Universal principles of perfect chaos

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The purpose of this work was to introduce strict, comprehensive definition of perfect chaos, to find out its basic properties in terms of phase transitions and give connections for uncertainties, lying in base of perfect chaos concept. Concept of perfect chaos as undetermined description was introduced basing on two formalized necessary and sufficient conditions: finite phase space resolution and instability of phase space trajectories. Properties of Kolmogorov system, including phase mixing, turned out to be consequences of chaotic state but not its comprehensive and sufficient conditions. Description relativity was defined as mandatory property of perfect chaos – the same areas of phase space may show regular and chaotic properties depending on description space - time accuracy. Herewith evolution of physical system in given generalized phase space can be represented by consequence of regular states and intermediate transitions. For chaotic state with uniform diffusion it was found out that nonlinear dispersion law is mandatory property. One in its turn necessarily leads to space – time instability of probability density and appearance of probability cavities in phase space - phase space attractors where particles density grows up. Case of chaotic state with fixed boundary and diffusion was considered. It turned out that Fourier decomposition allows to derive relations between coordinate – momentum and time - energy definition uncertainties. It was shown that chaos diffusion factor is the only parameter, limiting product of corresponding uncertainties.

Since discovery of turbulence as chaotic motion example several explanations of its appearance were proposed. Scientific thought generalized disorderly dynamic behavior and introduced abstract chaotic models. However as turbulence transitions scenarios and generalized chaotic mechanisms continued to compete and stayed not comprehensive separately. This led to necessity of testing areas of their application and correspondence to certain finite set of problems. For this reason goal of universal formulation of chaos and its basic properties became relevant. Results, represented in this article, allow receiving clear idea about chaos substance and ways of its appearance. They show that chaotic behavior depends on type of observation and has to be considered in the same space – time resolution for receiving comparable results. Typical problem of uniform chaotic motion in limited space was considered on basis of introduced formalized chaotic properties. It was found out that description of chaos may exist only under condition of certain resolution types and connections between space and time uncertainties.

I. PERFECT CHAOS AND RELATIVITY

Several scenarios of turbulence transition have been proposed since 1883 year when turbulence concept was introduced through experiments of English engineer Osborne Reynolds.

He has noticed dynamic phase transition in liquid stream, characterized by unstable vortex appearance and introduced two limit states of motion: laminar and turbulent. Since, several scenarios of turbulence transition have been developed. Among them Landau – Hopf instability mechanism (Ref.1), Lorenz attractor mechanism (Ref.2), scenario of Poincare – Feygebaum (Ref.3) and scenario of Kolmogorov - Arnold – Moser (Ref.4). Each of outlined mechanism has its individual area of application and basic assumptions. For this reason none of them is universal, moreover unambiguous connections between them are not stated yet.

Since introduction of turbulence concept its properties were investigated and generalized. For now concepts of dynamic limit states themselves were generalized and transformed into states of regular motion and perfect chaos state. Therefore determined motion corresponds to laminar stream while perfect chaos – to turbulent motion state. Let us consider second limit state - the concept of perfect chaos. One is defined as undetermined description in given phase space resolution. Unpredictability of motion is consequence of two conditions realization: a) finite resolution of generalized phase space; b) instability of phase space trajectories.

Concept of generalized phase space may be explained through system model consisting of M particles which have independent phase trajectories. If motion of each particle is determined in N dimensional phase space, then generalized phase is $M{\cdot}N$ dimensional and corresponding vector will be system characteristic vector in Hilbert space. If connections are introduced dimension of generalized space will be equal to $P=M{\cdot}N\text{-}d$, where d is number of connection equations. Then resolution finiteness in at least one direction of generalized phase space then leads to uncertainty in initial dynamic system state. Formally this condition may be represented in the following way:

$$\left({}_{\Delta}\Gamma\right)_i \geq \left({}_{\Delta}\Gamma_{\min}\right)_i \succ 0 \qquad i \in \{1, P\} \tag{1}$$

Here $\left({}_{\Delta}\Gamma\right)_i = \left({}_{\Delta}x_i\right)_{\min}$ is element of describing generalized phase space while x_i is characteristic vector projection, corresponding to i direction of Hilbert phase space. If we assume that minimal uncertainty is isotropic, $\left({}_{\Delta}\Gamma_{\min}\right)_i = \alpha$ then elementary cell volume of generalized phase space is expressed in the following way: ${}_{\Delta}\Gamma_{\min} = \prod_{i=1}^{P}\left({}_{\Delta}x_i\right)_{\min} = \alpha^P$.

Let's consider second condition of perfect chaos state under suggestion that first one is satisfied. If initial any two system parts (particles) have instable trajectories, diverging in phase space, determined dynamic description of their motion comes impossible and perfect chaos state is reached. Instability requirement may be expressed through sum of positive Lyapunov factors σ_i^+ for each dimension of generalized phase space:

$$h(\vec{x}(t)) \succ 0 \qquad h(\vec{x}(t)) = \sum_{i=1}^{K} \sigma_i^+(\vec{x}(t)) \tag{2}$$

