Action-Reaction Paradox Resolution

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Abstract

According to Newton's third law, in a collision between two isolated particles 'action equals reaction'. However, in classical electrodynamics, this law is violated. In general, in a collision between two isolated charged particles, the momentum of the particles is not conserved. Typically, it is necessary to combine field momentum with particle momentum in order to 'balance the scales'. A paradox arises from the fact that, generally, particle momentum is conserved in the center of mass frame, but not in the lab frame. Here, we offer a resolution to this paradox in which the third law remains valid for collisions between charged particles, in all situations and in all frames, without the need to invoke the momentum of the field. This article is a revision of my previous article by the same title.

1 Introduction

In Newton's third law, the momentum of an isolated system of 'particles' is conserved [1]. In other words, action equals reaction. Unfortunately, in classical electrodynamics, this is not the case. The momentum of an isolated system of 'charged' particles is *not* conserved, thus, action *does not* equal reaction, except in limited cases.

The usual remedy for this problem in classical electrodynamics is to add field momentum to particle momentum in order to 'balance the scales'. The solution instead, in my opinion, is not to add field momentum to particle momentum, but rather to modify the equations of motion - the Lorentz force equations. The result of this modification is that the equations of motion become consistent with Newton's third law for the particles, themselves, independent of the fields.

I will evaluate the forces on two isolated charged particles moving at constant (non-relativistic) velocities.[1] In the classical model, these forces are not always equal and opposite. I intend to show that this inequality of forces leads to a paradox, to which I offer a resolution.

In this article, the potential four-vector ($\mathbf{A}$,ϕ) will be substituted for the electric and magnetic field three-vectors $\mathbf{E}$ and $\mathbf{B}$ used in my previous article "Action-Reaction Paradox Resolution" [3]. SI units will be used throughout this article.

[1] My complete relativistic, four-dimensional force equations, can be seen in the section "The Force Density Four-vector" in [2].

2 The Paradox

Let us evaluate the forces on two isolated identical 'point' charges q and q' considered, for the purposes of this discussion, to be moving with constant velocities $\mathbf{v}$ and $\mathbf{v}'$, respectively. The Lorentz force $\mathbf{F}$ on q due to q' is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the conventional electric and magnetic field three-vectors at the position of q due to q' [4].

The Lorentz force $\mathbf{F}'$ on q' due to q is

$$\mathbf{F}' = q'(\mathbf{E}' + \mathbf{v}' \times \mathbf{B}') \tag{2}$$

where $\mathbf{E}'$ and $\mathbf{B}'$ are the conventional electric and magnetic field three-vectors at the position of q' due to q.

The electric forces $\mathbf{F}_e = q\mathbf{E}$ and $\mathbf{F}'_e = q'\mathbf{E}'$ on the particles are always equal in magnitude and opposite in direction, but the magnetic forces

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B} \tag{3}$$

and

$$\mathbf{F}'_m = q'\mathbf{v} \times \mathbf{B}' \tag{4}$$

are *not*, in general, equal and opposite. In fact, for every case except the cases, one or both particles at rest, or particles on parallel paths, the magnetic forces are *not* equal and opposite. Therefore, the total forces on the particles in the lab frame are not, in general, equal and opposite.

However, in the center of mass (cm) frame, the total forces on the particles *are* equal and opposite at all times, due to symmetry. Therefore, we have the paradox that the forces on the particles are equal and opposite in the cm frame, but not in the lab frame.

3 The Resolution

I would like to offer a resolution to this paradox. Consider the additional force,

$$\mathbf{F}_a = -q\mathbf{v}\left(\nabla \cdot \mathbf{A} - \frac{\partial \phi}{c^2 \partial t}\right) \tag{5}$$

on q due to q', where ϕ and $\mathbf{A}$ are the conventional electric potential and potential three-vector, respectively, at the position of q due to q', c is the speed of light and $\mathbf{A} = \mathbf{v}'\phi/c^2$.

The additional force on q' due to q is[2]

$$\mathbf{F}'_a = -q'\mathbf{v}'\left(\nabla \cdot \mathbf{A}' - \frac{\partial \phi'}{c^2 \partial t}\right) \tag{6}$$

[2]These additional forces are not *ad hoc* additions. They are due to the time component of my electric field four-vector which is part of my force density equations in [2] (these forces are not referred to as $\mathbf{F}_a$ and $\mathbf{F}'_a$ in my article).

where ϕ' and $\mathbf{A}'$ are the conventional electric potential and potential three-vector, respectively, at the position of q' due to q, and $\mathbf{A}' = \mathbf{v}\phi'/c^2$.

