Geometric Algebra of Quarks and some new Wedge Products*

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Abstract

Quarks are described mathematically by (3 x 3) matrices. To include these quarkonian mathematical structures into Geometric Algebra it is helpful to restate Geometric Algebra in the mathematical language of (3 x 3) matrices.

It will be shown in this paper how (3 x 3) permutation matrices can be interpreted as unit vectors. Special emphasis will be given to the definition of some wedge products which fit better to this algebra of (3 x 3) matrices than the usual Geometric Algebra wedge product. And as S_3 permutation symmetry is flavour symmetry a unified flavour picture of Geometric Algebra will emerge.

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1 Overview

The geometric product and its decomposition into dot and wedge product

$$ab = a \cdot b + a \wedge b \tag{1}$$

constitute the central core of geometric algebra and thus is of tremendous importance. It is central for understanding vectors [9, sec. V] in a didactical and in a mathematical sense: "The geometric product should be regarded as an essential part of the definition of vectors" [8]. Sobczyk even today vividly remembers his "sense of amazement", when David Hestenes wrote down this stunning equation (1), and asks himself: "Why hadn't I ever heard of this striking product?" [27, p. 1291].

It will be shown in this paper, that an alternative decomposition of the geometric product

$$ab = a \bullet b + a \wedge_{12} b + a \wedge_{21} b \tag{2}$$

might be of similar importance for understanding vectors of S_3 permutation algebra. This will be done in the second part of this paper, comprised of sections 12 to 18.

In the first part the basic ideas behind this alternative version of geometric algebra will be explained. This part, which comprises sections 2 to 11, can be found at the AGACSE 2012 flash

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drive [12] distributed by the AGACSE 2012 organizers. It was written before the conference took place in La Rochelle.

At the poster session it became clear that some features of this unorthodox geometric algebra perspective I presented should be explained in more detail. This is done with the help of the alternative decomposition (2). Therefore I hope that a little bit of this sense of amazement once felt at the first encounter with the geometric product will be felt again by some readers.

2 Warning

In this version of geometric algebra negative numbers are avoided. There will be only imaginary units i, positive scalars as multiples of 1, and matrices. This paper lives in a positive, but yet complex world.

Of course it is possible to include minus signs into geometric algebra of quarks as it is no crime against mathematics to write equations like

$$\begin{pmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

But for ontological reasons (see [11, sec. 8]) I prefer to write this equation without a minus sign as

$$\begin{aligned} \begin{pmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{pmatrix}^2 &\simeq \begin{pmatrix} 1 & 1 & 1+i \\ 1 & 1+i & 1 \\ 1+i & 1 & 1 \end{pmatrix}^2 \\ &= \begin{pmatrix} 2+2i & 3+2i & 3+2i \\ 3+2i & 2+2i & 3+2i \\ 3+2i & 3+2i & 2+2i \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{aligned} \tag{4}$$

3 Introduction

It is well known that generators of the symmetric group S_3, which is isomorphic to the dihedral group of order 6, can be represented by positive (3 x 3) matrices. In geometric algebra it is possible to consider these matrices as geometric objects with a clear geometric meaning.

And it is well known that permutation symmetry S_3 closely resembles flavour symmetry [20]. To prepare the scene for a unified geometric algebra picture of quarks (which will be constructed one day) permutation symmetry S_3 will be used in the following to restate geometric algebra in the language of (3 x 3) matrices. As Gell-Mann matrices are (3 x 3) matrices, a unification of geometric algebra and Gell-Mann matrix algebra (which will be found one day) is surely made easier to construct.

One possible way to identify (3 x 3) permutation matrices with geometric objects is presented in [11], where special emphasis is given to the fact that a purely positive world without negative and without complex numbers can be formulated. In the present paper a more direct relation to matrix representations of the symmetric group S_3 is drawn, now using imaginary units to describe directions perpendicular to the S_3-plane of [11]. For this reason imaginary numbers are included, while the representation of the null matrix (or nihilation matrix) is still applied by analogy to [11].

4 Ugliness in Geometric Group Theory

Geometric group theory [22] discusses among other things matrix representations of permutations. The six different permutations of three objects or positions or flavour families a, b, and c are represented by the six positive (3 x 3) matrices [23, p. 356] as operators of S_3. For example the second and third positions of a column vector are exchanged in [22, p. 180] by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ c \\ b \end{pmatrix} \tag{5}$$

This can be considered as geometric operation in three-dimensional space when we interprete the numbers a, b, and c as coordinates:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \\ y \end{pmatrix} \tag{6}$$

From a geometric algebra perspective this is a very, very ugly equation. The operator is represented by a (3 x 3) matrix while the operand is represented by a column vector or (1 x 3) matrix. Compared with the (3 x 3) matrix, a column vector is a totally different mathematical object. Thus we have an algebra of two different mathematical worlds: the world of (3 x 3) matrices and the world of (1 x 3) matrices.

In geometric algebra we have a more ambitious dream. Vianna, Trindade and Fernandes [28, p. 962] state this dream in the following way: ”We share with many authors the idea that operators and operands should be elements of the same space.” To fulfill this dream and to find an algebra which shows a ”proper conformity of the parts to one another and to the whole” (as Heisenberg [6] and Chandrasekhar [1] characterise mathematical beauty) it is tried in the following to use only (3 x 3) and later (9 x 9) matrices to de- scribe three-dimensional objects, operations, or geometric situations.

It is surely more beautiful to represent all parts of a mathematical system by the same mathematical structures. And if it is not considered as more beautiful by aesthetically desillusioned pragmatists, it should at least be considered as more consistent, practical or convenient.

5 Interpreting (3 x 3) Matrices

The (3 x 3) matrix of eq. (6) exchanges the y- and z-coordinates of three-dimensional Euclidean space. Therefore this (3 x 3) matrix acts like a reflection at a plane which is spanned by the x-axis and the diagonal line between the y- and z-axis (see figure 1(b)). In a first approach it can be checked whether it is possible to identify this matrix with the corresponding plane in geometric algebra.

In the following the (3 x 3) matrix representation will be given at the left side of the double sided arrow, while the standard Pauli matrix representation of geometric algebra can be found at the right side of the double sided arrow:

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \overset{??}{\longleftrightarrow} \quad \frac{1}{\sqrt{2}}(\sigma_y + \sigma_z)\sigma_x = \frac{1}{\sqrt{2}}(\sigma_z\sigma_x - \sigma_x\sigma_y) \tag{7}$$

Please note the question marks, because a problem arises. The square of bivectors or of linear combinations of bivectors in three-dimensional space of geometric algebra is negative. An evaluation of the right-hand side of the double sided arrow of eq. (7) consequently gives

$$\left(2^{-0.5}(\sigma_z\sigma_x - \sigma_x\sigma_y)\right)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

while the square of matrix e_1 at the left-hand side of the double sided arrow is positive:

$$e_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{9}$$

This leads to the conclusion that we have to identify the dual of e_1 (which will be called E_1) with the considered plane of eq. (7):

$$E_1 = ie_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \quad \longleftrightarrow \quad \frac{1}{\sqrt{2}}(\sigma_y + \sigma_z)\sigma_x = \frac{1}{\sqrt{2}}(\sigma_z\sigma_x - \sigma_x\sigma_y) \tag{10}$$

In this way we have identified a (3 x 3) matrix on the left side of the double sided arrow with a (2 x 2) matrix on the right side of the double sided arrow. It will be shown later that this (3 x 3) matrix E_1 indeed acts in the same way on a vector $r = x\sigma_x + y\sigma_y + z\sigma_z$ like the standard geometric algebra reflection matrix of eq. (10), which exchanges the y- and z-coordinates:

$$r' = \frac{1}{\sqrt{2}}(\sigma_y + \sigma_z)\sigma_x(x\sigma_x + y\sigma_y + z\sigma_z)\frac{1}{\sqrt{2}}(\sigma_y + \sigma_z)\sigma_x = x\sigma_x + z\sigma_y + y\sigma_z \tag{11}$$

But first we have to find the (3 x 3) matrix equivalent of vector r.