Undetermined characteristic trajectory is basic property of perfect chaos system which leads to two consequences. First one regards auto correlation function of dynamic value $f(\vec{x}(t))$. Here system evolution is defined by characteristic generalized function $\vec{x}(t)$ - reverse mapping $t(\vec{x})$ is not single valued in general case. According to relations (1) and (2) $g_1(\vec{x}(t)) = \lim_{t\to\infty}\left[f_1(\vec{x}(t))\right]$ and $g_2(\vec{x}(t)) = \lim_{t\to 0}\left[f_2(\vec{x}(t))\right]$ are independent functions (f_1 and f_2 are arbitrary dynamic functions), then auto correlation characteristic function $R\left[f(\vec{x}(t)), \tau\right]$ satisfies equation (3):

$$\lim_{\tau\to\infty} R\left[f(\vec{x}(t)), \tau\right] = 0 \tag{3}$$

This relation reflects called property of mixing according to terminology, introduced by G.M. Zaslavsky (Ref.5). In fact realization of (3) leads to execution of Slutsky criterion for ergodic system:

$$\lim_{\substack{T\to\infty \\ \tau\to\infty}} \int_0^T R[f(\vec{x}(t)),\tau]\cdot\left(1-\frac{\tau}{T}\right)d\tau = 0 \tag{4}$$

Here τ is delay time between start and the end of system evolution observation. According to (4) system becomes ergodic for $\tau\to\infty$. For physical systems this condition can be following expression:

$$_{\Delta}t_{\min} \succ {}_{\Delta}t_{ins} \qquad {}_{\Delta}t_{inst}[\vec{x}(t)] = \frac{1}{h(\vec{x}(t))} \tag{5}$$

Here $_{\Delta}t_{\min}$ is finite time resolution while $_{\Delta}t_{inst}$ is instability increment for $\vec{x}(t)$, that may be expressed through integrated Lyapunov factor (2). Satisfaction of third chaos condition allows receiving following equations for any dynamic function in frame of ergodic description:

$$\langle f[\vec{x}(t)]\rangle = \left[\int_{\Gamma} f[\vec{x}(t)]d\Gamma\right]\cdot\frac{1}{\Gamma} = \overline{f[\vec{x}(t)]} = \left[\int_0^{T(t)} f[\vec{x}(t)]dt\right]\cdot\frac{1}{T(t)} \tag{6}$$

In given relation $\Gamma(t)$ and $T(t)$ are phase space volume, occupied by phase trajectory during observation time and observation time itself. For integrated Lyapunov factor given property allows to outline consequence of relation (2):

$$h_d \succ 0 \qquad \langle h(\vec{x}(t))\rangle = \overline{h(\vec{x}(t))} = h_d \tag{7}$$

Here h_d is dynamic entropy of Kolmogorov – Sinai that may be expressed through entropy of system in given phase space representation (Ref.5):

$$h_d = \frac{\partial S}{\partial t} = \frac{1}{\Gamma(t)}\cdot\frac{\partial(\Gamma(t))}{\partial t} \tag{8}$$

Quantity $S = \ln(\Gamma(t))$ is Gibbs entropy of chaotic system with account of finite phase space resolution and condition (5). Satisfaction of chaos conditions (1) and (2) leads to mandatory growth of Gibbs entropy even in case when correspondent deterministic description is conservative.

Consequences (3), (6) and (7) for relations (1) and (2) in fact correspond to definition of Kolmogorov system (Ref.6) state (K – system) under condition that $_{\Delta}t_{\min} \succ {}_{\Delta}t_{ins}$. However we have to notice that K – system requirements are necessary but not sufficient for perfect chaos state (PCS) observation.

It may be useful to state another qualitative property of PCS – description relativity. As it was shown PCS is limit state of dynamic system, characterized by properties, outlined below:

$$({}_{\Delta}\Gamma)_i \geq ({}_{\Delta}\Gamma_{\min})_i \succ 0 \qquad i\in\{1,P\} \tag{9}$$

$$h(\vec{x}(t)) \succ 0 \qquad h(\vec{x}(t)) = \sum_{i=1}^{K}\sigma_i^{+}(\vec{x}(t)) \tag{10}$$

Satisfaction inequality depends on the description parameters $({}_{\Delta}\Gamma_{\min})_i$ and $h(\vec{x}(t))$. According to (9) and (10) magnitude of these parameters may lead to opposite limit states. They are perfect chaos state (PCS) and regular state (RS). Let's consider example of physical system. Then finiteness of $({}_{\Delta}\Gamma_{\min})_i$ is provided by quantum uncertainty relations.

In general case $\left({}_{\Delta}\Gamma_{\min}\right)_i$ is function of time resolution: $\left({}_{\Delta}\Gamma_{\min}\right)_i = f\left({}_{\Delta}t_{\min}\right)$. Finite magnitude of $\left({}_{\Delta}\Gamma_{\min}\right)_i$ allows to leave one control parameter - integrated Lyapunov factor. Therefore regular state of system will be represented by group of relations (10) and (11):

$$\left({}_{\Delta}\Gamma\right)_i \geq \left({}_{\Delta}\Gamma_{\min}\right)_i \succ 0 \qquad i \in \{1, P\} \tag{10}$$

$$h(\vec{x}(t)) \leq 0 \qquad h(\vec{x}(t)) = \sum_{i=1}^{K} \sigma_i^+(\vec{x}(t)) \tag{11}$$

Second relation contains time as parameter. In such a way generally transition between two limit states may occur at any instant of time. If evolution of physical system in given generalized phase space is represented by consequence of regular states and corresponding transitions, it can be defined as quasiregular state of motion (QRS). Transition between two regular trajectories (limit cycles) is realized through chaotic states. According to terminology of G.M.Zaslavsky (Ref.7) in phase space such type of motion is represented by "stochastic sea with stability islands". Time delay of two consequent transitions $R_j \Rightarrow R_{j+1}$ and $R_{j+1} \Rightarrow R_{j+2}$, also called bifurcations, $\Delta_j = t_{j+1} - t_j$ in general is function of time parameter and ${}_{\Delta}\Gamma_{\min}$: $\Delta_j = \Delta_j(t, {}_{\Delta}\Gamma_{\min})$. Let's consider phase trajectory in three generalized phase spaces Γ^1, Γ^2 and Γ^3 such that ${}_{\Delta}\Gamma^1{}_{\min} \succ {}_{\Delta}\Gamma^2{}_{\min} \succ {}_{\Delta}\Gamma^3{}_{\min}$. Then the same phase trajectory Γ^3, represented through Γ^1 and Γ^2, will have different fractions of regular state (stability islands) and transitional state (perfect chaos). Phenomenon of description relativity is explained by Fig.1 (a) and Fig.1 (b), where two dimensional phase spaces are supposed to have uniform resolution.