Therefore, our new law for the force on q is

$$\mathbf{F} = q\left[\mathbf{E} + \mathbf{v} \times \mathbf{B} - \mathbf{v}\left(\nabla \cdot \mathbf{A} - \frac{\partial \phi}{c^2 \partial t}\right)\right] \tag{7}$$

and for the force on q'

$$\mathbf{F}' = q'\left[\mathbf{E}' + \mathbf{v}' \times \mathbf{B}' - \mathbf{v}'\left(\nabla \cdot \mathbf{A}' - \frac{\partial \phi'}{c^2 \partial t}\right)\right] \tag{8}$$

For consistency, we will write the electric and magnetic fields in terms of the potential four-vectors ($\mathbf{A}$,ϕ) and ($\mathbf{A}'$,ϕ') as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{9}$$

$$\mathbf{E}' = -\nabla\phi' - \frac{\partial \mathbf{A}'}{\partial t}, \qquad \mathbf{B}' = \nabla \times \mathbf{A}' \tag{10}$$

Inserting (9) and (10) into (7) and (8), respectively, we get

$$\mathbf{F} = q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) - \mathbf{v}\left(\nabla \cdot \mathbf{A} - \frac{\partial \phi}{c^2 \partial t}\right)\right] \tag{11}$$

and

$$\mathbf{F}' = q'\left[-\nabla\phi' - \frac{\partial \mathbf{A}'}{\partial t} + \mathbf{v}' \times (\nabla \times \mathbf{A}') - \mathbf{v}'\left(\nabla \cdot \mathbf{A}' - \frac{\partial \phi'}{c^2 \partial t}\right)\right] \tag{12}$$

Then, using the vector identity

$$\mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A} \tag{13}$$

we can write (11) as

$$\mathbf{F} = q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A} - \mathbf{v}(\nabla \cdot \mathbf{A}) + \mathbf{v}\frac{\partial \phi}{c^2 \partial t}\right] \tag{14}$$

Making the following substitutions,[3] in (14)

$$\begin{aligned}
q &= q' \\
\frac{\partial \phi}{\partial t} &= \frac{\partial \phi'}{\partial t} \\
\nabla\phi &= -\nabla\phi' \\
\frac{\partial \mathbf{A}}{\partial t} &= \mathbf{v}'\frac{\partial \phi'}{c^2 \partial t} \ldots (A.1) \\
\nabla(\mathbf{v} \cdot \mathbf{A}) &= -\nabla(\mathbf{v}' \cdot \mathbf{A}') \ldots (A.2) \\
(\mathbf{v} \cdot \nabla)\mathbf{A} &= -\mathbf{v}'(\nabla \cdot \mathbf{A}') \ldots (A.3) \\
\mathbf{v}(\nabla \cdot \mathbf{A}) &= -(\mathbf{v}' \cdot \nabla)\mathbf{A}' \ldots (A.4) \\
\mathbf{v}\frac{\partial \phi}{c^2 \partial t} &= \frac{\partial \mathbf{A}'}{\partial t} \ldots (A.5)
\end{aligned}$$

[3]Please, click on the blue links to the right of the equations for their derivations in the Appendix.

we get

$$\mathbf{F} = -q' \left[-\nabla\phi' - \frac{\partial \mathbf{A}'}{\partial t} + \nabla(\mathbf{v}' \cdot \mathbf{A}') - (\mathbf{v}' \cdot \nabla)\mathbf{A}' - \mathbf{v}'(\nabla \cdot \mathbf{A}') + \mathbf{v}' \frac{\partial \phi'}{c^2 \partial t} \right] \tag{15}$$

From the vector identity

$$\mathbf{v}' \times (\nabla \times \mathbf{A}') = \nabla(\mathbf{v}' \cdot \mathbf{A}') - (\mathbf{v}' \cdot \nabla)\mathbf{A}' \tag{16}$$

we can convert (15) into

$$\mathbf{F} = -q' \left[-\nabla\phi' - \frac{\partial \mathbf{A}'}{\partial t} + \mathbf{v}' \times (\nabla \times \mathbf{A}') - \mathbf{v}' \left(\nabla \cdot \mathbf{A}' - \frac{\partial \phi'}{c^2 \partial t} \right) \right] \tag{17}$$

Now, notice that the right-hand side of (17) is the negative of the right-hand side of (12), so that we can write

$$\mathbf{F} = -\mathbf{F}' \tag{18}$$

Therefore, action equals reaction for the spatial part of the force. This is the same result we got in [3]. Thus, classical electrodynamics *can* be made consistent with Newton's third law for interactions between the particles, themselves, independent of the fields.