The two other (3 x 3) matrices E_2 and E_3 which exchange two other coordinates in each case can be interpreted in a similar way. The (3 x 3) matrix E_2 exchanges the x- and z-coordinates. Therefore it can be identified with a plane which is spanned by the y-axis and the diagonal line between the x- and z-axis (see figure 1(c)):

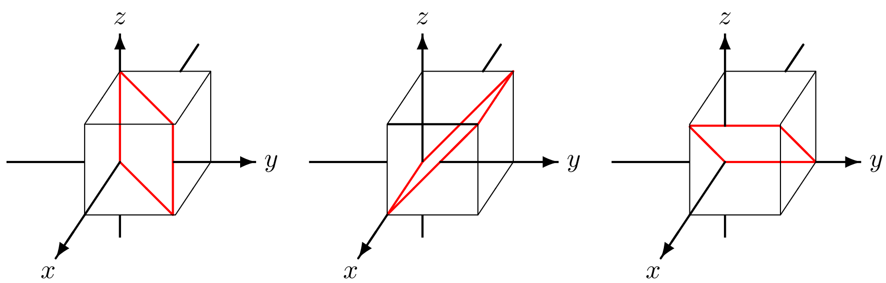
$$E_2 = ie_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{pmatrix} \quad \longleftrightarrow \quad \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)\sigma_y \tag{12}$$

And the (3 x 3) matrix E_3 can be identified with a plane which is spanned by the z-axis and the diagonal line between the x- and y-axis (see figure 1(a)):

$$E_3 = ie_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \quad \longleftrightarrow \quad \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y)\sigma_z \tag{13}$$

And it is clear that the red area elements of figure 1 have surface areas of $\sqrt{2}$ times the unit area.

A multiplication by the imaginary unit i in matrix algebra can be considered as a multiplication by the volume element $\sigma_x\sigma_y\sigma_z$ in geometric algebra. The three unit vectors e_1, e_2, and e_3 of geometric algebra of quarks can thus be identified with the following standard geometric algebra vectors:



(a) $\sqrt{2}\,E_3$ (b) $\sqrt{2}\,E_1$ (c) $\sqrt{2}\,E_2$

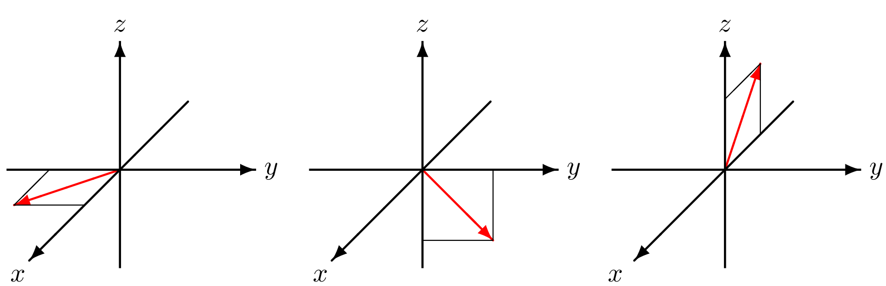
Figure 1: Imaginary permutation matrices E_1, E_2, and E_3 represent planes.

$$e_2 = i^3 E_2 = \frac{E_2}{i} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{array}{l} -\frac{1}{\sqrt{2}}(\sigma_x+\sigma_z)\sigma_y\,\sigma_x\sigma_y\sigma_z \\ = \frac{1}{\sqrt{2}}(\sigma_z-\sigma_x) \end{array} \quad (14)$$

$$e_3 = \frac{E_3}{i} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y) \quad (15)$$

$$e_1 = \frac{E_1}{i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{2}}(\sigma_y - \sigma_z) \quad (16)$$

In this way (3 x 3) matrices can be identified with vectors. This is an important message: S_3 permutation matrices represent vectors. These three vectors are shown in figure 2.



(a) $\sqrt{2}\,e_3$ (b) $\sqrt{2}\,e_1$ (c) $\sqrt{2}\,e_2$

Figure 2: Permutation matrices e_1, e_2, and e_3 represent vectors.

6 Nihilation Matrix and Identity

The three permutation vectors e_1, e_2, and e_3 are unit vectors because they square to one[1]. They are furthermore coplanar as e_1, e_2, and e_3 are located in the same plane. This has been tried to visualise in figure 3.

But figure 3 shows another important feature: The sum of the three vectors $e_1 + e_2 + e_3$ (see left picture of figure 3) or the double sum $2e_1 + 2e_2 + 2e_3$ (see right picture of figure 3) or every other multiple sum like $3(e_1 + e_2 + e_3)$ (see middle picture of figure 3) results in a vector of length zero. That is why we should identify the matrix of ones N geometrically *and* algebraically with the zero matrix O.

$$N = e_1 + e_2 + e_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O \tag{17}$$

This identification of N with zero is also justified when we compare the sum $e_1 + e_2 + e_3$ with its geometric algebra counterpart by adding eq. (14), (15), and (16).

$$e_1 + e_2 + e_3 \simeq O \quad \longleftrightarrow \quad \frac{1}{\sqrt{2}}(\sigma_z - \sigma_x + \sigma_x - \sigma_y + \sigma_y - \sigma_z) = 0 \tag{18}$$

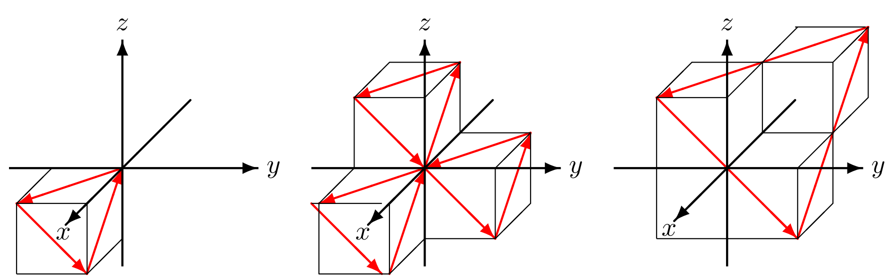


Figure 3: Some attempts to visualise that the vectors e_1, e_2, and e_3 lie in the same plane.

In the same way the sum of all three imaginary permutation matrices E_1, E_2, and E_3 which represent unit area elements has to be identified with nothingness, nihilation, null or zero.

$$\begin{aligned} E_1 + E_2 + E_3 = \begin{pmatrix} i & i & i \\ i & i & i \\ i & i & i \end{pmatrix} = iO \simeq N \\ \longleftrightarrow \quad \frac{1}{\sqrt{2}}(\sigma_x\sigma_y + \sigma_z\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z + \sigma_z\sigma_x + \sigma_y\sigma_x) = 0 \end{aligned} \tag{19}$$

In the literature the matrix of ones is sometimes called unit matrix (see e.g. [29]), which is rather confusing. The matrix of ones is not the identity matrix. And sometimes the matrix of ones is called democratic matrix (see e.g. [2]), which seems even more confusing and hides the structural meaning of N. If a (3 x 3) matrix is multiplied with N in geometric algebra of quarks,

[1] In a world with positive numbers only, it makes sense to call them base vectors, because they form a minimal set of vectors spanning a plane, see [11]. It is not possible to reach every point of a plane when there are just two base vectors with only positive coordinates.

it will be nihilated und becomes zero. Thus we have indefinitely many representations of matrices meaning zero. For example there are (with $r \in \mathbb{C}$):

$$N = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} r & r & r \\ r & r & r \\ r & r & r \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O \tag{20}$$

Hence every other (3 x 3) matrix possess indefinitely many representations too. The matrix Z_2' given by Dev, Gautam and Singh in [2, eq. (16)]

$$\begin{aligned} Z_2' &= \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{21}$$

is just another representation of the identity matrix. Therefore it is obvious that every mathematical structure should be invariant under Z_2' in geometric algebra of quarks.

And every vector r can be written in different ways:

$$\begin{aligned} r = x_1e_1 + x_2e_2 + x_3e_3 &\simeq (x_1 - x_3)\, e_1 + (x_2 - x_3)\, e_2 \\ &\simeq (x_2 - x_1)\, e_2 + (x_3 - x_1)\, e_3 \\ &\simeq (x_3 - x_2)\, e_3 + (x_1 - x_2)\, e_1 \\ \longleftrightarrow \quad & \frac{1}{\sqrt{2}}\left[(x_2 - x_1)\sigma_x + (x_3 - x_2)\sigma_y + (x_1 - x_3)\sigma_z\right] \end{aligned} \tag{22}$$

Therefore it is always possible to find a fundamental expression of vector r with only two unit vectors e_i and purely positive coefficients. For example, if $x_2, x_3 \geq x_1$ then it would make sense to use the second line of eq. (22) as the two coefficients are greater than or equal to zero.

Although it seems that we are living in a three-dimensional space with x-, y- and z-axes as shown in the previous figures, till now we are not able to reach points outside the plane indicated in figure 3. We are frozen in this plane. Every point we can reach till now is considered to be a linear combination of the three vectors e_1, e_2, and e_3. To reach points outside this plane it is crucial to identify a vector perpendicular[2] to the S_3-plane.