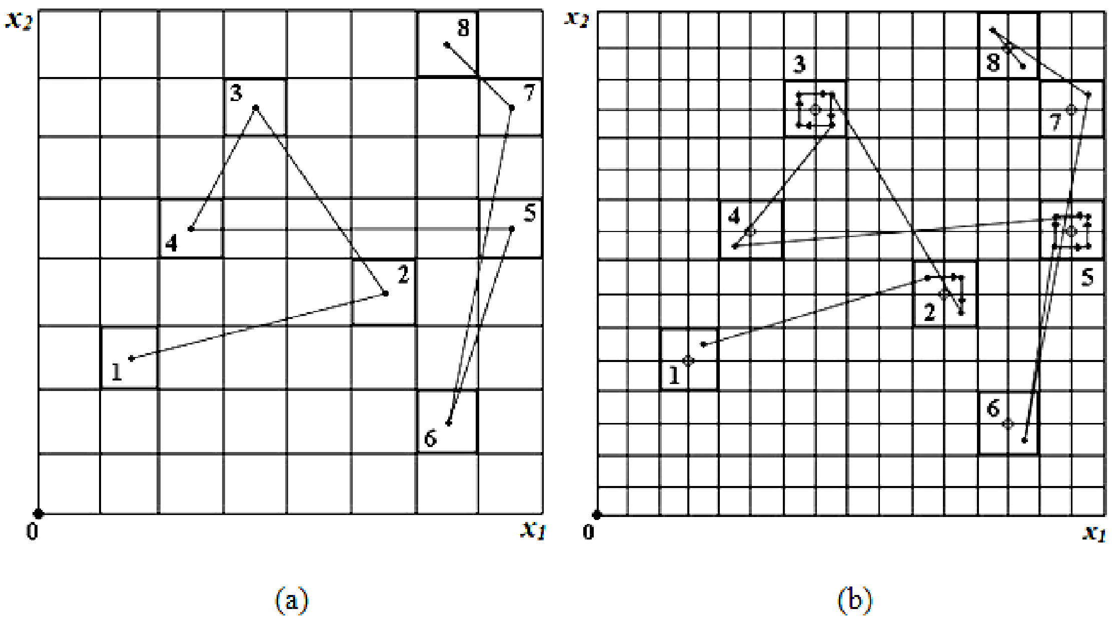

FIG.1. a) Γ^1 phase space representation. Chaotic phase trajectory 1→2→3→4→5→6→7→8; b) Γ^2 phase space representation. Quasi captures in segments 3 and 5 – regular motion areas with finite life time – quasi regular stability islands. Hollow circles duplicates state points in Γ^1 phase space representation.

Each system dynamic state is represented as point inside corresponding cell, which limits phase space uncertainty. Transitions between enumerated states are symbolically designated as straight line – we don't take into account phase ways of corresponding bifurcations.

In given figure the same segments *3* and *5* of phase trajectories are defined as chaotic motion - Fig.1 a) or quasiregular motion - Fig.1 (b) - with finite life time – quasi regular trajectories symbolically shown in Fig.1 (b) inside large cells. In general, duration of system existence, i.e. life time $\tau_i \succ 0$ (i=1,2,…,8), in any macroscopic dynamic state is arbitrary. Regular motion appearance may lead to space - time stabilization of system. If stabilization occurs for *i* state, then $\tau_i \to \infty$. In other case current stabilization is temporary and quasi capture is realized (Ref.7). In this case regular trajectory is stable during finite time length τ_i. After this time quasi regular torus comes unstable, deforms and may finally disappear.
Increase of generalized phase space resolution may lead to appearance of new quasi regular areas or overall space - time stabilization of trajectory. In first case some portion of particles in cells (representation of coarsened resolution) turns out to transform into toruses with finite or infinite life time. One is defined by total time of system observation – "infinite" life time will correspond in this case stable existence of regular area during all observation time. As we can see space – time relativity allows receiving qualitatively different chaotic (regular) properties for the same part of given dynamic system.

II. NONLINEARITY AS MANDATORY PROPERTY OF PERFECT CHAOS

In equation (3) deriving we used property of independence for arbitrary dynamic functions f_1 and f_2 if ${}_\Delta t_{\min} \succ {}_\Delta t_{ins}$. Let's assume that considered system consists of *M* subsystems – particles, characterized by corresponding probability densities $\rho_k(\vec{x}_k)$, $k = \overline{1,M}$ (k=1,…,M). Then, if $f_k = \rho_k(\vec{x}_k)$, for perfect chaos system we have generalized relation (3):

$$\lim_{\tau\to\infty} C[\rho_k(\vec{x}_k), \rho_l(\vec{x}_l), \tau] = 0 \quad l \neq k \tag{12}$$

Here *C* is correlation function. Property (12) may be called correlation decay or system memory loss.
One of approaches applied for characterization of transitional properties in given frame is based on Fokker - Plank - Kolmogorov model (Ref.8). One allows obtaining basic equation of transport from Chapman - Kolmogorov equation (13).