The result (18) takes into consideration only the spatial components of the force. I would, now, like to consider the time component of the force.

The new law for the time component of the force on q due to q' is

$$F_t = q \left[\frac{1}{c} \mathbf{v} \cdot \mathbf{E} + c \left(\nabla \cdot \mathbf{A} - \frac{\partial \phi}{c^2 \partial t} \right) \right] \tag{19}$$

Substituting from (9) into (19), we have

$$F_t = q \left[\frac{1}{c} \mathbf{v} \cdot \left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \right) + c \left(\nabla \cdot \mathbf{A} - \frac{\partial \phi}{c^2 \partial t} \right) \right] \tag{20}$$

or

$$F_t = q \left[-\frac{1}{c} \mathbf{v} \cdot \nabla\phi - \frac{1}{c} \mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial t} + c\nabla \cdot \mathbf{A} - \frac{\partial \phi}{c\, \partial t} \right] \tag{21}$$

The time component of the force on q' due to q is, therefore,

$$F'_t = q' \left[-\frac{1}{c} \mathbf{v}' \cdot \nabla\phi' - \frac{1}{c} \mathbf{v}' \cdot \frac{\partial \mathbf{A}'}{\partial t} + c\nabla \cdot \mathbf{A}' - \frac{\partial \phi'}{c\, \partial t} \right] \tag{22}$$

We now make the substitutions,[4] below into (21)

$$\begin{aligned} q &= q' \\ \frac{1}{c} \mathbf{v} \cdot \nabla\phi &= -c\nabla \cdot \mathbf{A}' \ldots (A.6) \\ \mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial t} &= \mathbf{v}' \cdot \frac{\partial \mathbf{A}'}{\partial t} \ldots (A.7) \\ c\nabla \cdot \mathbf{A} &= -\frac{1}{c} \mathbf{v}' \cdot \nabla\phi' \ldots (A.8) \\ \frac{\partial \phi}{c\, \partial t} &= \frac{\partial \phi'}{c\, \partial t} \end{aligned}$$

[4] Again, please, click on the blue links to the right of the equations for their derivations in the Appendix.

to get

$$F_t = q' \left[-\frac{1}{c}\mathbf{v}' \cdot \nabla\phi' - \frac{1}{c}\mathbf{v}' \cdot \frac{\partial \mathbf{A}'}{\partial t} + c\nabla \cdot \mathbf{A}' - \frac{\partial \phi'}{c\,\partial t} \right] \tag{23}$$

Now comparing (22) and (23), we see that

$$F_t = F'_t \tag{24}$$

which is, again, the same result we got in [3]. The time component of the force F_t on q and the time component of the force F'_t on q' are equal, but *not* opposite, for two isolated particles.

4 Final Notes

There are additional terms in the force equations in [2] that have been omitted in this discussion. Nevertheless, the inclusion of these terms would still leave $\mathbf{F} = -\mathbf{F}'$ and $F_t = F'_t$.

It would be interesting to conjecture, at this point, that the indication/reason that time and space are different might, in some way, be due to the difference in the results (18) and (24).

Equations (14) and (21) are gauge invariant under the substitution of $\mathbf{A}' = \mathbf{A} + \nabla\psi$ and $\phi' = \phi - \partial\psi/\partial t$ for $\mathbf{A}$ and ϕ, respectively, where ψ is some scalar field [5], providing that $\nabla^2\psi + \partial^2\psi/\partial t^2 = 0$. The vector potential $\mathbf{A}'$ and the scalar potential ϕ', here, should not be confused with the vector and scalar potentials at the position of q' due to q discussed earlier.