7 Products of Permutation Matrices

The following products of permutation matrices exist:

$$e_0 := e_1^2 = e_2^2 = e_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \tag{23}$$

$$\ominus := E_1^2 = E_2^2 = E_3^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \longleftrightarrow \quad -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1 \tag{24}$$

(25)

[2]As quarks should be regarded as entities having absolutely no rectangular symmetry e_4 will be ignored again in the second part of this paper.

$$e_{12} := e_1e_2 = e_2e_3 = e_3e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{2}(-1 + \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) \quad (26)$$

$$e_{21} := e_2e_1 = e_3e_2 = e_1e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{2}(-1 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x) \quad (27)$$

$$E_{12} := E_1E_2 = E_2E_3 = E_3E_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \longleftrightarrow \frac{1}{2}(1 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x) \quad (28)$$

$$E_{21} := E_2E_1 = E_3E_2 = E_1E_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \longleftrightarrow \frac{1}{2}(1 + \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) \quad (29)$$

These matrix products are geometric products. They thus bear geometrical meaning. The entities of eq. (23) and (24) can be identified with scalars. The entities of eq. (25) to (28) can be identified as linear combinations of a scalar and bivectors, parallelograms of sorts. The trivector or pseudoscalar can be found by the following permutation matrix multiplications:

$$I := E_1e_1 = E_2e_2 = E_3e_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \longleftrightarrow \sigma_x\sigma_y\sigma_z \quad (30)$$

If we restrict ourselves to the plane of figure 3, we can do everything in this plane using (3 x 3) matrices of geometric algebra of quarks what we are able to do with (2 x 2) Pauli matrices in conventional geometric algebra in a plane. This is important! **(2 x 2) matrices can be thought as and seen as (3 x 3) permutation matrices**. So it is no mathematical question, which system we use, it is instead a didactical question.

8 The Philosophy of Negative Numbers

As indicated in section 1 minus signs are avoided in this paper. Instead of this algebraic sign ”–” the geometric entity

$$e_{12} + e_{21} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \longleftrightarrow -1 \quad (31)$$

gives us a (3 x 3) matrix which does all that a minus sign usually does. Algebraically $e_{12} + e_{21}$ reduces every scalar ke_0 by one unit:

$$ke_0 + e_{12} + e_{21} = (k-1)e_0 + \underbrace{e_0 + e_{12} + e_{21}}_{N} \simeq (k-1)e_0 \quad (32)$$

But at the same time the matrix $e_{12} + e_{21}$ has a clear geometric meaning: It reverses the direction of vectors:

$$(e_{12} + e_{21})(x_1e_1 + x_2e_2 + x_3e_3) = x_1(e_2 + e_3) + x_2(e_3 + e_1) + x_3(e_1 + e_2) \tag{33}$$

This is indeed a complete reversion as for example the unit vector $e_2 + e_3$ is parallel to the unit vector e_1, but shows into the opposite direction.

Therefore this matrix $(e_{12} + e_{21})$ is called $\ominus$ in this paper, using the \ominus symbol of Latex like it is done in eq. (24). So $\ominus\, e_0$ is no multiplication of a negative sign with the scalar 1, but a matrix multiplication resulting in $\ominus\, e_0 = \ominus$. This avoidance of minus signs indicates that we might live in a mathematically purely positive world.

In this world negative entities do not exist. We just reverse directions. And sometimes we do not totally reverse a direction but change the direction only a little bit. This might have epistemological and ontological consequences for our physical world too. Do we really measure negative entities anywhere in physics? Or do we only measure positive entities in different or sometimes in opposite directions? The possibility of avoiding the minus sign might indicate that we not only might live in a mathematically positive world, but that we might live in a world which can be decribed in physics as a purely positive world too.

And as we actually speak about something like "reality" it is even possible that the world of physics not only can be but even must be described as purley positve, to understand it conceptully as "The Road to Reality" (see discussion in [21, §3.5]) is a mathematical road[3].

But whatever our ontological and epistemological positions are: We have reached here the true heart of geometric algebra: $\ominus$ can be interpreted as algebraic and as well as geometric operation. Algebra and geometry are deeply connected now. We live in both worlds: in the world of algebra and in the world of geometry. And as we can transfer from algebra to geometry and back to algebra at every moment, these worlds coalesce structurally.

9 Constructing e_4

After having found an appropriate entity to describe negativities in geometric algebra of quarks we are able to split up the geometric product of two vectors r_1r_2 into dot product and wedge product in analogy to eq. (1):

$$r_1r_2 = r_1 \cdot r_2 + r_1 \wedge r_2 \tag{34}$$

The dot product results in a scalar

$$r_1 \cdot r_2 = \frac{1}{2}(r_1r_2 + r_2r_1) \tag{35}$$

and is connected with the cosine of the angle α between the two vectors r_1 and r_2:

$$\cos\alpha = \hat{r}_1 \cdot \hat{r}_2 = \frac{1}{2}\left(\hat{r}_1\hat{r}_2 + \hat{r}_2\hat{r}_1\right) \tag{36}$$

where $\hat{r}$ is the unit vector of $r = x_1e_1 + x_2e_2 + x_3e_3$:

$$\hat{r} = \frac{r}{|r|} \tag{37}$$

[3] At the beginning of this discussion Penrose writes: "I think that it is clear that, unlike the natural numbers, there is no evident physical content to the notion of a negative number of physical objects" [21, p. 65]. But he later on revises this position slightly.

with

$$|r| = \sqrt{r^2} = \sqrt{x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1}\, e_0 \tag{38}$$

As indicated in figure 3 the angles between all unit vectors e_1, e_2, and e_3 indeed equal $2\pi/3$:

$$\cos\alpha = e_1 \cdot e_2 = e_2 \cdot e_3 = e_3 \cdot e_1 = \frac{1}{2}(e_1e_2 + e_2e_1) = \ominus\frac{1}{2}e_0 \quad \longleftrightarrow \quad -\frac{1}{2} \tag{39}$$

It surely makes sense to identify the arccosine of this expression with

$$\alpha = \arccos\left(\ominus\frac{1}{2}\right) \simeq \frac{2\pi}{3}e_0 = 120^\circ e_0 \tag{40}$$

Now the (standard) wedge product is defined as:

$$r_1 \wedge r_2 = \frac{1}{2}(r_1r_2 + \ominus r_2r_1) \tag{41}$$

Thus we get an expression for a bivector representing the plane A_{S_3} in which the unit vectors e_1, e_2, and e_3 are situated:

$$\begin{aligned} A_{S_3} &:= e_1 \wedge e_2 = e_2 \wedge e_3 = e_3 \wedge e_1 = \frac{1}{2}(e_1e_2 + \ominus e_2e_1) \\ &= \frac{1}{2}e_0 + e_{12} = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \quad \longleftrightarrow \quad \frac{1}{2}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) \end{aligned} \tag{42}$$

The magnitude of this area element is

$$\begin{aligned} |A_{S_3}| &= \sqrt{\ominus\left(e_{12} + \frac{1}{2}e_0\right)^2} = \sqrt{\ominus\left(\frac{1}{4}e_0 + e_{12} + e_{21}\right)} \\ &\simeq \sqrt{\ominus\frac{3}{4}\,(e_{12} + e_{21})} = \sqrt{\ominus^2\frac{3}{4}} \simeq \sqrt{\frac{3}{4}e_0} = \frac{1}{2}\sqrt{3}\, e_0 \end{aligned} \tag{43}$$

Therefore the unit area element E_4 which is perpendicular to the wanted unit vector e_4 equals

$$\begin{aligned} E_4 = \frac{A_{S_3}}{|A_{S_3}|} = \frac{1}{\sqrt{3}}(e_0 + 2e_{12}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \\ \longleftrightarrow \quad \frac{1}{\sqrt{3}}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) \end{aligned} \tag{44}$$

By analogy to eq. (14), (15), or (16) the unit vector e_4 perpendicular to all other unit vectors e_1, e_2, and e_4 can be found as

$$\begin{aligned} e_4 = \ominus iE_4 = \frac{1}{\sqrt{3}}i(e_0 + 2e_{21}) = \frac{1}{\sqrt{3}} \begin{pmatrix} i & 2i & 0 \\ 0 & i & 2i \\ 2i & 0 & i \end{pmatrix} \\ \longleftrightarrow \quad \frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z) \end{aligned} \tag{45}$$

As expected e_4 is a unit vector.

$$
\begin{aligned}
e_4^2 &= \frac{1}{3}\begin{pmatrix} i & 2i & 0 \\ 0 & i & 2i \\ 2i & 0 & i \end{pmatrix}^2 \simeq \frac{1}{3}\begin{pmatrix} i{+}2 & 2i{+}2 & 2 \\ 2 & i{+}2 & 2i{+}2 \\ 2i{+}2 & 2 & i{+}2 \end{pmatrix}^2 \\
&= \frac{1}{3}\begin{pmatrix} 12i{+}11 & 12i{+}8 & 12i{+}8 \\ 12i{+}8 & 12i{+}11 & 12i{+}8 \\ 12i{+}8 & 12i{+}8 & 12i{+}11 \end{pmatrix} \simeq \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{46}
$$