$$\rho(\vec{x}^3, t_3 | \vec{x}^1, t_1) = \int d\vec{x}^2 [\rho(\vec{x}^2, t_2 | \vec{x}^1, t_1) \cdot \rho(\vec{x}^3, t_3 | \vec{x}^2, t_2)] \qquad d\vec{x}^2 = \prod_{i=1}^{P} (dx_i^2) \tag{13}$$

Integration is made for phase volume occupied by system phase trajectory. Upper indexes of characteristic vector $\vec{x}(t)$ correspond to consequent time moments t_1, t_2, t_3 : $t_3 \succ t_2 \succ t_1$. Function $\rho(\vec{x}^r, t_r | \vec{x}^{r-1}, t_{r-1})$ ($r = 1,2,3$) is conditional probability density with fixed initial condition $\vec{x}^{r-1}, t_{r-1}$. Let's recall basic assumptions made for derivation of Fokker - Plank – Kolmogorov equation (Ref.8).

1. $\rho(\vec{x}^r, t_r | \vec{x}^{r-1}, t_{r-1}) = \rho(\vec{x}^r, | \vec{x}^{r-1}, t_{r-1} - t_r) = \rho(\vec{x}^r, | \vec{x}^{r-1}, {}_\Delta t_r)$. Given condition means that probability of bifurcation doesn't depend on absolute magnitude of initial time point: ${}_\Delta t_{\min} \succ {}_\Delta t_{ins}$.
This limitation is satisfied if (1), (2) and (5) relations for chaos are valid. Equation (5) is realized necessarily if we speak about formed instability;

2. $\rho(\vec{x}^{r},\vec{x}^{r-1},t_{r-1})=\rho(\vec{x}^{r},t_{r-1})$ - final conditional probability density doesn't depend on the initial coordinate vector. In terms of characteristic generalized function $\vec{x}(t)$ this condition is valid as well for the reasons given in Point 1;

3. $\rho(\vec{x}^{r},|\vec{x}^{r-1},t_{r-1}+{}_{\Delta}t_{r})=\rho(\vec{x}^{r},|\vec{x}^{r-1},t_{r-1})+{}_{\Delta}t_{r}\cdot\dfrac{\partial\rho(\vec{x}^{r},|\vec{x}^{r-1},t_{r-1})}{\partial t}$. For finite phase space cell and time account this expression can be realized for ${}_{\Delta}t_{\min}={}_{\Delta}t_{r}$ and $d\vec{x}={}_{\Delta}\vec{x}_{\min}$;

4. Initial distribution density is defined by Dirac delta function: $\rho(\vec{x})=\ \delta(0)$, i.e. initial coordinate can be defined accurately (in frame of phase space finite resolution Dirac delta function corresponds to rectangular function);

5. $\rho(\vec{x}^{r},|\vec{x}^{r-1},{}_{\Delta}t_{r})=\delta(\vec{x}^{r}-\vec{x}^{r-1})+a(\vec{x}^{r-1},{}_{\Delta}t_{r})\cdot\delta'(\vec{x}^{r}-\vec{x}^{r-1})+\frac{1}{2}\cdot b(\vec{x}^{r-1},{}_{\Delta}t_{r})\cdot\delta''(\vec{x}^{r}-\vec{x}^{r-1})$. Here for existence of second derivative of Dirac function it is necessary for ${}_{\Delta}t_{r}$ to satisfy following condition: ${}_{\Delta}t_{r}=2\cdot{}_{\Delta}t_{\min}$ in frame of certain resolution phase space (1). Coefficients $a(\vec{x}^{r-1},{}_{\Delta}t_{r})$ and $b(\vec{x}^{r-1},{}_{\Delta}t_{r})$ are defined by relations (14) and (15):

$$a(\vec{x}^{r-1},{}_{\Delta}t_{r})=\int(\vec{x}^{r}-\vec{x}^{r-1})\cdot\rho(\vec{x}^{r},|\vec{x}^{r-1},{}_{\Delta}t_{r})d\vec{x}^{r}=\left\langle\left\langle({}_{\Delta}\vec{x})\right\rangle\right\rangle \qquad (14)$$

$$b(\vec{x}^{r-1},{}_{\Delta}t_{r})=\int(\vec{x}^{r}-\vec{x}^{r-1})^{2}\cdot\rho(\vec{x}^{r},|\vec{x}^{r-1},{}_{\Delta}t_{r})d\vec{x}^{r}=\left\langle\left\langle({}_{\Delta}\vec{x})^{2}\right\rangle\right\rangle \qquad (15)$$

On basis of relation (15) second transport coefficient can be introduced:

$$B(\vec{x})=\lim_{{}_{\Delta}t_{r}\to0}\left(\frac{\left\langle\left\langle({}_{\Delta}\vec{x})^{2}\right\rangle\right\rangle}{{}_{\Delta}t_{r}}\right)=\frac{\left\langle\left\langle({}_{\Delta}\vec{x})^{2}\right\rangle\right\rangle}{2\cdot{}_{\Delta}t_{\min}} \qquad (16)$$

Given assumptions allow to formulate known, not parametric form of Fokker Plank Kolmogorov equation (FPK equation):

$$\frac{\partial\rho(\vec{x}^{r},t_{r-1})}{\partial t}=\frac{1}{2}\cdot\frac{\partial}{\partial\vec{x}^{r}}\left(B(\vec{x}^{r})\cdot\frac{\partial\rho(\vec{x}^{r},t_{r-1})}{\partial\vec{x}^{r}}\right) \qquad (17)$$

