Biot-Savart's Companion

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Abstract

We introduce a law that we believe is a natural companion to the Biot-Savart Law of classical electrodynamics. The forces resulting from these two laws compliment one another: the force due to the Biot-Savart Law changes the direction of the velocity of a test particle, but not its magnitude; the force due to the companion law changes the magnitude of the velocity, but not its direction.

1 The Companion Law

The Biot-Savart law states that the element of magnetic field $d\mathbf{B}$ produced by a short segment $d\mathbf{l}$ of wire of arbitrary shape carrying a steady line current I, in SI units, is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\mathbf{l}\times\hat{\mathbf{r}}}{r^2} \tag{1}$$

with magnitude

$$|d\mathbf{B}| = dB = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2} \tag{2}$$

where θ is the angle between d$\mathbf{l}$ and $\hat{\mathbf{r}}$.

I believe the Biot-Savart law has a companion law that, to my knowledge, has gone undiscovered until now. My companion laws to (1) and (2) introduce a 'scalar' field H whose element dH has similar form to (1) and (2).[1]

The companion law is derived from $\nabla\cdot\mathbf{A}$, where $\mathbf{A}$ is the vector potential due to a 'point' charge q' moving with constant velocity $\mathbf{v}'$.[2] We first note that $\mathbf{A} = \mathbf{v}'\phi/c^2$, where $\phi = q'/(4\pi\epsilon_0 r)$

[1] See http://www.softcom.net/users/der555/newtransform.pdf. The field H is actually part of my generalized electric field (however, it is not referred to as H, there). Note that, here, I am using a three-dimensional, non-relativistic treatment, as opposed to the four-dimensional, relativistic treatment used at the above link.

[2] $\nabla\cdot\mathbf{A}$ is physical and, in general, nonzero in my theory.

is the scalar electric potential, due to q', at a field point P a distance r from q'.[3] Thus, after substitution we get

$$\nabla \cdot \mathbf{A} = \nabla \cdot \left(\frac{\mathbf{v}'\phi}{c^2} \right) = -\,\frac{q'}{4\pi\epsilon_0 c^2}\,\frac{\mathbf{v}' \cdot \hat{\mathbf{r}}}{r^2} \tag{3}$$

where $\hat{\mathbf{r}}$ is a unit vector pointing from q' to P.

If we refer to $\nabla \cdot \mathbf{A}$ here as H, and note that $\epsilon_0\mu_0 = 1/c^2$, we can write (3) as

$$H = -\,\frac{\mu_0 q'}{4\pi}\,\frac{\mathbf{v}' \cdot \hat{\mathbf{r}}}{r^2} \tag{4}$$

Since $\mathbf{v}' \cdot \hat{\mathbf{r}}$ is a scalar quantity (in three-dimensional space), we could also write (4) as

$$H = -\,\frac{\mu_0 q'}{4\pi}\,\frac{v' \cos\theta}{r^2} \tag{5}$$

where v' is the magnitude of $\mathbf{v}'$ and θ is the angle between $\mathbf{v}'$ and $\hat{\mathbf{r}}$.

The element dH at P due to a 'point' charge dq' is

$$dH = -\,\frac{\mu_0 dq'}{4\pi}\,\frac{\mathbf{v}' \cdot \hat{\mathbf{r}}}{r^2} \tag{6}$$

or

$$dH = -\,\frac{\mu_0 dq'}{4\pi}\,\frac{v' \cos\theta}{r^2} \tag{7}$$

Now consider a small segment $d\mathbf{l}$ of wire carrying a steady current I, within which the 'point' charge dq' is moving with velocity $\mathbf{v}'$ parallel to $d\mathbf{l}$. In terms of the current I, substituting $dq' = I dt$ and $\mathbf{v}' = d\mathbf{l}/dt$, (6) and (7) become

$$dH = -\,\frac{\mu_0}{4\pi}\,\frac{I d\mathbf{l} \cdot \hat{\mathbf{r}}}{r^2} \tag{8}$$

or

$$dH = -\,\frac{\mu_0}{4\pi}\,\frac{I dl \cos\theta}{r^2} \tag{9}$$

where dl is the magnitude of $d\mathbf{l}$. The equations (8) and (9) are my companions to (1) and (2), respectively.

We can find the total field H at P by integrating (8) along the wire, so that

$$H = -\,\frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \cdot \hat{\mathbf{r}}}{r^2} \tag{10}$$

2 Force on a Test Charge in the Field H

The *additional force* $\mathbf{F}_a$ on a test charge q at point P moving with velocity $\mathbf{v}$ in the field H at P is[4]

$$\mathbf{F}_a = -\,q\mathbf{v}H \tag{11}$$

[3] It is important to note that the scalar potential ϕ, here, is the three-dimensional part of my four-dimensional potential at http://www.softcom.net/users/der555/newtransform.pdf, since this is a three-dimensional treatment.

[4] See http://www.softcom.net/users/der555/actreact.pdf.

