_{CB}=\begin{pmatrix}\sqrt[4]{M_uM_d}&0&0\\0&\sqrt{m_u}&0\\0&0&\sqrt{m_d}\end{pmatrix}\begin{pmatrix}\exp\left(\tfrac{1}{2}i\delta\right)&0&0\\0&\cos\tfrac{1}{2}\theta_1&\sin\tfrac{1}{2}\theta_1\\0&-\sin\tfrac{1}{2}\theta_1&\cos\tfrac{1}{2}\theta_1\end{pmatrix}$$

In the above, $\cos\theta_1=0.9474541242$ is the empirical number found in (5.28), and $\delta=0$ is identically true as found in (5.30). The above, E'_{AB} and $\overline{\mathrm{E}}'_{AB}$, are just the Koide triplet matrix

E_{AB} for the electron generation rotated into primed state by multiplying from the left and from the right via $\sqrt{U_1}_{AC}\mathrm{E}_{CB}$ and $\sqrt{U_1}_{AC}\mathrm{E}_{CB}$.

But we know from (3.4) and (3.5) that as soon as we have a Koide matrix, we can backtrack into a Lagrangian formulation. In this case, in (1.6) for a generalized Koide matrix K_{AB}, we are setting $m_1 = \sqrt{M_u M_d}$, $m_2 = m_u$ and $m_3 = m_d$, and the only new feature is that we are then rotating this matrix both from the left and the right via $K' = \sqrt{U}K$ and $\overline{K} = K\sqrt{U}$. Consequently, we may use (7.4) and (7.5) to write the mass sum $M_N + M_P$ in (7.3) in a Lagrangian formulation, using these rotated Koide matrices, via (3.4) and (3.5) as:

$$\begin{aligned}
M_N + M_P &= -(2\pi)^{\frac{3}{2}} \iiint \mathfrak{L} d^3x = \tfrac{1}{2}(2\pi)^{\frac{3}{2}} \mathrm{Tr}\iiint \overline{\mathcal{E}'}_{\mu\nu}\mathcal{E}'^{\mu\nu} d^3x = \tfrac{1}{2}(2\pi)^{\frac{3}{2}} \mathrm{Tr}\iiint \overline{\mathcal{E}'}_{AB}\cdot \mathcal{E}'_{BD} d^3x \\
&= \tfrac{1}{2}(2\pi)^{\frac{3}{2}} \iiint \overline{\mathcal{E}'}_{AB}\cdot \mathcal{E}'_{BA} d^3x = \overline{\mathrm{E}'}_{AB}\mathrm{E}'_{BA} \\
&= 3\mathrm{Tr}\begin{pmatrix} \sqrt[4]{M_u M_d}\exp\left(\frac{1}{2}i\delta\right) & 0 & 0 \\ 0 & \sqrt{m_u}\cos\frac{1}{2}\theta_1 & \sqrt{m_u}\sin\frac{1}{2}\theta_1 \\ 0 & -\sqrt{m_d}\sin\frac{1}{2}\theta_1 & \sqrt{m_d}\cos\frac{1}{2}\theta \end{pmatrix}\begin{pmatrix} \sqrt[4]{M_u M_d}\exp\left(\frac{1}{2}i\delta\right) & 0 & 0 \\ 0 & \sqrt{m_u}\cos\frac{1}{2}\theta_1 & \sqrt{m_d}\sin\frac{1}{2}\theta_1 \\ 0 & -\sqrt{m_u}\sin\frac{1}{2}\theta_1 & \sqrt{m_d}\cos\frac{1}{2}\theta_1 \end{pmatrix}, \quad (7.6) \\
&= 3\left(\exp(i\delta)\sqrt{M_u M_d} + m_u\cos\theta_1 + m_d\cos\theta_1\right) = M_N + M_P
\end{aligned}$$

by introducing new field strength tensors defined in the manner of (3.2), namely:

$$\mathrm{Tr}\mathcal{E}'^{\mu\nu} \equiv -i\left(\frac{\overline{\Psi'}_{ud}\left[\gamma^\mu,\gamma^\nu\right]\Psi'_{ud}}{\sqrt{M_u M_d}'} + \frac{\overline{\psi'}_u\left[\gamma^\mu,\gamma^\nu\right]\psi'_u}{m'_u} + \frac{\psi'_d\left[\gamma^\mu,\gamma^\nu\right]\psi'_d}{m'_d}\right), \quad (7.7)$$

$$\mathrm{Tr}\overline{\mathcal{E}'}^{\mu\nu} \equiv -i\left(\frac{\overline{\Psi'}_{ud}\left[\gamma^\mu,\gamma^\nu\right]\Psi'_{ud}}{\sqrt{M_u M_d}'} + \frac{\overline{\psi'}_u\left[\gamma^\mu,\gamma^\nu\right]\psi'_u}{m'_u} + \frac{\psi'_d\left[\gamma^\mu,\gamma^\nu\right]\psi'_d}{m'_d}\right), \quad (7.8)$$

where the "vacuum-amplified" masses M_u and M_d as well as the square root mass $\sqrt{M_u M_d}$ are defined as in (4.12) to (4.14), and where the Koide mass matrices are formed for $\mathcal{E}'^{\mu\nu}$ using left-multiplication (7.4) and for $\overline{\mathcal{E}'}^{\mu\nu}$ using right-multiplication (7.5). Referring back to sections 1 and 3, this means that for the we have now set $\psi'_1 = \Psi'_{ud}$, $\psi'_2 = \psi'_u$, $\psi'_3 = \psi'_d$ in the field strength tensor (3.2) and as just noted, $m_1 = \sqrt{M_u M_d}$, $m_2 = m_u$, $m_3 = m_d$ in the Koide matrix (1.6), then followed the remaining development of section 3 with the only addition being that we now are also employing the rotations (7.4) and (7.5) on these Koide triplet matrices. We also now have the knowledge which can be exploited for further future development, that (7.3) specifies a limiting case of the very general mass and mixing matrix Θ as specified in (5.17), see (5.20). So we have a hook into a Lagrangian formulation for other generations of fermion, and therefore, for formulating other charmed, strange, top and bottom-containing baryons.

As a consequence of the foregoing, the unrotated fermion eigenstates used to form the (7.8) are a triplet $(\Psi_{ud}, \psi_u, \psi_d)$ consisting of a wavefunction for a vacuum-enhanced fermion Ψ_{ud}, together with the ordinary fermion wavefunctions ψ_u, ψ_d for the up and down current quarks. It is the Ψ_{ud} wavefunction that is responsible for generating the vast preponderance of the *constituent* mass contributions to the neutron plus proton mass sum, see section 6, while ψ_u, ψ_d are responsible for the *current* mass contributions.

Lastly, as in (3.12) through (3.14), at the nuts and bolts level, we apply the Gaussian *ansatz* (3.12), in the form:

$$\psi_u(r) = u\left(\pi\bar{\lambda}_u^{\ 2}\right)^{-\frac{3}{4}} \exp\left(-\frac{1}{2}\frac{(r-r_0)^2}{\bar{\lambda}_u^{\ 2}}\right), \tag{7.9}$$

$$\psi_d(r) = d\left(\pi\bar{\lambda}_d^{\ 2}\right)^{-\frac{3}{4}} \exp\left(-\frac{1}{2}\frac{(r-r_0)^2}{\bar{\lambda}_d^{\ 2}}\right), \tag{7.10}$$

$$\Psi_{ud}(r) = V\left(\pi\bar{\Lambda}_{ud}^{\ 2}\right)^{-\frac{3}{4}} \exp\left(-\frac{1}{2}\frac{(r-r_0)^2}{\bar{\Lambda}_{ud}^{\ 2}}\right), \tag{7.11}$$

and for the reduced Compton wavelengths, converting to $\hbar = c = 1$ units, we specify:

$$\bar{\lambda}_u = \hbar / m_u c = 1/m_u, \tag{7.12}$$

$$\bar{\lambda}_u = \hbar / m_u c = 1/m_u, \tag{7.13}$$

$$\bar{\Lambda}_{ud} = \hbar / \sqrt{M_u M_d}\, c = 1/\sqrt{M_u M_d}\ . \tag{7.14}$$