It can be shown that in relations (16) and (17) time is hidden parameter (Ref.8). Let's represent energy of system mass unit:

$$\varepsilon(\vec{x}^{r},\vec{x}^{r-1},t)=\frac{\left[{}_{\Delta}\vec{x}(\vec{x}^{r},\vec{x}^{r-1},t_{r})\right]^{2}}{\left[{}_{\Delta}t_{\min}\right]^{2}} \qquad (18)$$

According to (17) second transport factor can be expressed in the modified form (19) - superscripts are omitted.

$$B(\vec{x},t)=\frac{\left\langle\left\langle({}_{\Delta}\vec{x})^{2}\right\rangle\right\rangle}{2\cdot{}_{\Delta}t_{\min}}=2\cdot{}_{\Delta}t_{\min}\cdot\int\varepsilon(\vec{x}^{r},\vec{x}^{r-1},t)\cdot\rho(\vec{x}^{r},\vec{x}^{r-1},{}_{\Delta}t_{r})d\vec{x}^{r-1}=2\cdot{}_{\Delta}t_{\min}\cdot\left\langle\left\langle\varepsilon(\vec{x}^{r},\vec{x}^{r-1},t)\right\rangle\right\rangle \qquad (19)$$

In relation (18) $\vec{x},t$ are generally independent arguments for energy expression. Indeed, because of phase trajectory mixing (3) specific energy and coordinate may not have mutual correspondence.

Then for conditional probability density we have modified equation:

$$\rho(\vec{x}^{r},|\vec{x}^{r-1},{}_{\Delta}t_{r}) = \delta(\vec{x}^{r} - \vec{x}^{r-1}) + a(\vec{x}^{r-1},{}_{\Delta}t_{r},t_{r}) \cdot \delta'(\vec{x}^{r} - \vec{x}^{r-1}) + \frac{1}{2} \cdot b(\vec{x}^{r-1},{}_{\Delta}t_{r},t_{r}) \cdot \delta''(\vec{x}^{r} - \vec{x}^{r-1}) \quad (20)$$

At the same time derivative of probability $\rho(\vec{x},t)$ can be represented, using Chapman Kolmogorov equation (21) in the following way:

$$\frac{\partial \rho(\vec{x}^{r},t_{r})}{\partial t} = \lim_{{}_{\Delta}t \to 0}\left[\frac{1}{{}_{\Delta}t_{r}} \cdot \left(\int d\vec{x}^{r-1} \cdot \rho(\vec{x}^{r-1},t_{r}) \cdot \left(\rho(\vec{x}^{r},\vec{x}^{r-1},{}_{\Delta}t_{r}) - \delta(\vec{x}^{r} - \vec{x}^{r-1})\right)\right)\right] \quad (21)$$

In this equation $\rho(\vec{x}^{r},\vec{x}^{r-1},{}_{\Delta}t_{r})$ is transitional probability density. Substitution of (20) into relation (21) gives extended *FPK* equation (*EFPK*) (Ref.8):

$$\frac{\partial \rho(\vec{x}^{r},t_{r-1})}{\partial t} = \frac{1}{2} \cdot \frac{\partial}{\partial \vec{x}^{r}}\left(B(\vec{x}^{r},t_{r-1}) \cdot \frac{\partial \rho(\vec{x}^{r},t_{r-1})}{\partial \vec{x}^{r}}\right) \quad (22)$$

Variation of ${}_{\Delta}t$ such that ${}_{\Delta}t \succ\succ {}_{\Delta}t_{\min}$ allows representing equation (19) in asymptotic form for ${}_{\Delta}t \to \infty$ and receiving abnormal transport equation:

$$\left\langle\left\langle \left({}_{\Delta}\vec{x}\right)^{2} \right\rangle\right\rangle = 2 \cdot B(\vec{x},t) \cdot {}_{\Delta}t = B'(\vec{x},t) \cdot (t - t_{0}) \quad (23)$$

Root extraction of equation both parts leads to law of abnormal diffusion (Ref.9):

$$\sigma_{x} = D(\vec{x},t) \cdot \sqrt{t - t_{0}} \quad (24)$$

In this relation $D(\vec{x},t) = \sqrt{2 \cdot B(\vec{x},t)}$ is anomalous diffusion factor. Traditionally abnormal diffusion law is explained, artificially introducing fractal *FPK* equation – *FFPK* (Ref.9).

Let's consider uniform state for averaged characteristic energy of chaotic system: $\left\langle\left\langle \varepsilon(\vec{x},t) \right\rangle\right\rangle = f(t)$. Expression (19) allows receiving correspondent form of transport coefficient: $B(t) = 2 \cdot {}_{\Delta}t_{\min} \cdot f(t)$. In this case Fourier decomposition of one dimensional local *EFPK* equation (22) may be represented in the following way:

$$\int i\omega \cdot \rho_{2}(x_{j},\omega) \cdot \exp(-i\omega \cdot t) d\omega = -\int B_{0}^{j}(t) \cdot (k_{j})^{2} \rho_{1}(k_{j},t) \cdot \exp(-ik_{j} \cdot x_{j}) dk_{j} \qquad j = \overline{1,P} \quad (25)$$

Here $B_{0}^{j}(t) = \frac{1}{2} \cdot B^{j}(t)$ is corresponding modified transport coefficient for j dimension. Amplitudes of Fourier decomposition are outlined through expressions (26) and (27):

$$\rho_{1}(k_{j},t) = \frac{1}{\sqrt{2\pi}} \cdot \int_{\Gamma_{j}} \rho_{1}(x_{j},t) \cdot \exp(ik_{j} \cdot x_{j}) dx_{j} \quad (26)$$

$$\rho_{2}(x_{j},\omega) = \frac{1}{\sqrt{2\pi}} \cdot \int_{0}^{T} \rho(x_{j},t) \cdot \exp(it \cdot x_{j}) dt \quad (27)$$