Or written in vector notation instead of matrix notation:

$$
\begin{aligned}
e_4^2 = \left(\frac{1}{\sqrt{3}}i(e_0 + 2e_{21})\right)^2 &= \frac{1}{3}(e_{12} + e_{21})(3e_{12} + 3e_{21}) \simeq e_0 \\
\longleftrightarrow \quad & \left(\frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z)\right)^2 = 1
\end{aligned} \tag{47}
$$

And e_4 is perpendicular to all other unit vectors:

$$
\begin{aligned}
\cos\alpha_{14} = e_1 \cdot e_4 = e_2 \cdot e_4 = e_3 \cdot e_4 &= \frac{1}{2}(e_1 e_4 + e_4 e_1) \\
= \frac{1}{2\sqrt{3}}i\,(2e_1 + 2e_2 + 2e_3) &= \frac{1}{\sqrt{3}}iN \simeq 0 \quad \longleftrightarrow \quad \cos\alpha_{14} = 0
\end{aligned} \tag{48}
$$

$$
\Rightarrow \qquad \alpha_{14} = \frac{\pi}{2}e_0 = 90^\circ e_0 \tag{49}
$$

Consequently the products of e_4 with the other unit vectors are:

$$
\begin{aligned}
e_{14} := e_1 e_4 = \frac{1}{\sqrt{3}}i(e_1 + 2e_3) = \frac{1}{\sqrt{3}}&\begin{pmatrix} i & 2i & 0 \\ 2i & 0 & i \\ 0 & i & 2i \end{pmatrix} = \ominus\, e_4 e_1 \\
\longleftrightarrow \quad & \frac{1}{\sqrt{6}}(-\sigma_x\sigma_y + 2\sigma_y\sigma_z - \sigma_z\sigma_x)
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
e_{41} := e_4 e_1 = \frac{1}{\sqrt{3}}i(e_1 + 2e_2) = \frac{1}{\sqrt{3}}&\begin{pmatrix} i & 0 & 2i \\ 0 & 2i & i \\ 2i & i & 0 \end{pmatrix} = \ominus\, e_1 e_4 \\
\longleftrightarrow \quad & \frac{1}{\sqrt{6}}(\sigma_x\sigma_y - 2\sigma_y\sigma_z + \sigma_z\sigma_x)
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
e_{24} := e_2 e_4 = \frac{1}{\sqrt{3}}i(e_2 + 2e_1) = \frac{1}{\sqrt{3}}&\begin{pmatrix} 2i & 0 & i \\ 0 & i & 2i \\ i & 2i & 0 \end{pmatrix} = \ominus\, e_4 e_2 \\
\longleftrightarrow \quad & \frac{1}{\sqrt{6}}(-\sigma_x\sigma_y - \sigma_y\sigma_z + 2\sigma_z\sigma_x)
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
e_{42} := e_4 e_2 = \frac{1}{\sqrt{3}}i(e_2 + 2e_3) = \frac{1}{\sqrt{3}}&\begin{pmatrix} 0 & 2i & i \\ 2i & i & 0 \\ i & 0 & 2i \end{pmatrix} = \ominus\, e_2 e_4 \\
\longleftrightarrow \quad & \frac{1}{\sqrt{6}}(\sigma_x\sigma_y + \sigma_y\sigma_z - 2\sigma_z\sigma_x)
\end{aligned} \tag{53}
$$

$$e_{34} := e_3 e_4 = \frac{1}{\sqrt{3}} i(e_3 + 2e_2) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & i & 2i \\ i & 2i & 0 \\ 2i & 0 & i \end{pmatrix} = \ominus\, e_4 e_3$$
$$\longleftrightarrow \quad \frac{1}{\sqrt{6}}(2\sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x) \tag{54}$$

$$e_{43} := e_4 e_3 = \frac{1}{\sqrt{3}} i(e_3 + 2e_1) = \frac{1}{\sqrt{3}} \begin{pmatrix} 2i & i & 0 \\ i & 0 & 2i \\ 0 & 2i & i \end{pmatrix} = \ominus\, e_3 e_4$$
$$\longleftrightarrow \quad \frac{1}{\sqrt{6}}(-2\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) \tag{55}$$

Reflecting the unit vectors e_1, e_2, e_3 or e_4 at each other then results in:

$$e_1e_1e_1 = e_1 \qquad e_2e_1e_2 = e_3 \qquad e_3e_1e_3 = e_2 \qquad e_4e_1e_4 = e_2 + e_3 \tag{56}$$
$$e_1e_2e_1 = e_3 \qquad e_2e_2e_2 = e_2 \qquad e_3e_2e_3 = e_1 \qquad e_4e_2e_4 = e_3 + e_1 \tag{57}$$
$$e_1e_3e_1 = e_2 \qquad e_2e_3e_2 = e_1 \qquad e_3e_3e_3 = e_3 \qquad e_4e_3e_4 = e_1 + e_2 \tag{58}$$
$$e_1e_4e_1 = \ominus e_4 \qquad e_2e_4e_2 = \ominus e_4 \qquad e_3e_4e_3 = \ominus e_4 \qquad e_4e_4e_4 = e_4 \tag{59}$$

10 Pauli Matrices

In a last step to reach a full identification of Pauli matrices with (3 x 3)-matrices explicit formulae for them can be found using eq. (14), (15), (16), and (44):

$$e_x = \frac{1}{3}\left(\sqrt{2}(e_1 + 2e_3) + \sqrt{3}e_4\right) \quad \longleftrightarrow \quad \sigma_x \tag{60}$$
$$e_y = \frac{1}{3}\left(\sqrt{2}(e_2 + 2e_1) + \sqrt{3}e_4\right) \quad \longleftrightarrow \quad \sigma_y \tag{61}$$
$$e_z = \frac{1}{3}\left(\sqrt{2}(e_3 + 2e_2) + \sqrt{3}e_4\right) \quad \longleftrightarrow \quad \sigma_z \tag{62}$$

Reflections of these (3 x 3) Pauli vectors e_x, e_y, and e_z at unit vector e_1 then are:

$$e_1e_xe_1 = \frac{1}{3}\left(\sqrt{2}(e_1 + 2e_2) + \ominus\sqrt{3}e_4\right) = \ominus e_x \quad \longleftrightarrow \quad -\sigma_x \tag{63}$$
$$e_1e_ye_1 = \frac{1}{3}\left(\sqrt{2}(e_3 + 2e_1) + \ominus\sqrt{3}e_4\right) = \ominus e_z \quad \longleftrightarrow \quad -\sigma_z \tag{64}$$
$$e_1e_ze_1 = \frac{1}{3}\left(\sqrt{2}(e_2 + 2e_3) + \ominus\sqrt{3}e_4\right) = \ominus e_y \quad \longleftrightarrow \quad -\sigma_y \tag{65}$$

or

$$\ominus e_1e_xe_1 = \frac{1}{3}\left(\sqrt{2}(e_2 + e_3 + 2e_1 + 2e_3) + \sqrt{3}e_4\right) \tag{66}$$
$$\simeq \frac{1}{3}\left(\sqrt{2}(e_1 + 2e_3) + \sqrt{3}e_4\right) = e_x \quad \longleftrightarrow \quad \sigma_x \tag{67}$$
$$\ominus e_1e_ye_1 = \frac{1}{3}\left(\sqrt{2}(e_1 + e_2 + 2e_2 + 2e_3) + \sqrt{3}e_4\right) \tag{68}$$
$$\simeq \frac{1}{3}\left(\sqrt{2}(e_3 + 2e_2) + \sqrt{3}e_4\right) = e_z \quad \longleftrightarrow \quad \sigma_z \tag{69}$$
$$\ominus e_1e_ze_1 = \frac{1}{3}\left(\sqrt{2}(e_1 + e_3 + 2e_1 + 2e_2) + \sqrt{3}e_4\right) \tag{70}$$
$$\simeq \frac{1}{3}\left(\sqrt{2}(e_2 + 2e_1) + \sqrt{3}e_4\right) = e_y \quad \longleftrightarrow \quad \sigma_y \tag{71}$$

Now we can construct a (3 x 3) matrix expression which is equivalent to eq. (11). A reflection of vector

$$r = xe_x + ye_y + ze_z \quad \longleftrightarrow \quad x\sigma_x + y\sigma_y + z\sigma_z \tag{72}$$

at plane $E_1 = ie_1$ is given by

$$\begin{aligned} r' &= ie_1(xe_x + ye_y + ze_z)ie_1 \\ &= \ominus x\, e_1e_xe_1 + \ominus y\, e_1e_ye_1 + \ominus z\, e_1e_ze_1 \\ &= x\, e_x + z\, e_y + y\, e_z \quad \longleftrightarrow \quad x\sigma_x + z\sigma_y + y\sigma_z \end{aligned} \tag{73}$$

and exchangs indeed the y- and z-coordinates. This and similar equations for reflections at plane E_2 and E_3 show the inherent linkage between a geometric algebra of (3 x 3) matrices and permutation symmetry S_3. And as S_3 seems to describe important features of flavour symmetry I hope that this will indeed help us to understand quarks and neutrions one day in a geometrically convincing manner.