So, referring back to the discussion at the end of section 3, as was the case with the short range of the nuclear interaction, we can indeed use the Gaussian *ansatz* to model fermion wavefunctions as Gaussians and obtain the fully-dressed neutron and proton masses. But to do so, in the above we are using the undressed "current" quarks ψ_u, ψ_d which yielded binding energies in [1] and [3], together in the same Koide triplet with a vacuum-amplified quark wavefunction Ψ_{ud} and associated masses and wavelengths. So here too, it is not a question of *whether* we can use a Gaussian *ansatz*, but rather, it is a question of *which* wavefunctions with *which* masses and wavelengths we need to use in the Gaussian *ansatz*, in order to obtain a precise concurrence with empirical data.

So, insofar as fully covered protons and neutrons are concerned, it looks as if the *vacuum-amplified* quarks in combination with the *current* quarks, are behaving as free fermions, as specified in detail in all of the foregoing. This underscores the role of the Gaussian *ansatz* as a modeling tool used to derive effective concurrence with empirical data, rather than as a part of the theory per se. The theory is centered on baryons being Yang-Mills magnetic monopoles, and nucleons which release or retain binding energies based on their resonant properties which in turn depend upon the current quark content of those nucleons. For calculations which involve the components and emissions of protons and neutrons such as their quarks and their binding energies, the *current* quarks can be modeled as free fermions to obtain empirically-accurate results. For other calculations which involve the bulk behavior of protons and neutrons, accurate results may be obtained by modeling vacuum-enhanced quarks together with the current quarks as free fermions, in the manner outlined above.

The whole point of the discussion in this section has been to make clear that the neutron plus proton mass sum (and thus the individual neutron and proton masses) developed in this paper is not just the result of developing formulas which fit the empirical data but have unclear, opaque origins, in the way that the Koide relations have, until the development here, see sections 2 and 3, also had unclear origins. Rather, as shown in (7.6) this mass sum can be formulated as the energy $M_N + M_P = -(2\pi)^{\frac{3}{2}} \iiint \mathfrak{L} d^3x = \frac{1}{2}(2\pi)^{\frac{3}{2}} \mathrm{Tr}\iiint \overline{\mathcal{E}'}_{\mu\nu}\mathcal{E}'^{\mu\nu} d^3x$ arising from integrating a Lagrangian density $\mathfrak{L} = -\frac{1}{2}\overline{\mathcal{E}'}_{\mu\nu}\mathcal{E}'^{\mu\nu}$ over the entirety of a three-space volume element d^3x. This puts the neutron and proton masses (and by implication via Θ as specified in (5.17) other baryon masses as well) into the context of fundamental, Lagrangian-based physics, and gives much more credence to the proposition that these mass formulas are not just lucky numeric coincidences of unexplained origin, but truly are real physics relationships.

8. Conclusion

We have shown how the Koide relationships and associated triplet mass matrices can be generalized to derive the observed sum of the free neutron and proton rest masses in terms of the up and down current quark masses and the Fermi vev to six parts in 10,000, which sum can then be solved for the separate neutron and proton masses using the neutron minus proton mass difference earlier derived in [3]. The opposite charges of the up and down quarks are responsible for the appearance of a complex phase $\exp(i\delta)$ and real rotation angle which leads on an independent basis to mass and mixing matrices similar to that of Cabibbo, Kobayashi and Maskawa (CKM) and which can be used to specify the neutron and proton mass relationships to unlimited accuracy and which are shown within experimental errors to be related to the CKM mixing angles. The Koide generalizations developed here enable these neutron and proton mass relationships to be given a Lagrangian formulation based on neutron and proton field strength tensors that contain vacuum-amplified and current quark wavefunctions and masses. In the course of development, we also uncover new Koide relationships for the neutrinos, the up quarks, and the down quarks.

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