Second Fourier decomposition gives relations (28) and (29) with equivalent operator's kernels $\rho'(k_{j},\omega)$ - (28), (29).

$$\rho_{1}(k_{j},t) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \rho'(k_{j},\omega) \cdot \exp(-i\omega \cdot t) d\omega \quad (28)$$

$$\rho_{2}(x_{j},\omega) = \frac{1}{\sqrt{2\pi}} \int_{-K}^{K} \rho'(k_{j},\omega) \cdot \exp(-ik_{j} \cdot x_{j}) dk_{j} \quad (29)$$

Integrals limits are defined according to Kotelnikov theorem: $\Omega = \dfrac{1}{2 \cdot {}_{\Delta}t_{\min}}$, $K = \dfrac{1}{2 \cdot \left({}_{\Delta}\Gamma_j\right)_{\min}}$.

Substitution of expressions (28) and (29) into equation (25) gives wave packet form:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} -i\omega \cdot \rho'(k_j,\omega)\cdot \exp(i\omega\cdot t + ik_j \cdot x_j) dk_j d\omega = -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} B_0^j(t)\cdot (k_j)^2 \rho'(k_j,\omega)\cdot \exp(i\omega\cdot t + ik_j\cdot x_j) dk_j d\omega \quad (30)$$

General arbitrariness of integration limits finally allows representing $\omega(k_i)$ law:

$$\omega(k_j) = i \cdot B_0^j(t)\cdot (k_j)^2 \quad (31)$$

As it follows from outlined expression nonlinear dispersion law (31) is mandatory property of uniform chaotic state. Allocation of $\omega(k_i)$ real part leads to equation (32):

$$\mathrm{Re}[\omega(k_j)] = -2\cdot B_0^j(t)\cdot [\mathrm{Im}(k_j)]\cdot[\mathrm{Re}(k_j)] = -B^j(t)\cdot[\mathrm{Im}(k_j)]\cdot[\mathrm{Re}(k_j)] \quad (32)$$

Positiveness of physically measured quantities $\mathrm{Re}(\omega)$ and $\mathrm{Re}(k)$ allows receiving following property of complex wave number: $\mathrm{Im}(k_j) \le 0$. Here positiveness of specific energy $\varepsilon(\vec{x},t)$ and consequently transport coefficient $B^j(t)$ are taken into account. First Fourier decomposition of probability density then can be given by relation (33):

$$\rho(x_j,t) = \int \rho_1(k_j,t)\cdot \exp[|\mathrm{Im}(k_j)|\cdot x_j]\exp[-i\,\mathrm{Re}(k_j)\cdot x_j]dk_j \quad (33)$$

Here $|\mathrm{Im}(k_j)|$ as positive space increment shows existence of space instability for probability density amplitude $\rho_1(k_j,t)$. Let's consider the imaginary relation for both parts of equation (31):

$$\mathrm{Im}[\omega(k_j)] = B_0^j(t)\cdot\left([\mathrm{Im}(k_j)]^2 + [\mathrm{Re}(k_j)]^2\right) = \alpha(t) \quad (34)$$

Positiveness of time increment shows time instability of probability density:

$$\rho(x_j,t) = \int \rho_2(x_j,\omega)\cdot \exp[|\alpha|\cdot t]\exp[-i\,\mathrm{Re}(k_j)\cdot x_j]dk_j \quad (35)$$

As we see space – time instability of probability density is defined by mandatory nonlinear dispersion law (32) of chaotic system. Given instability leads to appearance of probability cavities in phase space Γ_i - phase space attractors where particles density grows up. This process continues up to the moment when specific energy and transport factor achieves space inhomogeneity: $\langle\langle\varepsilon(t)\rangle\rangle \Rightarrow \langle\langle\varepsilon(\vec{x},t)\rangle\rangle$, $B^j(t) \Rightarrow B^j(\vec{x},t)$. Since that local EFPK equation has to be considered in general form of relation (22).

III. UNCERTAINTY RELATION OF PHASE STATE

It was mentioned above, that two possible types of phase trajectories are possible in frame of characteristic vector description: bijection $\vec{x} \leftrightarrow t$ and multivalued mapping. Each type is characterized by specific energy in form of $\langle\langle\varepsilon(\vec{x}(t))\rangle\rangle$ and $\langle\langle\varepsilon(\vec{x},t)\rangle\rangle$ correspondingly. Given division allows introducing qualitative properties of dynamic system basing on transport parameter $B(\vec{x},t) = 2\cdot{}_{\Delta}t_{\min}\cdot\langle\langle\varepsilon(\vec{x},t)\rangle\rangle$. We shall designate phase states as bijection states of constant averaged energy $\langle\langle\varepsilon(\vec{x})\rangle\rangle$, i.e. energy without explicit time dependence. Then multivalued mapping corresponds to transitional motion with phase trajectory mixing.

Appearance of transitional state is defined by first return of characteristic vector. Phase transitions are described by *EFPK* equation (22). In terms of diffusion factors given types of motion are also designated as normal and abnormal diffusion (Ref.9).