References

[1] Richard P Feynman, RB Leighton, and M Sands. "The Feynman Lectures on Physics. Volume 2. Massachusetts, Palo Alto". In: London: Addison-Wesley Publishing Company, Inc, 1964, II-10-1.

[2] David E. Rutherford. "New Transformation Equations and the Electric Field Four-vector". In: *viXra* (2013). URL: https://vixra.org/pdf/1301.0112v1.pdf.

[3] David E. Rutherford. "Action-Reaction Paradox Resolution, v1". In: *viXra* (2013). URL: https://vixra.org/pdf/1301.0156v1.pdf.

[4] Richard P Feynman, RB Leighton, and M Sands. "The Feynman Lectures on Physics. Volume 2. Massachusetts, Palo Alto". In: London: Addison-Wesley Publishing Company, Inc, 1964, II-13-1.

[5] Richard P Feynman, RB Leighton, and M Sands. "The Feynman Lectures on Physics. Volume 2. Massachusetts, Palo Alto". In: London: Addison-Wesley Publishing Company, Inc, 1964, II-18-10.

A Appendix

These derivations are done recalling that $\mathbf{A} = \mathbf{v}'\phi/c^2$ and $\mathbf{A}' = \mathbf{v}\phi'/c^2$, and that $\mathbf{v}$ and $\mathbf{v}'$ are assumed to be constant, thus their derivatives vanish.

A.1 Derivation of $\partial\mathbf{A}/\partial t = \mathbf{v}'(\partial\phi'/c^2\partial t)$

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial t} &= \frac{\partial(\mathbf{v}'\phi)}{c^2\partial t}\\ &= \mathbf{v}'\frac{\partial \phi}{c^2\partial t}\\ &= \mathbf{v}'\frac{\partial \phi'}{c^2\partial t}\end{aligned}$$

A.2 Derivation of $\nabla(\mathbf{v}\cdot\mathbf{A}) = -\nabla(\mathbf{v}'\cdot\mathbf{A}')$

The x-component of $\nabla(\mathbf{v}\cdot\mathbf{A})$ is

$$\begin{aligned}\left[\nabla(\mathbf{v}\cdot\mathbf{A})\right]_x &= \frac{\partial}{\partial x}\left(v_x A_x + v_y A_y + v_z A_z\right)\\ &= v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\\ &= \frac{1}{c^2}\left[v_x\frac{\partial(v'_x\phi)}{\partial x} + v_y\frac{\partial(v'_y\phi)}{\partial x} + v_z\frac{\partial(v'_z\phi)}{\partial x}\right]\\ &= \frac{1}{c^2}\left(v_x v'_x\frac{\partial\phi}{\partial x} + v_y v'_y\frac{\partial\phi}{\partial x} + v_z v'_z\frac{\partial\phi}{\partial x}\right)\\ &= \frac{1}{c^2}\left(-v_x v'_x\frac{\partial\phi'}{\partial x} - v_y v'_y\frac{\partial\phi'}{\partial x} - v_z v'_z\frac{\partial\phi'}{\partial x}\right)\\ &= -\frac{1}{c^2}\left[v'_x\frac{\partial(v_x\phi')}{\partial x} + v'_y\frac{\partial(v_y\phi')}{\partial x} + v'_z\frac{\partial(v_z\phi')}{\partial x}\right]\\ &= -\left(v'_x\frac{\partial A'_x}{\partial x} + v'_y\frac{\partial A'_y}{\partial x} + v'_z\frac{\partial A'_z}{\partial x}\right)\\ &= -\frac{\partial}{\partial x}\left(v'_x A'_x + v'_y A'_y + v'_z A'_z\right)\\ &= -\left[\nabla(\mathbf{v}'\cdot\mathbf{A}')\right]_x,\end{aligned}$$

therefore,

$$\nabla(\mathbf{v}\cdot\mathbf{A}) = -\nabla(\mathbf{v}'\cdot\mathbf{A}')$$

A.3 Derivation of $(\mathbf{v}\cdot\nabla)\mathbf{A} = -\mathbf{v}'(\nabla\cdot\mathbf{A}')$