And again: we are able do everything in three-dimensional Euclidean space using (3 x 3) matrices of geometric algebra of quarks what we are able to do with (2 x 2) Pauli matrices in conventional geometric algebra. It is no mathematical question, which system we use, it is a didactical one.

11 Interlude

This AGACSE paper has been reviewed by two reviewers whom I wish to thank for their very constructive and helpful remarks. But there are two comments I do not agree with, and I think this should be discussed openly.

First of all one of the reviewers wrote: "Please avoid the use of words like crime, ugly and ugliness in a scientific paper." I want to clarify why I didn't follow this proposition. As I am a physics teacher and a physics education researcher my daily work is to analyse the process of physics and mathematics learning. Categories like 'beauty' or 'ugliness' consciously and unconsciously influence learning processes [19]. Not only students but we all evaluate ideas and concepts according to such categories – even if we do not speak about them.

But not to speak about them does not mean that theses categories are not there. Scientific research is learning too: We learn something new about nature, and we do that on the basis of our preconceptions. These preconceptions about whether an idea in physics or mathematics is beautiful or ugly are important features of our understanding of a subject. The more we learn about a subject the more we care about its inherent beauty or bother about its inherent lack of beauty.

This care about beauty even is a sign of professionality, and Dirac once explained: "With increasing knowledge of a subject, when one has a great deal of support to work from, one can go over more and more towards the mathematical procedure. One then has as one's underlying motivation the striving for mathematical beauty. Theoretical physicists accept the need for mathematical beauty as an act of faith. There is no compelling reason for it, but it has proved a very profitable objective in the past" [3, p. 21]. Therefore the statement that the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & c & b \\ c & b & a \\ b & a & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \tag{74}$$

is much more beautiful than equation (5) is important for me. And it directly adresses the main point of this paper as another comment of the reviewers shows.

This comment was: "Note that the (3 x 3) matrices 'describing' quarks are the 8 infinitesimal generators of SU(3). One representation are the Gell-Mann matrices. They act on columns of 3 spinors, the spinors represent quarks." This describes the standard procedure given in standard physics books like [4, p. 51/52].

To find a unified geometric algebra picture of quarks it might be helpful to use a representation of space by (3 x 3) matrices instead of (2 x 2) Pauli matrices. But in addition to that it seems inevitable to get rid of these columns of spinors used today (which are in my eyes as ugly as eq. (5)) and to construct (3 x 3) matrices of spinors similar to the second matrix of (73).

There exist more ideas how to construct a geometric algebra picture of quarks in the literature. For example Hestenes [7], Schmeikal [24], [25] or Keller [15, chap. 4.6] present some of these. But it seems that all these ideas go into more or less different directions and we are still in need of a really unified geometric algebra picture of quarks.

12 La Rochelle Poster Session

The AGACSE conference 2012 at the University of La Rochelle was an excellent scientific event. It was very well organized and a great success. But at the same time it was a depressing and discouraging experience for me: I had a message, and nobody seemed to understand it. The message was: If we use negative numbers or minus signs, we will not work in a coordinate independent way. There is a geometric meaning encoded in minus signs which affects and biases mathematical structures.

This is of course strong stuff. One of the main criteria Hestenes identified for a geometric design of a unified mathematical language for physics is indeed working with "coordinate-free methods to formulate and solve basic equations of physics" [9, p. 106]. Thus working and thinking without reference to preferred or priviledged coordinates is (or at least should be) a key advantage of geometric algebra.

But *coordinate-free* does not automatically mean *coordinate independent*. We can (unconsciously or openly) weave coordinate-related structures into a mathematical language written in a coordinate-free way. This happens when we are using minus signs and negative numbers.

If minus signs are used in geometric algebra, coordinate systems with a reflexion symmetry or a symmetry of π are automatically preferred, as other coordinate systems with oblique axes will look artifical and facticious. Thus the presence of minus signs in formulae is an indication that a group of coordinates (especially rectangular coordinates) is favoured and other possible groups of coordinates are severly disadvantaged in this mathematical language.

I told this message and got nothing but sort of silence and disbelief. Therefore I go on defending my position by discussing a different way of wedge product construction which will be presented in the following.

13 Wedge Products of S_3 Permutation Matrices

The canoncial desomposition of the geometric product (1) into standard dot product (34) and standard wedge product (40) is found by adding the nihilation matrix $ba + \ominus ba = baN \simeq O$ or zero $ba - ba = 0$ to the geometric product ab.

$$ab = \frac{1}{2}(ab + ba + ab + \ominus ba) \quad \longleftrightarrow \quad ab = \frac{1}{2}(ab + ba + ab - ba) \tag{75}$$

The obvious use of $\ominus = e_{12} + e_{21}$ in geometric algebra of quarks (left-hand side) or the minus sign in standard geometric algebra (right-hand side of eq. (74)) for constructing the wedge product clearly indicates that orthogonal axes are priviledged.

And this results in the well-known phase shift of $\pi/2$ between the magnitude of dot and wedge product (see figure 4). This diagram shows the canonical decomposition of the geometric product of the unit vector e_1 with a second unit vector $r = e_1 \cos\alpha + \frac{1}{\sqrt{3}}(e_1 + 2e_2)\sin\alpha$:

$$e_1 r = \cos\alpha\, e_0 + \frac{1}{\sqrt{3}} \sin\alpha\, (e_0 + 2e_{12}) \tag{76}$$

The standard dot product of this expression

$$e_1 \cdot r = \frac{1}{2}(e_1 r + r e_1) = \cos\alpha\, e_0 \tag{77}$$

represents a scalar and is given in red color in figure 4 while the standard wedge product

$$e_1 \wedge r = \frac{1}{2}(e_1 r + \ominus r e_1) = \frac{1}{\sqrt{3}} \sin\alpha (e_0 + 2e_{12}) \tag{78}$$

represents a bivector. As $\frac{1}{\sqrt{3}}(e_0 + 2e_{12})$ is a unit area element, the following graph is shown in figure 4 in blue color:

$$|e_1 \wedge r| = \left| \frac{1}{\sqrt{3}}(e_0 + 2e_{12}) \right| \sin\alpha \tag{79}$$

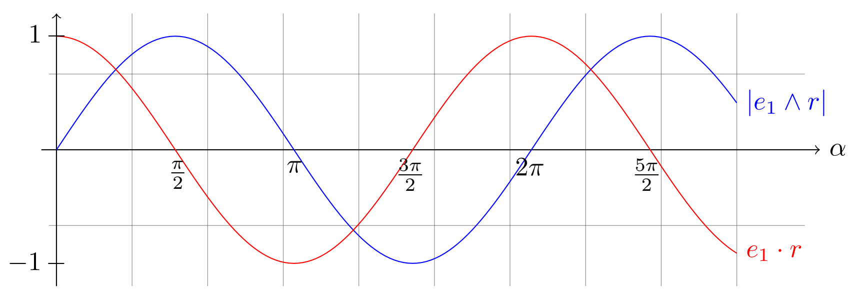


Figure 4: Canonical decomposition of the geometric product.

Now we try to priviledge the coordinate system of S_3 permutation algebra by avoiding the canonical decomposition into two different products (1). Since $\ominus = e_{12} + e_{21}$ is itself a mathematical structure made up of two elements e_{12} and e_{21} (30), an alternative decomposition into three different products can be constructed easily in analogy to (74):

$$ab = \frac{1}{3}(ab + ba + ab + e_{12}ba + ab + e_{21}ba) \tag{80}$$

These three products are a modified dot product

$$a \bullet b = \frac{1}{3}(ab + ba) \tag{81}$$

and the two new wedge products

$$\begin{aligned} a \wedge_{12} b &= \frac{1}{3}(ab + e_{12}\, ba) \\ &\longleftrightarrow \frac{1}{3}\left(ab - {}^1/_2\,(1 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x)\, ba\right) \end{aligned} \tag{82}$$

$$a \wedge_{21} b = \frac{1}{3}(ab + e_{21}\, ba) \qquad \longleftrightarrow \quad \frac{1}{3}\left(ab - {}^1/_2\,(1 + \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)\, ba\right) \tag{83}$$