Let's consider case of uniform phase state with fixed boundary: $\langle\langle \varepsilon(\vec{x})\rangle\rangle = const$, $\Gamma = const$. This phenomenon appears under condition of phase space time stability of probability cavity, as it was shown in Section II. Description of corresponding system state can be realized in frame of normal diffusion *FPK* equation (17) for life time of phase state: $t \in \left[t_1^f, t_2^f\right]$. For selected dimension j we can represent (17) as uniform linear diffusion equation:

$$\frac{\partial \rho(x_j,t)}{\partial t} = \frac{1}{2}\cdot B_j^l \frac{\partial^2 \rho(x_j,t)}{\partial x_j^2} \quad x_j \in [0, L_j] \quad t \in \left[t_1^f, t_2^f\right] \tag{36}$$

Solution can be searched in form of Fourier expansion series (37),(38) which satisfies boundary condition and initial state: $\rho(0,t) = \rho(L_j,t)$, $\rho(x_j,0) = \rho_0(x_j)$.

$$\rho(x_j,t) = \sum_{j=1}^{N} c_j^l(t)\cdot \sin\left[\left(\frac{\pi\cdot l}{L_j}\right)\cdot x_j\right] \tag{37}$$

$$c_j^l(t) = \frac{2}{L_j}\cdot \int_0^{L_j} \overline{\rho}(\zeta,t)\cdot \sin\left[\left(\frac{\pi\cdot l}{L_j}\right)\cdot \xi\right] d\xi \tag{38}$$

Substitution of (37) into (36) gives equations (39) and (40) for Fourier coefficients:

$$\sum_{l=1}^{N} \sin\left[\left(\frac{\pi\cdot l}{L}\right)\cdot x_j\right]\cdot \left(\frac{\partial c_j^l(t)}{\partial t} + \frac{B_j^l\cdot c_j^l(t)}{2}\cdot \left(\frac{\pi\cdot l}{L_j}\right)^2\right) = 0 \tag{39}$$

$$\frac{\partial c_j^l(t)}{\partial t} = -\frac{B_j^l\cdot c_j^l(t)}{2}\cdot \left(\frac{\pi\cdot l}{L_j}\right)^2 \tag{40}$$

Corresponding values of transport factor are represented by relation (41):

$$B_j^l = -\frac{2}{c_j^l(t)}\cdot \frac{dc_j^l(t)}{dt}\cdot \left(\frac{L}{\pi\cdot l}\right)^2 \tag{41}$$

According to (41) coefficients $c_j^l(t)$ satisfies following condition: $\frac{1}{c_j^l(t)}\cdot \frac{dc_j^l(t)}{dt} = -\phi_j^l = const$,

$\phi_j^l = \frac{B_j^l}{2}\cdot \left(\frac{\pi\cdot l}{L_j}\right)^2$ .Consequently for $c_j^l(t)$ we have: $c_j^l(t) = c_j^l(0)\cdot \exp\left(-\phi_j^l\cdot t\right)$.

Taking into account relation (19) for averaged specific energy we have got following expression for discrete energy spectrum:

$$\langle\langle \varepsilon_j^l\rangle\rangle = \frac{2\cdot \phi_j^l}{{}_{\Delta}t_{\min}}\cdot \left(\frac{L_j}{\pi\cdot l}\right)^2 \tag{42}$$

Let's designate $\omega = \frac{2\pi}{{}_{\Delta}t_{\min}}$, then for energy derivative we have relation (43), given below.

$$\frac{\partial\langle\langle\varepsilon_j^l\rangle\rangle}{\partial\omega}=\frac{\phi_j^l}{\pi^3}\cdot\left(\frac{L_j}{l}\right)^2 \qquad (43)$$

Under conditions of finite phase space and time resolution (1), (5) for chaotic system we can modify given relation into form of (44).

$$_\Delta\omega=\frac{\pi^3}{\phi_j^l}\cdot\left(\frac{l}{L_j}\right)^2\cdot{}_\Delta\langle\langle\varepsilon_j^l\rangle\rangle \qquad (44)$$

For dynamic description with perfect accuracy initial probability density is represented as Dirac function (Section II, Item 4):

$$\rho(x_j(t_0))=\delta(x_j(t)-x_j(t_0)) \qquad (45)$$

In vicinity of t_0 ($t\to t_0$) projection of characteristic vector $x_j(t_0)$ is bijection $x_j\leftrightarrow t$. Normalization condition for $\rho(x_j(t_0))$ then can be represented in the following way:

$$\int_0^{L_j}\delta(x_j(t)-x_j(t_0))dx_j=\int_0^{L_j}\left(\sum_k\frac{\delta(t-t_0)}{\left|\partial x_j^k(t)/\partial t\right|}\right)dx \qquad (46)$$

Dirac functional is represented here through time argument. Index *k* corresponds to zeros of function $x_j(t)$. In considered case we have only one value of argument, corresponding to zero - t_0. Then relation (46) can be modified in the following way:

$$\lim_{t\to t_0}\left[\int_0^{L_j}\left(\sum_k\frac{\delta(t-t_0)}{\left|\partial x_j^k(t)/\partial t\right|}\right)dx\right]=\frac{\partial x_j(t)/\partial t}{\left|\partial x_j(t)/\partial t\right|}\int_0^{T(L_j)}\delta(t-t_0)dt \qquad (47)$$

As we can see in vicinity of t_0 ($t\to t_0$, $x_j\to x_j(t_0)$) space - time bijection allows introducing probability density correspondence:

$$\delta(x_j(t)-x_j(t_0))\leftrightarrow\delta(t-t_0)\cdot sign(\partial x_j(t)/\partial t) \qquad (48)$$

Finite space - time resolution allows substitution of Delta function by its discrete alternative – rectangular pulse. Without loosing of generality we may assume that $t_0=0$:

$$x_j(t)=\begin{cases}C_1 & x\prec\left|({}_\Delta x_j)_{\min}\right|\\ 0 & x\ge\left|({}_\Delta x_j)_{\min}\right|\end{cases}\qquad t(x_j)=\begin{cases}C_2 & t\prec\left|{}_\Delta t_{\min}\right|\\ 0 & t\ge\left|{}_\Delta t_{\min}\right|\end{cases} \qquad (49)$$