The x-component of $(\mathbf{v}\cdot\nabla)\mathbf{A}$ is

$$\begin{aligned}
[(\mathbf{v}\cdot\nabla)\mathbf{A}]_x &= \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)A_x \\
&= v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z} \\
&= \frac{1}{c^2}\left[v_x\frac{\partial(v'_x\phi)}{\partial x} + v_y\frac{\partial(v'_x\phi)}{\partial y} + v_z\frac{\partial(v'_x\phi)}{\partial z}\right] \\
&= \frac{1}{c^2}\left(v_xv'_x\frac{\partial\phi}{\partial x} + v_yv'_x\frac{\partial\phi}{\partial y} + v_zv'_x\frac{\partial\phi}{\partial z}\right) \\
&= \frac{1}{c^2}\left(-v_xv'_x\frac{\partial\phi'}{\partial x} - v_yv'_x\frac{\partial\phi'}{\partial y} - v_zv'_x\frac{\partial\phi'}{\partial z}\right) \\
&= -\frac{1}{c^2}\left[v'_x\frac{\partial(v_x\phi')}{\partial x} + v'_x\frac{\partial(v_y\phi')}{\partial y} + v'_x\frac{\partial(v_z\phi')}{\partial z}\right] \\
&= -\left(v'_x\frac{\partial A'_x}{\partial x} + v'_x\frac{\partial A'_y}{\partial y} + v'_x\frac{\partial A'_z}{\partial z}\right) \\
&= -v'_x\left(\frac{\partial A'_x}{\partial x} + v'_x\frac{\partial A'_y}{\partial y} + v'_x\frac{\partial A'_z}{\partial z}\right) \\
&= -\left[\mathbf{v}'\left(\nabla\cdot\mathbf{A}'\right)\right]_x,
\end{aligned}$$

therefore,

$$(\mathbf{v}\cdot\nabla)\mathbf{A} = -\mathbf{v}'\left(\nabla\cdot\mathbf{A}'\right)$$

A.4 Derivation of $\mathbf{v}(\nabla\cdot\mathbf{A}) = -(\mathbf{v}'\cdot\nabla)\mathbf{A}'$

The x-component of $\mathbf{v}(\nabla\cdot\mathbf{A})$ is

$$\begin{aligned}
[\mathbf{v}(\nabla\cdot\mathbf{A})]_x &= v_x\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)\\
&= v_x\frac{\partial A_x}{\partial x}+v_x\frac{\partial A_y}{\partial y}+v_x\frac{\partial A_z}{\partial z}\\
&= \frac{1}{c^2}\left[v_x\frac{\partial(v'_x\phi)}{\partial x}+v_x\frac{\partial(v'_y\phi)}{\partial y}+v_x\frac{\partial(v'_z\phi)}{\partial z}\right]\\
&= \frac{1}{c^2}\left(v_xv'_x\frac{\partial\phi}{\partial x}+v_xv'_y\frac{\partial\phi}{\partial y}+v_xv'_z\frac{\partial\phi}{\partial z}\right)\\
&= \frac{1}{c^2}\left(-v_xv'_x\frac{\partial\phi'}{\partial x}-v_xv'_y\frac{\partial\phi'}{\partial y}-v_xv'_z\frac{\partial\phi'}{\partial z}\right)\\
&= -\frac{1}{c^2}\left[v'_x\frac{\partial(v_x\phi')}{\partial x}+v'_y\frac{\partial(v_x\phi')}{\partial y}+v'_z\frac{\partial(v_x\phi')}{\partial z}\right]\\
&= -\left(v'_x\frac{\partial A'_x}{\partial x}+v'_y\frac{\partial A'_x}{\partial y}+v'_z\frac{\partial A'_x}{\partial z}\right)\\
&= -\left(v'_x\frac{\partial}{\partial x}+v'_y\frac{\partial}{\partial y}+v'_z\frac{\partial}{\partial z}\right)A'_x\\
&= -\left[(\mathbf{v}'\cdot\nabla)\,\mathbf{A}'\right]_x,
\end{aligned}$$

therefore,

$$\mathbf{v}(\nabla\cdot\mathbf{A}) = -\left(\mathbf{v}'\cdot\nabla\right)\mathbf{A}'$$

A.5 Derivation of $\mathbf{v}(\partial\phi/c^2\partial t) = \partial\mathbf{A}'/\partial t$

$$\begin{aligned}
\mathbf{v}\frac{\partial\phi}{c^2\partial t} &= \mathbf{v}\frac{\partial\phi'}{c^2\partial t}\\
&= \frac{\partial(\mathbf{v}\phi')}{c^2\partial t}\\
&= \frac{\partial\mathbf{A}'}{\partial t}
\end{aligned}$$