This time the decomposition results in a phase shift of $2\pi/3$ between the magnitudes of dot and wedge products (see figure 5). This diagram shows the new quarkonian decomposition of the geometric product (75) of unit vector e_1 with unit vector $r = e_1 \cos\alpha + \frac{1}{\sqrt{3}}(e_1 + 2e_2)\sin\alpha$. The modified dot product of this expression

$$e_1 \bullet r = \frac{1}{3}(e_1 r + r e_1) = \frac{2}{3}\cos\alpha\, e_0 \tag{84}$$

represents a scalar and is given in red color in figure 5 while the first new wedge product

$$\begin{aligned} e_1 \wedge_{12} r = \frac{1}{3}(e_1 r + e_{12}\, r e_1) &= \frac{1}{3}(e_0 + e_{12})(\cos\alpha + \sqrt{3}\sin\alpha) \\ &= \ominus\frac{2}{3}(e_0 + e_{12})\cos\left(\alpha + \frac{2\pi}{3}\right) \\ &= \frac{2}{3}\, e_{21} \cos\left(\alpha + \frac{2\pi}{3}\right) \end{aligned} \tag{85}$$

represents a parallelogram with an angle of $2\pi/3 = 120°$. As the magnitude of e_{21} is

$$|e_{21}| = \sqrt{e_{21}\, e_{12}} = e_0 \tag{86}$$

the following graph is shown in figure 5 in blue color:

$$|e_1 \wedge_{12} r| = \left|\frac{2}{3}\, e_{21}\right| \cos\left(\alpha + \frac{2\pi}{3}\right) \tag{87}$$

The second new wedge product

$$\begin{aligned} e_1 \wedge_{21} r = \frac{1}{3}(e_1 r + e_{21}\, r e_1) &= \frac{1}{3}(e_0 + e_{21})(\cos\alpha + \ominus\sqrt{3}\sin\alpha) \\ &= \frac{1}{3}\, e_{12}\, (\ominus\cos\alpha + \sqrt{3}\sin\alpha) \\ &= \frac{2}{3}\, e_{12}\, \cos\left(\alpha + \frac{4\pi}{3}\right) \end{aligned} \tag{88}$$

represents again a parallelogram with an opening angle of $2\pi/3 = 120°$ and is shown by the following graph in figure 5 in green color:

$$|e_1 \wedge_{21} r| = \left|\frac{2}{3}\, e_{12}\right| \cos\left(\alpha + \frac{4\pi}{3}\right) \tag{89}$$

While the canonical decompostion of the geometric product results in projections on two orthogonal axes, this alternative decomposition is equivalent to projections on three axes at an angle of $2\pi/3$.

Figure 4 and standard geometric algebra surely is useful to explain the mathematics of alternating current. But as figure 5 clearly indicates: Geometric algebra of S_3 permuation matrices might be useful to explain the mathematics of three-phase current, as it possesses an appropriate inner structure which makes it easier to discuss problems with three constituents. This inner mathematical structure becomes more apparent when a three-sided coin is thrown (see following section) or quarks are modelled (see sections 15 & 17).

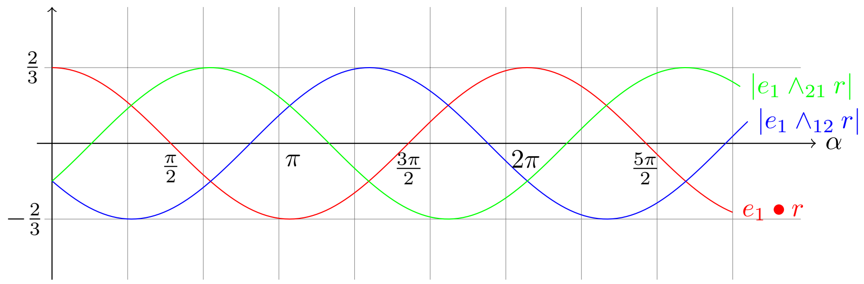


Figure 5: Alternative decomposition of the geometric product.

14 Throwing Quarkonian Coins

If we throw a two-sided coin n times, the probabilities for getting the $n+1$ possible different results will be represented by binomial coefficients:

$$p_{(n,k)} = \frac{1}{2^n}\binom{n}{k} \tag{90}$$

Theses coefficients are the coefficients of the Taylor expansion of an ordinary scalar binom

$$(e_0 + x\,e_0)^n \quad \longleftrightarrow \quad (1+x)^n \tag{91}$$

and can be found in the three Pascal triangles. When we have to deal with two opposing elements in physics, this makes sense: "There are positive and there are negative electric charges, there are magnetic north and there are magnetic south poles, there is attraction and there is repulsion. There is no third form of electric charge, there is no third form of a magnetic pole, and there is no third form of force which is the opposite of attraction and at the same time the opposite of repulsion, too. (...) But sometimes this strategy of choosing only two basic elements fails in physics. Obviously baryons like neutrons and protons can be described in a mathematically appropriate way only, if we use three distinct and somehow opposing basic elements. There are three quarks, two are not enough," I explained in [13, p. 2/3].

In these situations where we have to deal with three 'opposing' elements, we need the Taylor expansion of an ordinary scalar trinom

$$(e_0 + x\,e_0 + x^2\,e_0)^n \quad \longleftrightarrow \quad (1+x+x^2)^n \tag{92}$$

to get the probabilities for the possible differents results which are represented by Eulerian trinomial coefficients:

$$p^\star_{(n,k)} = \frac{1}{3^n}\binom{n}{k}_2 \tag{93}$$

We have to throw a three-sided coin n times, and these coefficients can be found in the three trinomial triangles [13, fig. 1]. But does nature really use three-sided coins to construct these results? A second option would be to throw instead a two-sided quarkonian coin $2n$ times. Then the three trinomial triangles are constructed with the quarkonian geometric algebra binom [13, eq. (40)]

$$(e_1 + \ominus x\,e_2)^{2n} \quad \longleftrightarrow \quad \frac{1}{2^n}\left(\sigma_y - \sigma_z + x(\sigma_z - \sigma_x)\right)^{2n} \tag{94}$$

The coefficients of the Taylor expansion of this quarkonian binom (93) can be arranged as the three quarkonian trinomial triangles. Geometric algebra of quarks thus transfers (or translates) trinomial mathematical structures into binomial ones. The positive quarkonian trinomial triangle is shown in figure 6. For a complete picture of all three quarkonian trinomial triangles please look at [13, fig. 4].

$$e_0$$
$$e_1 \qquad \ominus e_2$$
$$e_0 \qquad e_0 \qquad e_0$$
$$e_1 \qquad e_1 + \ominus e_2 \qquad e_1 + \ominus e_2 \qquad \ominus e_2$$
$$e_0 \qquad 2e_0 \qquad 3e_0 \qquad 2e_0 \qquad e_0$$
$$e_1 \qquad 2e_1 + \ominus e_2 \quad 3e_1 + \ominus 2e_2 \quad 2e_1 + \ominus 3e_2 \quad e_1 + \ominus 2e_2 \qquad \ominus e_2$$
$$e_0 \qquad 3e_0 \qquad 6e_0 \qquad 7e_0 \qquad 6e_0 \qquad 3e_0 \qquad e_0$$

Figure 6: The positive quarkonian trinomial triangle.

15 Fundamental Building Blocks of Quarks

We do not live in two- or three-dimensional space. We live in four-dimensional spacetime. Everyday experience tells us that this spacetime possesses three spacelike directions. And everyday experience tells us that there is one timelike direction. All theses directions can be represented by Dirac matrices in geometric algebra.

To construct Dirac-like matrices we will return to the ideas of [11] and eschew orthogonal unit vectors e_4 or $\ominus e_4$. This makes sense if we consider quarks as objects (or entities) with no orthogonal structural composition, but with a totally $2\pi/3$ triangularial structure. In the spirit of [10] we are then able to construct Dirac-like matrices with the help of the Zehfuss-Kronecker product or direct product using only unit vectors of a plane. The direct products of base scalar, the two base vectors, and the base bivector of a plane automatically yield the geometric base entities of four-dimensional spacetimes (see discussion in [10]).