According to normalization condition coefficients C_1 and C_2 can be expressed in the following way: $C_1=\frac{1}{({}_\Delta x_j)_{\min}}$, $C_2=\frac{1}{({}_\Delta t)_{\min}}$. For given video pulse relation, connecting characteristic width of spectrum and pulse width can be written in the following way:

$$_\Delta\omega\cdot{}_\Delta t\ge 2\pi \qquad (50)$$

Substitution of expression (44) into (50) gives relation (51).

$$_\Delta\langle\langle\varepsilon_j^l\rangle\rangle\cdot{}_\Delta t=\frac{\phi_j^l}{\pi^3}\cdot\left(\frac{L_j}{l}\right)^2\cdot{}_\Delta\omega\cdot{}_\Delta t \qquad (51)$$

One allows receiving connection between energy and time resolution – (52).

$$ {}_{\Delta}\left\langle\left\langle \varepsilon_j^l \right\rangle\right\rangle \cdot {}_{\Delta} t \geq \frac{2\phi_j^l}{\left(k_j^l\right)^2} \tag{52}$$

Here with accordance to (37) wave number $k_j^l = \dfrac{\pi \cdot l}{L_j}$ is introduced.

Expression for auxiliary function is represented below:

$$ \phi_j^l = \frac{B_j^l}{2} \cdot \left(\frac{\pi \cdot l}{L_j}\right)^2 = \frac{\left\langle\left\langle \left({}_{\Delta} x_j\right)^2 \right\rangle\right\rangle}{4 \cdot {}_{\Delta} t_{\min}} \cdot \left(\frac{\pi \cdot l}{L_j}\right)^2 \tag{53}$$

Then relation (52) can be modified in following way:

$$ {}_{\Delta}\left\langle\left\langle \varepsilon_j^l \right\rangle\right\rangle \cdot {}_{\Delta} t \geq \left(B_j^l\right)_{\min} \tag{54}$$

Here $\left(B_j^l\right)_{\min}$ is minimal transport factor for j dimension. In frame of diffusion representation expression (54) can be represented in given form (lower indexes are omitted):

$$ {}_{\Delta}\left\langle\left\langle \varepsilon^l \right\rangle\right\rangle \cdot {}_{\Delta} t \geq \frac{(D_0)^2}{2} \tag{55}$$

D_0 is minimal diffusion factor for j dimension of phase state.

Let's receive connection between space and time uncertainties. Satisfaction of ergodicity condition for chaotic state allows gives ability to modify expression (19):

$$ \left\langle\left\langle \varepsilon(\vec{x}^r, \vec{x}^{r-1}, t) \right\rangle\right\rangle = \int_{\Gamma(T)} \varepsilon(\vec{x}^r, \vec{x}^{r-1}, t) \cdot \rho(\vec{x}^r, \vec{x}^{r-1}, {}_{\Delta} t_r) d\vec{x}^{r-1} = \frac{1}{T} \cdot \int_0^T \varepsilon(\vec{x}^r, \vec{x}^{r-1}, t) dt = \overline{\varepsilon(\vec{x}^r, \vec{x}^{r-1}, t)} \tag{56}$$

Upper underscore here means time averaging. Space – time independence of phase state leads to space independence in $\overline{\varepsilon(\vec{x}^r, \vec{x}^{r-1}, t)}$. For arbitrariness of integration time this means that relation (56) can be simplified in the following way:

$$ \varepsilon(\vec{x}^r, t) = \left\langle\left\langle \varepsilon(\vec{x}^r, t) \right\rangle\right\rangle \tag{57}$$

Finite differential for energy then can be expressed through momentum: ${}_{\Delta}\left\langle\left\langle \varepsilon(\vec{x}^r, t) \right\rangle\right\rangle = 2p(\vec{x}^r, t) \cdot {}_{\Delta} p(\vec{x}^r, t)$. Substitution of given relation in (55) allows receiving differential equation for momentum:

$$ p^l(\vec{x}, t) \cdot {}_{\Delta} p^l(\vec{x}, t) \cdot {}_{\Delta} t \geq \frac{(D_0)^2}{4} \tag{58}$$

Momentum is expressed in finite form: $p^l(\vec{x}, t) = \dfrac{{}_{\Delta} \vec{x}^l(t)}{{}_{\Delta} t}$. Substitution of this expression in relation (58) gives connection between ${}_{\Delta} p^l$ and ${}_{\Delta} \vec{x}^l$:

$$ {}_{\Delta} p^l \cdot {}_{\Delta} x^l \geq \frac{(D_0)^2}{4} \tag{59}$$

Less strict form of relation (59) allows uniform representing of (59) and (55), given below.

$$ {}_{\Delta} p^l \cdot {}_{\Delta} x^l \geq \frac{(D_0)^2}{2} \tag{60}$$

$$ {}_{\Delta} \varepsilon^l \cdot {}_{\Delta} t \geq \frac{(D_0)^2}{2} \tag{61}$$

Equations (60) and (61) show connections between uncertainties of coordinate – momentum and time - energy definition correspondingly. It may be useful to note that any of given uncertainties may be determined as corresponding standard deviations: ${}_\Delta p^l = \sigma_{p^l}$, ${}_\Delta x^l = \sigma_{x^l}$, ${}_\Delta \varepsilon^l = \sigma_{\varepsilon^l}$, ${}_\Delta t = \sigma_t$.

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