If H at P is due to a 'point' charge q' moving with constant velocity $\mathbf{v}'$, we can substitute (4) or (5) into (11), to obtain

$$\mathbf{F}_a = -\,q\mathbf{v}\left(-\,\frac{\mu_0 q'}{4\pi}\,\frac{\mathbf{v}'\cdot\hat{\mathbf{r}}}{r^2}\right) = \frac{\mu_0 q q'}{4\pi}\,\frac{\mathbf{v}\,(\mathbf{v}'\cdot\hat{\mathbf{r}})}{r^2} \tag{12}$$

or

$$\mathbf{F}_a = \frac{\mu_0 q q'}{4\pi}\,\frac{\mathbf{v}\,(v'\cos\theta)}{r^2} \tag{13}$$

respectively.

For a steady line current I, the element of force $d\mathbf{F}_a$ on q due to a short segment $d\mathbf{l}$ of wire, using (8), is

$$d\mathbf{F}_a = -\,q\mathbf{v}dH = \frac{\mu_0 q\mathbf{v}}{4\pi}\,\frac{Id\mathbf{l}\cdot\hat{\mathbf{r}}}{r^2} \tag{14}$$

The total force $\mathbf{F}_a$ on q can be found by integrating (14) along the wire, resulting in

$$\mathbf{F}_a = \frac{\mu_0 qI\mathbf{v}}{4\pi}\int\frac{d\mathbf{l}\cdot\hat{\mathbf{r}}}{r^2} \tag{15}$$

It is interesting to note that the magnetic force $\mathbf{F}_m = q\mathbf{v}\times\mathbf{B}$ changes the *direction* of the velocity of a test particle, but not the magnitude of its velocity. In contrast, my additional force $\mathbf{F}_a = -\,q\mathbf{v}H$ changes the *magnitude* of the velocity, but not its direction. These two forces go hand-in-hand, along with the electric force $\mathbf{F}_e = q\mathbf{E}$, to complete the three-dimensional, non-relativistic equations of motion of a test particle.[5]

These equations predict new physics, but the most interesting prediction involves the time component of the force. Analyzing the time component of the force is outside of the three-dimensional treatment here, however due to its implications I would like to mention it.

The time component of the force on q, due to q', is[6]

$$F_t = q\left(\frac{1}{c}\,\mathbf{v}\cdot\mathbf{E} + c\,\nabla\cdot\mathbf{A}\right) \tag{16}$$

The last term is the time component of the additional force

$$F_{ta} = qc\,\nabla\cdot\mathbf{A} \tag{17}$$

or

$$F_{ta} = qcH \tag{18}$$

Substituting (4) into (18), we obtain

$$F_{ta} = -\,\frac{\mu_0 q q' c}{4\pi}\,\frac{\mathbf{v}'\cdot\hat{\mathbf{r}}}{r^2} \tag{19}$$

For a steady line current I, using the same methods as above, we find that the element of force dF_{ta} on q due to a short segment $d\mathbf{l}$ of wire, is

$$dF_{ta} = -\,\frac{\mu_0 qc}{4\pi}\,\frac{Id\mathbf{l}\cdot\hat{\mathbf{r}}}{r^2} \tag{20}$$

[5]See http://www.softcom.net/users/der555/actreact.pdf.
[6]See http://www.softcom.net/users/der555/actreact.pdf.

The total time component of the additional force F_{ta} on q can be found by integrating (20) along the wire, to obtain

$$F_{ta} = -\frac{\mu_0 qcI}{4\pi} \int \frac{d\mathbf{l} \cdot \hat{\mathbf{r}}}{r^2} \tag{21}$$

If we extend Newton's Second Law $\mathbf{F} = m\mathbf{a}$ to include the time component of the force F_t, that is $F_t = ma_t$, where a_t is the time component of the acceleration, then we can write the time component of the additional force as $F_{ta} = ma_{ta}$, where a_{ta} is the acceleration due to the time component of the additional force F_{ta}.[7] Thus we can now write the time component of the acceleration a_{ta} of q, due to F_{ta}, as

$$a_{ta} = -\frac{\mu_0 qcI}{4\pi m} \int \frac{d\mathbf{l} \cdot \hat{\mathbf{r}}}{r^2} \tag{22}$$

where m is the mass of q.

Note that c in F_{ta} and dF_{ta} is the time component of the velocity of q, thus q is *stationary* in space.[8] Also note that, for a nonzero F_{ta} (or dF_{ta}), there is a *nonzero* acceleration in time. In other words, (22) describes a means of transporting q in time.

[7]There are terms in addition to the time rate of change of four-momentum in my four-dimensional, relativistic treatment at http://www.softcom.net/users/der555/newtransform.pdf. These terms are considered to be relativistic, however, and are not considered here.

[8]I'm focusing, here, on the case where q is stationary in F_t. There might also be an acceleration in time due to $(q/c)\,\mathbf{v} \cdot \mathbf{E}$ if q is not stationary in space.