Let's start with two orthogonal Pauli-like (3 x 3) matrices which represent two space-like base vectors of the $e_1 e_2$-plane:

$$\sigma_x = e_1 \tag{95}$$
$$\sigma_y = \frac{1}{\sqrt{3}}(e_1 + 2e_2) \tag{96}$$

(9 x 9) Dirac-like matrices which can be interpreted as base vectors of a four-dimensional spacetime with one time direction and three space directions can be found with the help of the Zehfuss-Kronecker product in analogy to [10, p. 9, eq. 35 to 38] by:

$$\gamma_x = e_0 \otimes \sigma_x = e_0 \otimes e_1 \tag{97}$$
$$\gamma_y = e_0 \otimes \sigma_y = \frac{1}{\sqrt{3}}\, e_0 \otimes (e_1 + 2e_2) \tag{98}$$
$$\gamma_z = (\sigma_x \sigma_y) \otimes (\sigma_x \sigma_y) = \frac{1}{3}\, (e_0 + 2e_{12}) \otimes (e_0 + 2e_{12}) \tag{99}$$
$$\gamma_t = \sigma_x \otimes (\sigma_x \sigma_y) = \frac{1}{\sqrt{3}}\, e_1 \otimes (e_0 + 2e_{12}) \tag{100}$$

Using the identies [14, sec. V]

$$1 = \ominus \otimes \ominus = e_0 \otimes e_0 \tag{101}$$

$$(-1) = \bar{1} = \ominus \otimes e_0 = e_0 \otimes \ominus \tag{102}$$

it can be seen that these base vectors square to

$$\gamma_x^2 = \gamma_y^2 = \gamma_z^2 = e_0 \otimes e_0 \tag{103}$$

$$\gamma_t^2 = \ominus \otimes e_0 \tag{104}$$

Space-like base vectors thus square to the (9 x 9) identity matrix (100) of plus one while the time-like base vector squares to the (9 x 9) matrix of minus one (101), as it is not possible to construct the inverse signature of $(+, -, -, -)$ with the Zehfuss-Kronecker product of two space-like base vectors [10, p. 9]. And of course all base vectors (96) to (99) anti-commute:

$$\gamma_i \gamma_j = (e_0 \otimes \ominus)\, \gamma_j \gamma_i = (\ominus \otimes e_0)\, \gamma_j \gamma_i \tag{105}$$

Thus the following conclusion can be drawn: The Zehfuss-Kronecker products or direct products

$$\begin{array}{lll} e_{0,1} = e_0 \otimes e_1 & e_{0,2} = e_0 \otimes e_2 & e_{0,3} = e_0 \otimes e_3 \\ e_{1,0} = e_1 \otimes e_0 & e_{2,0} = e_2 \otimes e_0 & e_{3,0} = e_3 \otimes e_0 \end{array} \tag{106}$$

form fundamental building blocks of our world as the Dirac-like (9 x 9) matrices (96) to (99) representing four-dimensional base vectors are nothing else than linear combinations of several of these fundamental building blocks or of products (117) of these fundamental building blocks:

$$\gamma_x = e_{0,1} \tag{107}$$

$$\gamma_y = \frac{1}{\sqrt{3}}\,(e_{0,1} + 2e_{0,2}) \tag{108}$$

$$\gamma_z = \frac{1}{3}\,(e_{0,0} + 2e_{0,1}e_{0,2} + 2e_{1,0}e_{2,0} + 4e_{1,1}e_{2,2}) \tag{109}$$

$$\gamma_t = \frac{1}{\sqrt{3}}\,(e_{1,0} + 2e_{1,1}e_{0,2}) \tag{110}$$

Every four-dimensional spacetime vector r can then be written as usual as

$$r = ct\gamma_t + x\gamma_x + y\gamma_y + z\gamma_z \tag{111}$$

And the other way round we are able to identify

$$e_{0,1} = \gamma_x \tag{112}$$

$$e_{0,2} = \frac{1}{2}\,(\bar{1}\gamma_x + \sqrt{3}\gamma_y) \tag{113}$$

$$e_{0,3} = \frac{1}{2}\,(\bar{1}\gamma_x + \bar{1}\sqrt{3}\gamma_y) \tag{114}$$

$$e_{1,0} = \bar{1}\gamma_x\gamma_y\gamma_t \tag{115}$$

$$e_{2,0} = \frac{1}{2}\,(\gamma_x\gamma_y\gamma_t + \sqrt{3}\gamma_z\gamma_t) \tag{116}$$

$$e_{3,0} = \frac{1}{2}\,(\gamma_x\gamma_y\gamma_t + \bar{1}\sqrt{3}\gamma_z\gamma_t) \tag{117}$$

And part of the same four-dimensional world then are the following building blocks which are constructed by simple (9 x 9) matrix multiplications

$$e_{i,j} = e_{i,0}e_{0,j} = e_{0,j}e_{i,0} \tag{118}$$

using the fundamental building blocks $e_{i,0}$ and $e_{0,j}$ of eq. (105):

$$\begin{array}{lll} e_{1,1} = e_1 \otimes e_1 & e_{1,2} = e_1 \otimes e_2 & e_{1,3} = e_1 \otimes e_3 \\ e_{2,1} = e_2 \otimes e_1 & e_{2,2} = e_2 \otimes e_2 & e_{2,3} = e_2 \otimes e_3 \\ e_{3,1} = e_3 \otimes e_1 & e_{3,2} = e_3 \otimes e_2 & e_{3,3} = e_3 \otimes e_3 \end{array} \tag{119}$$

There is nothing rectangular any more in these mathematical entities (118). Consequently these building blocks are the building blocks of quarks as they show quarkonian symmetry (see section 17). They are four-dimensional spacetime entities and obviously no entities of higher dimensional worlds.

The philosophy behind this is the philosophy of Gull, Lasenby and Doran, who once wrote as second last sentence of a paper: "We have no objections to the use of higher dimensions as such; it just seems to us to be unnecessary at present, when the algebra of the space that we *do* observe contains so many wonders that are not yet generally appreciated" [5].

Furthermore this philosophy is an extended version of the philosophy of Snygg, when he describes the history of the electron and its algebra with the words: "It was necessary to attribute to the electron a spin of ½ and a periodicity of 4π. In recent years, it has become more widely recognized that objects larger than electrons also have 4π periodicities" [26, p. 11]. In the same way the Dirac belt trick demonstrates that extended macroscopic objects "in some sense loosely attached to its surroundings" [26, p. 12] show the 4π symmetry of electrons, I am convinced another belt-like trick will show us quark symmetry for objects larger than quarks one day. Finding such a trick for extended objects should be only a matter of time revealing the geometrical simplicity of quark algebra.

16 Quarkonian Wedge Products

The (9 x 9) direct nihilation product $N \otimes N$ can be decomposed into

$$\begin{aligned} N \otimes N = {} & e_0 \otimes e_0 + e_{12} \otimes e_0 + e_{21} \otimes e_0 + e_0 \otimes e_{12} + e_0 \otimes e_{21} \\ & + e_{12} \otimes e_{12} + e_{12} \otimes e_{21} + e_{21} \otimes e_{12} + e_{21} \otimes e_{21} \\ = {} & 1 + e_{12,0} + e_{21,0} + e_{0,12} + e_{0,21} + e_{12,12} + e_{12,21} \\ & + e_{21,12} + e_{21,21} \end{aligned} \tag{120}$$

The decomposition (119) can be used for constructing a quarkonian decomposition of the (9 x 9) geometric product of two vectors by adding $\frac{1}{9}\, ba(N \otimes N)$ to the geometric product ab.

$$\begin{aligned} ab = \frac{1}{9}(&9ab + ba + e_{12,0}ba + e_{21,0}ba + e_{0,12}ba + e_{0,21}ba \\ & + e_{12,12}ba + e_{12,21}ba + e_{21,12}ba + e_{21,21}ba) \end{aligned} \tag{121}$$

This produces a modified dot product

$$a \bullet b = \frac{1}{9}(ab + ba) \tag{122}$$

and eight new wedge products

$$a \wedge_{12,0} b = \frac{1}{9}(ab + e_{12,0}\, ba) \qquad a \wedge_{21,0} b = \frac{1}{9}(ab + e_{21,0}\, ba) \tag{123}$$

$$a \wedge_{0,12} b = \frac{1}{9}(ab + e_{0,12}\, ba) \qquad a \wedge_{0,21} b = \frac{1}{9}(ab + e_{0,21}\, ba) \tag{124}$$

$$a \wedge_{12,12} b = \frac{1}{9}(ab + e_{12,12,}\, ba) \qquad a \wedge_{12,21} b = \frac{1}{9}(ab + e_{12,21}\, ba) \tag{125}$$

$$a \wedge_{21,12} b = \frac{1}{9}(ab + e_{21,12}\, ba) \qquad a \wedge_{21,21} b = \frac{1}{9}(ab + e_{21,21}\, ba) \tag{126}$$

These eight wedge products should describe quarkonian symmetry much better than the orthogonally constructed standard wedge product of eq. (40).

17 Greetings from Vienna

Ten days after the AGACSE conference in La Rochelle I took part at the ICNPAA 2012 congress in Vienna which included an organised Clifford algebra session. Directly after my talk (about five-dimensional cosmological special relativity) Suzuki gave a very impressive account about the construction of quarks with the help of nonion algebra and Galois extensions [18].

And there are good reasons to relate the (3 x 3) nonions of [18, p. 1010, eq. 22]

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \bar{Q}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{127}$$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} \quad \bar{Q}_2 = \begin{pmatrix} 0 & 0 & 1 \\ j & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix} \tag{128}$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix} \quad \bar{Q}_1 = \begin{pmatrix} 0 & 0 & 1 \\ j^2 & 0 & 0 \\ 0 & j & 0 \end{pmatrix} \quad Q_1 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & j^2 \\ 1 & 0 & 0 \end{pmatrix} \tag{129}$$

Kerner [16], [17], Larynowicz, Nouno, Nagayama and Suzuki [18] use to model quarks with the basic (9 x 9) building blocks of quarks given in eq. (111). This can be done by identifying

$$j \,\hat{=}\, e_{12} \tag{130}$$

$$j^2 \,\hat{=}\, e_{12}^2 = e_{21} \tag{131}$$

$$1 = j^3 \,\hat{=}\, e_{12}^3 = e_0 \tag{132}$$

and hence according to [16, slides 21, 42, 83], [17, p. 154 & 159]

$$j + j^2 = -1 \,\hat{=}\, \ominus \qquad \text{or} \qquad j + j^2 + 1 = 0 \,\hat{=}\, N \tag{133}$$

by rearranging these matrices.