A.6 Derivation of $(1/c)\mathbf{v} \cdot \nabla\phi = -c\nabla \cdot \mathbf{A}'$

$$
\begin{aligned}
\frac{1}{c}\mathbf{v} \cdot \nabla\phi &= \frac{1}{c}\left(v_x\frac{\partial\phi}{\partial x} + v_y\frac{\partial\phi}{\partial y} + v_z\frac{\partial\phi}{\partial z}\right) \\
&= \frac{1}{c}\left(-v_x\frac{\partial\phi'}{\partial x} - v_y\frac{\partial\phi'}{\partial y} - v_z\frac{\partial\phi'}{\partial z}\right) \\
&= -\frac{1}{c}\left[\frac{\partial(v_x\phi')}{\partial x} + \frac{\partial(v_y\phi')}{\partial y} + \frac{\partial(v_z\phi')}{\partial z}\right] \\
&= -c\left(\frac{\partial A'_x}{\partial x} + \frac{\partial A'_y}{\partial y} + \frac{\partial A'_z}{\partial z}\right) \\
&= -c\nabla \cdot \mathbf{A}'
\end{aligned}
$$

A.7 Derivation of $\mathbf{v} \cdot \partial\mathbf{A}/\partial t = \mathbf{v}' \cdot \partial\mathbf{A}'/\partial t$

$$
\begin{aligned}
\mathbf{v} \cdot \frac{\partial\mathbf{A}}{\partial t} &= v_x\frac{\partial A_x}{\partial t} + v_y\frac{\partial A_y}{\partial t} + v_z\frac{\partial A_z}{\partial t} \\
&= \frac{1}{c^2}\left[v_x\frac{\partial(v'_x\phi)}{\partial t} + v_y\frac{\partial(v'_y\phi)}{\partial t} + v_z\frac{\partial(v'_z\phi)}{\partial t}\right] \\
&= \frac{1}{c^2}\left(v_x v'_x\frac{\partial\phi}{\partial t} + v_y v'_y\frac{\partial\phi}{\partial t} + v_z v'_z\frac{\partial\phi}{\partial t}\right) \\
&= \frac{1}{c^2}\left(v_x v'_x\frac{\partial\phi'}{\partial t} + v_y v'_y\frac{\partial\phi'}{\partial t} + v_z v'_z\frac{\partial\phi'}{\partial t}\right) \\
&= \frac{1}{c^2}\left[v'_x\frac{\partial(v_x\phi')}{\partial t} + v'_y\frac{\partial(v_y\phi')}{\partial t} + v'_z\frac{\partial(v_z\phi')}{\partial t}\right] \\
&= v'_x\frac{\partial A'_x}{\partial t} + v'_y\frac{\partial A'_y}{\partial t} + v'_z\frac{\partial A'_z}{\partial t} \\
&= \mathbf{v}' \cdot \frac{\partial\mathbf{A}'}{\partial t}
\end{aligned}
$$

A.8 Derivation of $c\nabla \cdot \mathbf{A} = -(1/c)\mathbf{v}' \cdot \nabla\phi'$

$$\begin{aligned}
c\nabla \cdot \mathbf{A} &= c\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \\
&= \frac{1}{c}\left[\frac{\partial (v'_x\phi)}{\partial x} + \frac{\partial (v'_y\phi)}{\partial y} + \frac{\partial (v'_z\phi)}{\partial z}\right] \\
&= \frac{1}{c}\left(v'_x\frac{\partial \phi}{\partial x} + v'_y\frac{\partial \phi}{\partial y} + v'_z\frac{\partial \phi}{\partial z}\right) \\
&= \frac{1}{c}\left(-v'_x\frac{\partial \phi'}{\partial x} - v'_y\frac{\partial \phi'}{\partial y} - v'_z\frac{\partial \phi'}{\partial z}\right) \\
&= -\frac{1}{c}\left(v'_x\frac{\partial \phi'}{\partial x} + v'_y\frac{\partial \phi'}{\partial y} + v'_z\frac{\partial \phi'}{\partial z}\right) \\
&= -\frac{1}{c}\mathbf{v}' \cdot \nabla\phi'
\end{aligned}$$