One central idea of the AGACSE poster [12] and my paper [11] is to strictly base this alternative geometric algebra approach on the three S_3 permutation algebra vectors e_1, e_2, and e_3. These vectors are the starting point, not the geometric products e_1e_2, e_2e_1, or $e_1^2 = e_0$ which only constitute a subgroup of S_3 permuation algebra. In the spirit of this conceptual foundation it is reasonable to start with vector-like matrices (133) to (135) which structurally resemble e_1, e_2, and e_3.

$$q_{1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad q_{1,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & j \\ 0 & j^2 & 0 \end{pmatrix} \quad q_{1,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & j^2 \\ 0 & j & 0 \end{pmatrix} \tag{134}$$

$$q_{2,1} = \begin{pmatrix} 0 & 0 & j \\ 0 & 1 & 0 \\ j^2 & 0 & 0 \end{pmatrix} \quad q_{2,2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad q_{2,3} = \begin{pmatrix} 0 & 0 & j^2 \\ 0 & 1 & 0 \\ j & 0 & 0 \end{pmatrix} \tag{135}$$

$$q_{3,1} = \begin{pmatrix} 0 & j & 0 \\ j^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad q_{3,2} = \begin{pmatrix} 0 & j^2 & 0 \\ j & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad q_{3,3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{136}$$

Please note: These matrices are only seemingly vector-like matrices. Even though they square to the base scalar $e_0 \otimes e_0$, they do not represent vectors.

The nonion matrices of eq. (126), (127), and (128) can now be expressed as the following matrix products:

$$\begin{aligned} R_1 &= q_{1,1}q_{1,1} = q_{1,2}q_{1,2} = q_{1,3}q_{1,3} = q_{2,1}q_{2,1} = q_{2,2}q_{2,2} \\ &= q_{2,3}q_{2,3} = q_{3,1}q_{3,1} = q_{3,2}q_{3,2} = q_{3,3}q_{3,3} \end{aligned} \tag{137}$$

$$\begin{aligned} R_2 &= q_{1,1}q_{1,3} = q_{1,2}q_{1,1} = q_{1,3}q_{1,2} = j\, q_{2,1}q_{2,3} = j\, q_{2,2}q_{2,1} \\ &= j\, q_{2,3}q_{2,2} = j^2 q_{3,1}q_{3,3} = j^2 q_{3,2}q_{3,1} = j^2 q_{3,3}q_{3,2} \end{aligned} \tag{138}$$

$$\begin{aligned} R_3 &= q_{1,1}q_{1,2} = q_{1,2}q_{1,3} = q_{1,3}q_{1,1} = j^2 q_{2,1}q_{2,2} = j^2 q_{2,2}q_{2,3} \\ &= j^2 q_{2,3}q_{2,1} = j\, q_{3,1}q_{3,2} = j\, q_{3,2}q_{3,3} = j\, q_{3,3}q_{3,1} \end{aligned} \tag{139}$$

$$\begin{aligned} Q_1 &= j^2 q_{1,1}q_{3,2} = j\, q_{1,2}q_{3,3} = q_{1,3}q_{3,1} = j\, q_{2,1}q_{1,2} = q_{2,2}q_{1,3} \\ &= j^2 q_{2,3}q_{1,1} = q_{3,1}q_{2,2} = j^2 q_{3,2}q_{2,3} = j\, q_{3,3}q_{2,1} \end{aligned} \tag{140}$$

$$\begin{aligned} Q_2 &= j\, q_{1,1}q_{3,1} = q_{1,2}q_{3,2} = j^2 q_{1,3}q_{3,3} = j\, q_{2,1}q_{1,1} = q_{2,2}q_{1,2} \\ &= j^2 q_{2,3}q_{1,3} = j\, q_{3,1}q_{2,1} = q_{3,2}q_{2,2} = j^2 q_{3,3}q_{2,3} \end{aligned} \tag{141}$$

$$\begin{aligned} Q_3 &= q_{1,1}q_{3,3} = j^2 q_{1,2}q_{3,1} = j\, q_{1,3}q_{3,2} = j\, q_{2,1}q_{1,3} = q_{2,2}q_{1,1} \\ &= j^2 q_{2,3}q_{1,2} = j^2 q_{3,1}q_{2,3} = j\, q_{3,2}q_{2,1} = q_{3,3}q_{2,2} \end{aligned} \tag{142}$$

$$\begin{aligned} \bar{Q}_1 &= j\, q_{1,1}q_{2,3} = j^2 q_{1,2}q_{2,1} = q_{1,3}q_{2,2} = j^2 q_{2,1}q_{3,3} = q_{2,2}q_{3,1} \\ &= j\, q_{2,3}q_{3,2} = q_{3,1}q_{1,3} = j\, q_{3,2}q_{1,1} = j^2 q_{3,3}q_{1,2} \end{aligned} \tag{143}$$

$$\begin{aligned} \bar{Q}_2 &= j^2 q_{1,1}q_{2,1} = q_{1,2}q_{2,2} = j\, q_{1,3}q_{2,3} = j^2 q_{2,1}q_{3,1} = q_{2,2}q_{3,2} \\ &= j\, q_{2,3}q_{3,3} = j^2 q_{3,1}q_{1,1} = q_{3,2}q_{1,2} = j\, q_{3,3}q_{1,3} \end{aligned} \tag{144}$$

$$\begin{aligned} \bar{Q}_3 &= q_{1,1}q_{2,2} = j\, q_{1,2}q_{2,3} = j^2 q_{1,3}q_{2,1} = j^2 q_{2,1}q_{3,2} = q_{2,2}q_{3,3} \\ &= j\, q_{2,3}q_{3,1} = j\, q_{3,1}q_{1,2} = j^2 q_{3,2}q_{1,3} = q_{3,3}q_{1,1} \end{aligned} \tag{145}$$

Thus Kerner's and Suzuki's nonions (126), (127), (128) are constructed by using the more basic building blocks (133), (134), (135).

Another central idea of this paper is that nasty and ugly transformations should be avoided (see section 4). In the spirit of this perhaps naive principle one-sided matrix multiplications are ignored and rejected as legitimate transformations. Only beautiful transformations like eq. (73) with right-hand and left-hand matrix multiplications are considered and accepted as genuine and truly geometric algebraic transformations.

Following this principle the basic building blocks of quarks $e_{i,j}$ with $i, j = 1, 2, 3$ (neglecting direct products of e_4) and the basic building blocks of nonions $q_{i,j}$ can be considered as isomorphic as obey the same reflection laws. The basic building blocks of quarks transform as

$$e_{i,j}\, e_{k,l}\, e_{i,j} = e_{iki,jlj} \qquad e_{k,l}\, e_{i,j}\, e_{k,l} = e_{iki,jlj} \tag{146}$$

$$e_{k,l}\, e_{iki,jlj}\, e_{k,l} = e_{i,j} \qquad e_{iki,jlj}\, e_{k,l}\, e_{iki,jlj} = e_{i,j} \tag{147}$$

$$e_{iki,jlj}\, e_{i,j}\, e_{iki,jlj} = e_{k,l} \qquad e_{i,j}\, e_{iki,jlj}\, e_{i,j} = e_{k,l} \tag{148}$$

The basic building blocks of nonions transform in a totally identical way:

$$q_{i,j}\, e_{k,l}\, q_{i,j} = q_{iki,jlj} \qquad q_{k,l}\, q_{i,j}\, q_{k,l} = q_{iki,jlj} \tag{149}$$

$$q_{k,l}\, q_{iki,jlj}\, q_{k,l} = q_{i,j} \qquad q_{iki,jlj}\, q_{k,l}\, q_{iki,jlj} = e_{i,j} \tag{150}$$

$$q_{iki,jlj}\, q_{i,j}\, q_{iki,jlj} = q_{k,l} \qquad q_{i,j}\, q_{iki,jlj}\, q_{i,j} = q_{k,l} \tag{151}$$

Thus both sets of basic building blocks $e_{i,j}$ and $q_{i,j}$ possess the same mathematical structure concerning the beautiful parts of mathematics. Consequently it should be possible to describe the same phenomena of physics – provided the personal categories of mathematical beauty mentioned in this paper coincide (at least partly) with the formative principles of nature. It is my innermost conviction that this is indeed the case.

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